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Method of the Integral Error Functions for the Solution of the One- and Two-Phase Stefan Problems and its Application

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Abstract. The analytical solutions of the one- and two-phase Stefan problems are found in the form of series containing linear combinations of the integral error functions which satisfy a priori the heat equation. The unknown coefficients are defined from the initial and boundary conditions by the comparison of the like power terms of the series using the Faa di Bruno formula. The convergence of the series for the temperature and for the free boundary is proved. The approximate solution is found using the replacement of series by the corresponding finite sums and the collocation method. The presented test examples confirm a good approximation of such approach. This method is applied for the solution of the Stefan problem describing the dynamics of the interaction of the electrical arc with electrodes and corresponding erosion.

1. Introduction

A wide range of transient phenomena in the field of heat and mass transfer, low-temperature plasma, filtration and other dynamical processes associated with the phase transformation of materials are considered in many papers [1-12]. From the theoretical point of view, these problems are among the most challenging problems in the theory of non-linear parabolic equations (Stefan type problems). As a rule, the numerical methods are predominant for the solution of the concrete problems. However, development of new analytical and approximate methods is very important especially for various applications because it enables one to analyze an interrelationship of different input parameters and their influence on the dynamics of the investigated phenomena. Moreover, the analytical methods can give a good idea for the elaboration and development of new numerical methods.

In some specific cases the Stefan problem can be reduced to integral equations using the heat potentials [3,4]. However, if the domain degenerates at the initial time, an additional difficulty appears because of the singularity of the final integral equations. The method presented in this paper is effective exactly for such kind of Stefan problems with degenerating initial domains.

One of the most important areas of application of the proposed method is the mathematical modeling of electric contact phenomena at electrical arcing. The investigated processes are of very short duration (micro- and nanosecond range) such that their experimental study is very difficult [13-15]. In this case only mathematical modeling can give an idea of their dynamics. Thus, the need of modeling is due not only to

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the need to optimize the planning experiment, but also due to the impossibility to use a different approach since experimental investigation is very difficult or even impossible.

1.1. Integral error functions and heat polynomials

The integral error functions are determined by the recurrent formulas

$$\mathrm{i}^n\mathrm{erfc}(x) = \int_x^\infty \mathrm{i}^{n-1}\mathrm{erfc}(v)dv, \quad n = 1,\ 2,\ ...,\ \mathrm{i}^{\,0}\mathrm{erfc}(x) \equiv \mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-v^2)dv. \tag{1}$$

Their combination can be expressed by polynomials

$$\mathrm{i}^n\mathrm{erfc}(-x) + (-1)^n\mathrm{i}^n\mathrm{erfc}(x) = \sum_{m=0}^{[\frac{n}{2}]} \frac{x^{n-2m}}{2^{2m-1}m!(n-2m)!} \tag{2}$$

which are called the heat polynomials.

Integral error functions and heat polynomials are very useful for investigation of heat transfer, diffusion and other phenomena which can be described by the equation

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} \tag{3}$$

in a region $D : (t > 0, 0 < x < \alpha(t))$ with free boundary $x = \alpha(t)$.

The functions

$$u(x,t) = \sum_{n=0}^{\infty}[A_n u_n(x,t) + B_n u_n(-x,t)] \tag{4}$$

where

$$u_n(x,t) = t^{\frac{n}{2}}\mathrm{i}^n\mathrm{erfc}(\frac{\pm x}{2a\sqrt{t}})$$

satisfy the equation (3) for any constants A_n,B_n. We can choose these constants to satisfy the boundary conditions at $x = 0$ and $x = \alpha(t)$, if the given boundary functions can be expanded into Taylor series with powers of t or $\sqrt{t}$.

Using expression (2) we can represent (4) in the form

$$u(x,t) = \sum_{n=0}^{\infty}\{A_{2n}\sum_{m=0}^{n} x^{2n-2m}t^m\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n} x^{2n-2m+1}t^m\beta_{2n+1,m}\} \tag{5}$$

where

$$\beta(n,m) = \frac{1}{2^{n+m-1}m!(n-2m)!}.$$

1.2. Approximate solution of test problem by the heat polynomials using collocation method

Let the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{6}$$

be given on the domain with the moving boundary $D:\ \{0 < x < t, \quad 0 < t < 1\}$ subjected to the boundary conditions

$$u|_{x=0} = e^t, \tag{7}$$

$$u|_{x=t} = 1 \tag{8}$$

and the fitting condition

$$u|_{t=0} = 1. \tag{9}$$

We can consider the approximate solution in the form (5) which satisfies a priori the heat equation (3).

Satisfying the boundary conditions (7) we get

$$e^t = \sum_{n=0}^{k} A_{2n}\beta_{2n,n}t^n \tag{10}$$

and from (8)

$$1 = \sum_{n=0}^{k} \left\{ A_{2n} \sum_{m=0}^{n} \beta_{2n,m}t^{2n-m} + A_{2n+1} \sum_{m=0}^{n} \beta_{2n+1,m}t^{2n-m+1} \right\}, \tag{11}$$

respectively.

To find the unknown coefficients A_{2n},A_{2n+1} we use the method of collocations. Taking $k = 5$ and satisfying the equations (10) and (11) at $t = t_i = \frac{i}{5}, i = 0, 1, 2, 3, 4, 5$ we can find the values for A_{2n},A_{2n+1}. Figure 1 depicts the graphs of approximate function

$$v(t) = \left\{ \sum_{n=0}^{5} \left\{ A_{2n} \sum_{m=0}^{n} x^{2n-2m}t^m\beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1}t^m\beta_{2n+1,m} \right\} \right\} \Bigg|_{x=0}$$

and the original function $e^t = \exp(t)$ at the boundary $x = 0$, which are almost identical.

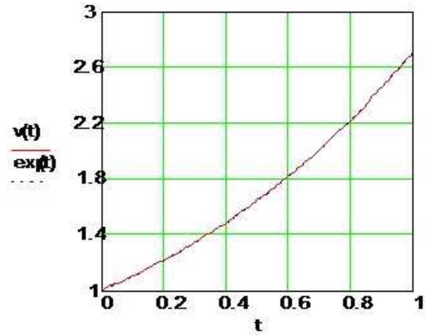

Figure 1: The graph of approximate function

The greatest error of approximation is in the neighborhood of zero. The graphs for this neighborhood are presented in Figure 2.

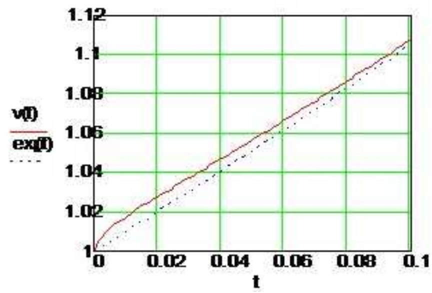

Figure 2: The graph for neighborhood of zero

One can see that the error of approximation is less than 1%.

A similar situation can be observed at the second boundary $x = t$. The graphs of the functions $g(t) = 1$ and $W(t) = \left\{\sum_{n=0}^{5} \left\{A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + \quad A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m}\right\}\right\}\Big|_{x=t}$ are presented in Figure 3 and Figure 4.

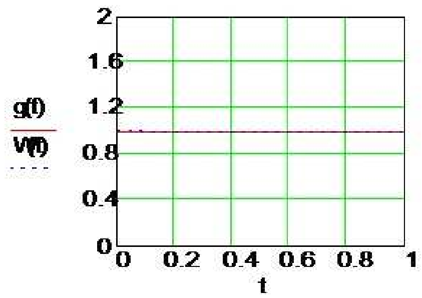

Figure 3: The graphs of the functions $g(t)$ and $W(t)$

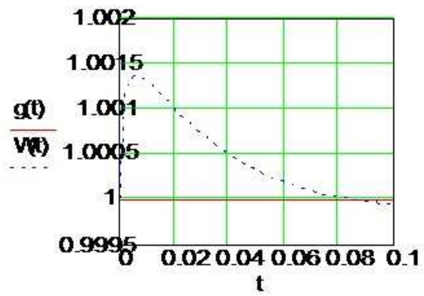

Figure 4: The graphs of the functions $g(t)$ and $W(t)$

The greatest error of approximation is less than 0.15 %.

Thus if we replace the original functions e^t and 1 by approximate functions, then according to the maximum principle for the heat equation the error of approximation of the solution in the whole domain is not greater than the error on the boundaries.

2. Analytical Solution of the One-Phase Stefan Problem by Heat Polynomials and Integral Error Functions

Let us consider the problem for the heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, 0 < x < \alpha(t), 0 < t < \infty, \tag{12}$$

with the boundary conditions

$$-\lambda \frac{\partial u}{\partial x}\Big|_{x=0} = P(t), 0 < t < \infty, \tag{13}$$

$$u_{x=\alpha(t)} = U_m, 0 < t < \infty, \tag{14}$$

the Stefan condition

$$-\lambda \frac{\partial u}{\partial x}\bigg|_{x=\alpha(t)} = L\frac{d\alpha(t)}{dt}, 0 < x < \alpha(t), \tag{15}$$

and the concordance condition

$$u(0,0) = 0. \tag{16}$$

Here $P(t)$ is the heat flux entering the electrode from the electric arc and U_m is the melting temperature of the electric contact material. Power balance is described by Stefan's condition (15). It is supposed that the arc heat flux is consumed only for melting of the solid region. The temperature of the solid domain is assumed to be constant, that is valid for such refractory metals like wolfram.

We represent the temperature distribution for the problem (12 – 16) in the form of a combination of the heat polynomials and the integral error functions

$$u(x,t) = \sum_{n=0}^{\infty} C_n \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1} + \sum_{n=0}^{\infty} A_n (2a\sqrt{t})^n \left[\mathrm{i}^n \mathrm{erfc}\left(-\frac{x}{2a\sqrt{t}}\right) + \mathrm{i}^n \mathrm{erfc}\left(\frac{x}{2a\sqrt{t}}\right) \right] \tag{17}$$

while the free boundary is represented in the form of a power series

$$\alpha(t) = \sum_{n=1}^{\infty} \alpha_n t^{\frac{n}{2}}. \tag{18}$$

Taking k times derivatives of (17) at $t = 0$, we get

$$C_k = -\frac{1}{2\lambda\beta_{2k+1}} P^{(k)}(0). \tag{19}$$

Making the substitution $\sqrt{t} = \tau$ for the heat polynomial of (17) we get

$$\sum_{n=0}^{\infty} C_n \sum_{m=0}^{n} [\alpha(\tau)]^{2n-2m+1} \tau^{2m} \beta_{2n+1} = \sum_{n=0}^{\infty} C_n \sum_{s_1+s_2+\ldots+s_k=2n-2m+1} \begin{pmatrix} 2n-2m+1 \\ s_1, s_2, \ldots, s_k \end{pmatrix} \alpha_1^{s_1} \alpha_2^{s_2} \ldots \alpha_k^{s_k} \tau^{(s_1+2s_2+\ldots+ks_k+2m)}.$$

To find A_n coefficients we utilize Leibniz rule for the k-th derivative of product and Faa Di Bruno formula for the k-th derivative of the composite function, thus

$$\left[\sum_{n=0}^{\infty} A_n (2a\tau)^n \, \mathrm{i}^n \mathrm{erfc}\left(\pm\frac{\alpha(\tau)}{2a\tau}\right)\right]^{(k)}\bigg|_{\tau=0} = \sum_{n=0}^{k} \frac{2^{\frac{n}{2}} k!}{(k-n)!} [\mathrm{i}^n \mathrm{erfc}(\pm\delta)]^{(k-n)}$$
$$= \sum_{n=0}^{k} \frac{2^{\frac{n}{2}} k!}{(k-n)!} \sum_{m=1}^{k-n} [\mathrm{i}^n \mathrm{erfc}(\pm\delta)]^{(m)} B_{k-n,m}\left((\pm\delta)', (\pm\delta)''^{(k-n-m+1)}\right)\bigg|_{\delta=0}$$

where $B_{k-n,m}$ is Bell's polynomial

$$B_{k-n,m} = \sum \frac{(k-n)!}{j_1! j_2! \ldots j_{k-n-m+1}!} (\pm\delta_1)^{j_1} (\pm\delta_2)^{j_2} \ldots (\pm\delta_{k-n-m+1})^{j_{k-n-m+1}}.$$

Here $\delta = \frac{\alpha(\tau)}{2a\tau}$, $\delta_n = \frac{\alpha_n}{2a}, n = 1, 2, 3, \ldots, j_1 + j_2 + \ldots + j_{k-n-m+1} = m$, $j_1 + 2j_2 + \ldots + (k-n-m+1)j_{k-n-m+1} = k-n$, and $[\mathrm{erfc}(\pm\delta)]^{(m)}\big|_{\delta=0} = (\pm 1)^m \frac{\Gamma\left(\frac{n-m+1}{2}\right)}{(n-m)!\sqrt{n}}$.

Ultimately, taking k times derivative of (17) at $\tau = 0$ we get

$$\left[\sum_{n=0}^{\infty} A_n (2a\tau)^n [\mathrm{i}^n \mathrm{erfc}(-\delta(\tau)) + \mathrm{i}^n \mathrm{erfc}(\delta(\tau))]\right]^{(k)}\Bigg|_{\tau=0} = \left[u_m - \sum_{n=0}^{\infty} C_n \sum_{m=0}^{n} (\alpha(\tau))^{2n-2m+1} \tau^{2m} \beta_{2n-2m+1}\right]^{(k)}\Bigg|_{\tau=0}$$

and

$$
\begin{aligned}
&\sum_{n=0}^{k} A_n \frac{2^{\frac{n}{2}} k!}{(k-n)!} \left(\sum_{m=1}^{k-n} (-1)^m \frac{\Gamma\left(\frac{n-m+1}{2}\right)}{(n-m)!\sqrt{n}} \sum \frac{(k-n)!}{j_1! j_2! \ldots j_{k-n-m+1}!} (-\delta)_1^{j_1} (-\delta)_2^{j_2} \ldots (-\delta)_{k-n-m+1}^{j_{k-n-m+1}} \right. \\
&\left. + \sum_{m=1}^{k-n} \frac{\Gamma\left(\frac{n-m+1}{2}\right)}{(n-m)!\sqrt{n}} \sum \frac{(k-n)!}{j_1! j_2! \ldots j_{k-n-m+1}!} \delta_1^{j_1} \delta_2^{j_2} \ldots \delta_{k-n-m+1}^{j_{k-n-m+1}} \right) \\
&= -\sum_{n=0}^{k} C_n \sum_{m=0}^{n} \sum_{s_1+s_2+\ldots+s_k=2n-2m+1}^{n} \binom{2n-2m+1}{s_1, s_2, \ldots, s_k} \alpha_1^{s_1} \alpha_2^{s_2} \ldots \alpha_k^{s_k} \tau^{(s_1+2s_2+\ldots+ks_k+2m)}.
\end{aligned} \tag{20}
$$

Thus A_k can be determined from the formula (20).

Taking both sides of (18) at $\tau = 0$, in the same manner, we determine α_k coefficients from the recurrent formula that is produced by the equation

$$
\alpha_{k+1} = \left.\left[-\frac{\lambda}{L} \left.\frac{\partial u}{\partial x}\right|_{x=\alpha(\tau)} \right]^{(k)} \right|_{\tau=0}, k = 0, 1, 2, \cdots . \tag{21}
$$

2.1. Convergence

Let $\alpha(t_0) = \alpha_0$ for any time $t = t_0$. Then the series $\sum_{n=0}^{\infty} A_n \left(2a\sqrt{t_0}\right)^n \left[\mathrm{i}^n \mathrm{erfc}\left(-\frac{\alpha_0}{2a\sqrt{t_0}}\right) + \mathrm{i}^n \mathrm{erfc}\left(\frac{\alpha_0}{2a\sqrt{t_0}}\right)\right]$ should be convergent because of the identity $u = U_m$ on the interphase. Therefore, there exists a constant C_1, independent of n, such that

$$
|A_n| < C_1 / \left(2a\sqrt{t_0}\right)^n \left[\mathrm{i}^n \mathrm{erfc}\left(-\delta_0\right) + \mathrm{i}^n \mathrm{erfc}\left(\delta_0\right)\right], \qquad \delta_0 = \frac{\alpha_0}{2a\sqrt{t_0}}.
$$

The function $\mathrm{i}^n\mathrm{erfc}(-\delta)+\mathrm{i}^n\mathrm{erfc}(\delta)$ is a monotonically increasing positive function, therefore $\mathrm{i}^n\mathrm{erfc}(-\delta)+\mathrm{i}^n\mathrm{erfc}(\delta) < \mathrm{i}^n\mathrm{erfc}(-\delta_0)+\mathrm{i}^n\mathrm{erfc}(\delta_0)\,, 0 < \delta < \delta_0$.

Thus

$$
\left| A_n \left(2a\sqrt{t_0}\right)^n \left[\mathrm{i}^n \mathrm{erfc}(-\delta) + \mathrm{i}^n \mathrm{erfc}\left(\delta\right)\right] \right| < C_1 \sum_{n=0}^{\infty} \left(\frac{t}{t_0}\right)^{n/2} \frac{\mathrm{i}^n \mathrm{erfc}(-\delta) + \mathrm{i}^n \mathrm{erfc}\left(\delta\right)}{\mathrm{i}^n \mathrm{erfc}(-\delta_0) + \mathrm{i}^n \mathrm{erfc}\left(\delta_0\right)} < C_1 \sum_{n=0}^{\infty} \left(\frac{t}{t_0}\right)^{n/2}.
$$

These are geometric series and the series for $u(x,t)$ converges for all $x < \alpha_0$ and $t < t_0$.

The series for $\alpha(t)$ can be estimated similarly.

This means that $u(x,t)$ is bounded, thus the series for $\alpha(t)$ converges for all $t < t_0$.

3. The Two-Phase Spherical Stefan Problem

Let us consider the two-phase spherical Stefan problem, which enables us to describe the heat transfer phenomena in electrical contacts during arcing. The heat flux *P(t)* entering the sphere of radius b melts the contact material (liquid zone $b < r < \alpha(t)$) and passes further through the solid zone $\alpha(t) < r < \infty$.

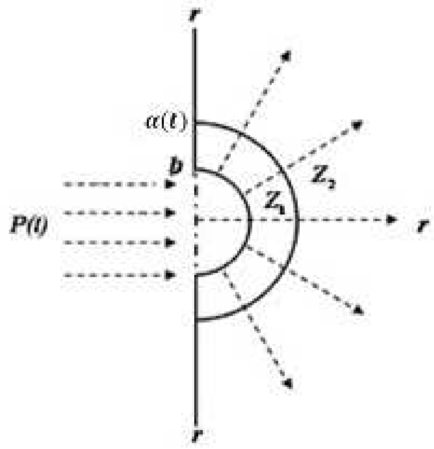

Figure 5: The heat flux $P(t)$ entering the sphere of the radius b

The heat equations for each zone are

$$\frac{\partial \theta_1}{\partial t} = a_1^2 \left(\frac{\partial^2 \theta_1}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial \theta_1}{\partial r} \right),\ b < r < \alpha(t), \tag{22}$$

$$\frac{\partial \theta_2}{\partial t} = a_2^2 \left(\frac{\partial^2 \theta_2}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial \theta_2}{\partial r} \right),\ \alpha(t) < r < \infty. \tag{23}$$

They should be solved for the conditions

$$\theta_1(b, 0) = T_m, \tag{24}$$

$$\theta_2(r, 0) = f(r), \tag{25}$$

$$f(b) = T_m, \tag{26}$$

$$\alpha(0) = b, \tag{27}$$

$$f(\infty) = 0, \theta_2(\infty, t) = 0, \tag{28}$$

$$r = b : -\lambda_1 \frac{\partial \theta_1(b, t)}{\partial r} = P(t), \tag{29}$$

$$r = \alpha(t) : \theta_1(\alpha(t), t) = T_m, \tag{30}$$

$$\theta_2(\alpha(t), t) = T_m, \tag{31}$$

$$-\lambda_1 \frac{\partial u_1}{\partial r} = -\lambda_2 \frac{\partial u_2}{\partial r} + L\gamma \frac{d\alpha\,(t)}{dt}. \tag{32}$$

By making the substitution $\theta_i = \frac{U_i}{r} + T_m$ and $r = x + b, \beta(t) = \alpha(t) - b$ in $(22 - 32)$ we reduce this problem to the following problem:

$$\frac{\partial U_1}{\partial t} = a_1^2 \frac{\partial^2 U_1}{\partial x^2}, \tag{33}$$

$$\frac{\partial U_2}{\partial t} = a_2^2 \frac{\partial^2 U_2}{\partial x^2}, \tag{34}$$

$$U_1(0, 0) = 0, \tag{35}$$

$$U_2(x+b,0) = [f(x+b) - T_m](x+b), \tag{36}$$

$$f(0) = T_m, \tag{37}$$
$$\beta(0) = 0, \tag{38}$$
$$f(\infty) = 0, \tag{39}$$

$$x=0: \quad -\lambda_1 \left[b\frac{\partial U_1}{\partial x} - U_1 \right]\Bigg|_{x=0} = b^2 P(t), \tag{40}$$

$$-\lambda_1 \left[\beta(t)\frac{\partial U_1}{\partial x} - U_1 \right]_{x=\beta(t)} = -\lambda_2 \left[\beta(t)\frac{\partial U_2}{\partial x} - U_2 \right]_{x=\beta(t)} + \beta^2(t)\frac{\partial \beta(t)}{\partial t} L\gamma, \tag{41}$$

$$x=\beta(t): \quad U_1(\beta(t),t) = U_2(\beta(t),t) = 0, \tag{42}$$

$$U_2(\infty,t) = 0. \tag{43}$$

We have the Stefan condition:

$$\lambda_1 \left[\beta(t)\frac{\partial U_1}{\partial x} - U_1 \right]_{x=\beta(t)} = -\lambda_2 \left[\beta(t)\frac{\partial U_2}{\partial x} - U_2 \right]_{x=\beta(t)} + \beta^2(t)\frac{\partial \beta(t)}{\partial t} L\gamma. \tag{44}$$

We represent the solution in the following form:

$$\begin{aligned} U_1(x,t) &= \sum_{n=0}^{\infty} A_{2n}(2a_1t)^n \left[\mathrm{i}^{2n}\mathrm{erfc}\left(\tfrac{-x}{2a_1\sqrt{t}}\right) + \mathrm{i}^{2n}\mathrm{erfc}\left(\tfrac{x}{2a_1\sqrt{t}}\right) \right] \\ &+ \sum_{n=0}^{\infty} A_{2n+1}(2a_1t)^{\frac{2n+1}{2}} \left[\mathrm{i}^{2n+1}\mathrm{erfc}\left(\tfrac{-x}{2a_1\sqrt{t}}\right) - \mathrm{i}^{2n}\mathrm{erfc}\left(\tfrac{x}{2a_1\sqrt{t}}\right) \right], \end{aligned} \tag{45}$$

$$U_2(x,t) = \sum_{n=1}^{\infty} B_n(2a_2t)^{\frac{n}{2}} \left[\mathrm{i}^n\mathrm{erfc}\left(\frac{-x}{2a_2\sqrt{t}}\right) \right] + \sum_{n=1}^{\infty} C_n(2a_2t)^{\frac{n}{2}} \left[\mathrm{i}^n\mathrm{erfc}\left(\frac{x}{2a_2\sqrt{t}}\right) \right], \tag{46}$$

where A_n,B_n,C_n,D_n,β_n can be found from the conditions $(35-43)$ and $\beta(t)$ from the condition (44) by expansion of all considered functions in Taylor series and equating coefficients at the like powers. The convergence of the series for the solution may be proved using a similar method presented in the paper [17]. However this exact solution is of little use for numerical calculation and applications, so it is very important to consider another approximate method.

4. The Approximate Solution

To construct the approximate solution we take finitely many terms in the above series

$$U_1(r,t) = \frac{1}{r}\sum_{n=0}^{m}(2a_1\sqrt{t})^n \left(A_n \mathrm{i}^n\mathrm{erfc}\left(\frac{r-r_0}{2a_1\sqrt{t}}\right) + B_n \mathrm{i}^n\mathrm{erfc}\left(\frac{r_0-r}{2a_1\sqrt{t}}\right)\right), \tag{47}$$

$$U_2(r,t) = \frac{1}{r}\sum_{n=0}^{m}(2a_2\sqrt{t})^n \left(C_n \mathrm{i}^n\mathrm{erfc}\left(\frac{r-r_0}{2a_2\sqrt{t}}\right) + D_n \mathrm{i}^n\mathrm{erfc}\left(\frac{r_0-r}{2a_2\sqrt{t}}\right)\right). \tag{48}$$

We consider now the interval $0 < t < t_a$ instead of $0 < t < \infty$ where t_a is the arc duration, and the interval $\alpha(t) < r < R$ instead of $\alpha(t) < r < \infty$ where R is the radius of the cross-section of the cylinder.

It is not so difficult to find from the boundary conditions (24) , (26) that

$$A_0 + B_0 = bT_m, \tag{49}$$

$$C_0 + D_0 = bT_m. \tag{50}$$

To satisfy the condition (25) we use the collocation method. Accordingly to this method we take k points $r_1, r_2, ..., r_k$ on the interval $[b, R]$ to satisfy the equation at these points. Thus taking into account that

$$\begin{aligned} f(r) = U_2(r,0) = \lim_{t\to 0}\textstyle\sum_{n=0}^{m}(2a_2\sqrt{t})^n\left(C_n \mathrm{i}^n\mathrm{erfc}\left(\frac{r-b}{2a_2\sqrt{t}}\right) + D_n\mathrm{i}^n\mathrm{erfc}\left(\frac{b-r}{2a_2\sqrt{t}}\right)\right) \\ = \lim_{t\to 0}\textstyle\sum_{n=0}^{m}(2a_2\sqrt{t})^n C_n\mathrm{i}^n\mathrm{erfc}\left(\frac{r-b}{2a_2\sqrt{t}}\right) + \lim_{t\to 0}\textstyle\sum_{n=0}^{m}(2a_2\sqrt{t})^n D_n\mathrm{i}^n\mathrm{erfc}\left(\frac{b-r}{2a_2\sqrt{t}}\right), \end{aligned} \tag{51}$$

$$\lim_{t\to 0}\sum_{n=0}^{m}(2a_2\sqrt{t})^n C_n\mathrm{i}^n\mathrm{erfc}\left(\frac{r-b}{2a_2\sqrt{t}}\right) = 0, \tag{52}$$

$$\begin{aligned} \lim_{t\to 0}\sum_{n=0}^{m}(2a_2\sqrt{t})^n D_n\mathrm{i}^n\mathrm{erfc}\left(\tfrac{b-r}{2a_2\sqrt{t}}\right) = \sum_{n=0}^{m} D_n(2a_2\sqrt{t})^n \lim_{t\to 0}\frac{\mathrm{i}^n\mathrm{erfc}\left(\frac{b-r}{2a_2\sqrt{t}}\right)}{\left(\frac{b-r}{2a_2\sqrt{t}}\right)^n}\cdot\left(\tfrac{b-r}{2a_2\sqrt{t}}\right)^n \\ = \frac{1}{r}\sum_{n=0}^{m}\frac{2}{n!}D_n(r-b)^n. \end{aligned} \tag{53}$$

We have

$$\frac{1}{r_i}\sum_{n=0}^{m}\frac{2}{n!}D_n(r_i-b)^n = f(r_i),\ i = 1, 2, ..., k. \tag{54}$$

Let us divide the time interval $0 \le t \le t_a$ into s subintervals $(0, t_1), (t_1, t_2), ..., (t_{s-1}, t_s), t_s = t_a$.

Integration of the Stefan condition (32) over the first interval gives

$$\alpha(t_1) - b = \frac{1}{L\gamma}\int_0^{t_1}\left[-\lambda_1\frac{\partial U_1(\alpha(t),t)}{\partial r} + \lambda_2\frac{\partial U_2(\alpha(t),t)}{\partial r}\right]dt.$$

If t_1 is sufficiently small we can put

$$-\lambda_1\frac{\partial U_1(\alpha(t),t)}{\partial r} \approx -\lambda_1\frac{\partial U_1(b,t)}{\partial r} = P(t),$$

$$\lambda_2\frac{\partial U_2(\alpha(t),t)}{\partial r} \approx \lambda_2\frac{\partial U_2(b,t)}{\partial r} = \lambda_2 f'(b).$$

Then

$$\alpha(t_1) = b + \frac{1}{L\gamma}\int_0^{t_1}\left[P(t) + \lambda_2 f'(b)\right]dt. \tag{55}$$

Furthermore we can put a simpler approximation

$$\alpha(t) = b + \alpha_1 t,\ 0 \le t \le t_1, \tag{56}$$

where

$$\alpha_1 = \frac{1}{L\gamma}\left[P(0) + \lambda_2 f'(b)\right]. \tag{57}$$

The boundary condition (29) gives the equation

$$-\lambda_1\left[-\frac{1}{b^2}\sum_{n=0}^{m}\left(2a_1\sqrt{t}\right)^n(A_n+B_n)\mathrm{i}^n\mathrm{erfc}\,(0) + \frac{1}{b}\sum_{n=0}^{m}\left(2a_1\sqrt{t}\right)^{n-1}(-A_n+B_n)\mathrm{i}^{n-1}\mathrm{erfc}\,(0)\right] = P(t) \tag{58}$$

or

$$\frac{-\lambda_1}{b}(-A_0+B_0)\frac{\mathrm{i}^{-1}\mathrm{erfc}\,(0)}{2a_1\sqrt{t}} + \frac{\lambda_1}{b^2}\sum_{n=0}^{m}\left(2a_1\sqrt{t}\right)^n\left(A_n+B_n+b(A_{n+1}-B_{n+1})\right)\mathrm{i}^n\mathrm{erfc}\,(0) = P(t)$$

To avoid the singularity at $t = 0$ we put

$$A_0 = B_0. \tag{59}$$

Satisfying the equation at the discrete points $t_1, t_2, ..., t_s$ we get

$$\sum_{n=0}^{m} \left(2a_1 \sqrt{t_j}\right)^n \mathrm{i}^n\mathrm{erfc}\,(0)\,(A_n + B_n + b(A_{n+1} - B_{n+1})) = \frac{b^2}{\lambda_1} P(t_j) \tag{60}$$

where $j = 1, 2, \cdots, s$.

Now from (49) and (59),

$$A_0 = B_0 = \frac{1}{2} b T_m. \tag{61}$$

Putting $k = m$ we can write

$$\begin{cases} D_0 + (r_1 - b)D_1 + ... + \frac{(r_1-b)^m}{m!} D_m = \frac{r_1}{2} f(r_1), \\ D_0 + (r_2 - b)D_1 + ... + \frac{(r_2-b)^m}{m!} D_m = \frac{r_2}{2} f(r_2), \\ \vdots \\ D_0 + (r_m - b)D_1 + ... + \frac{(r_m-b)^m}{m!} D_m = \frac{r_m}{2} f(r_m). \end{cases} \tag{62}$$

Solving this system we get the values D_j. Satisfying the condition (32) in the interval $(0, t_1)$ we get

$$U_1(b + \alpha_1 t, t) = T_m$$

or using (47)

$$\frac{1}{b + \alpha_1 t} \sum_{n=0}^{m} (2a_1 \sqrt{t})^n \left(A_n \mathrm{i}^n\mathrm{erfc}\left(\frac{\alpha_1 \sqrt{t}}{2a_1}\right) + B_n \mathrm{i}^n\mathrm{erfc}\left(\frac{-\alpha_1 \sqrt{t}}{2a_1}\right)\right) = T_m$$

At the discrete points $t_1, t_2, t_3, ..., t_s$ we get

$$\frac{1}{b + \alpha_1 t_j} \sum_{n=0}^{m} (2a_1 \sqrt{t})^n \left(A_n \mathrm{i}^n\mathrm{erfc}\left(\frac{\alpha_1 \sqrt{t_j}}{2a_1}\right) + B_n \mathrm{i}^n\mathrm{erfc}\left(\frac{-\alpha_1 \sqrt{t_j}}{2a_1}\right)\right) = T_m, \tag{63}$$

where $j = 1, 2, \ldots, s$.

Setting $s = 2(m + 1)$ one can find the constants A_n and B_n from the system of the equations (60) and (63). Similarly from the condition (41) we get

$$\frac{1}{b + \alpha_1 t_j} \sum_{n=0}^{m} (2a_2 \sqrt{t_j})^n \left(C_n \mathrm{i}^n\mathrm{erfc}\left(\frac{\alpha_1 \sqrt{t_j}}{2a_2}\right) + D_n \mathrm{i}^n\mathrm{erfc}\left(\frac{-\alpha_1 \sqrt{t_j}}{2a_2}\right)\right) = T_m. \tag{64}$$

We find now C_n and D_n from (62) and (64). Thus the solution in the interval $(0, t_1)$ is found and can be determined by the expressions (47) and (48).

Let us consider now the next interval (t_1, t_2). Integrating Stefan's condition (32) along this interval we get

$$\alpha(t_2) = \alpha(t_1) + \frac{1}{L\gamma} \int_{t_1}^{t_2} \left[-\lambda_1 \frac{\partial U_1(\alpha(t), t)}{\partial r} + \lambda_2 \frac{\partial U_2(\alpha(t), t)}{\partial r} \right] dt.$$

Putting inside, the integral $\alpha(t) \approx \alpha_1$ and using the obtained above values U_1 and U_2 for $0 \le t \le t_1$ we can find the values $\alpha_2 \approx \alpha(t_2)$ and put

$$\alpha(t) = \alpha_1 + (\alpha_2 - \alpha_1) \frac{t - t_1}{t_2 - t_1}, \ t_1 \le t \le t_2.$$

The procedure of definition of the coefficients A_n, B_n, C_n, D_n is similar to the described above. The next steps are also similar. This stage of heating will be completed at the step $t_{b-1} \le t \le t_b$ such that $U_1(b, t_b) = T_b$ (boiling temperature).

Error estimation can be found by the maximum principle as it was shown in subsection 1.2.

5. Experimental verification of the model

Experiments have been carried out for AgCdO contacts in air at 1 atmosphere pressure and for Ni contacts in a chamber at varied pressure. Electrical circuit diagram of the test rig is presented in Figure 6.

The values of measured parameters for both contact materials are given in the Table 1.

	AgCdO	Ni
Supplied voltage U_0, V	*100*	*250*
Initial current I_0, A	*2.0*	*3.0*
Load resistance R, Ω	*100*	*220*
Load inductance L, mH	*340*	*2300*
Circuit capacitance C, nF	*9.0*	*0*
Wires resistance R_W, mΩ	*100*	*0*
Wires inductance L_W, μH	*5*	*0*
Opening velocity V, m/sec	*0.75*	*0.3*
Pressure P, 10^5 Pa	*1.0*	*1.0*
Arc radius r_a , μm	*45*	*15*
Arc duration t_a , μsec	*7.3*	*6.7*

Table 1: Parameters of the electrical circuit

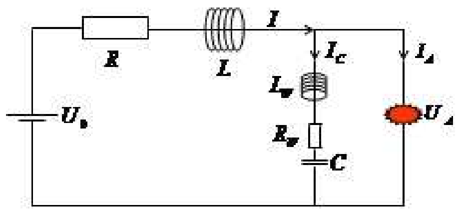

Figure 6: Electrical circuit

The corresponding arc power $P_A(t)$ for this electrical circuit is calculated in the paper [15] and presented for AgCdO in Figure 7.

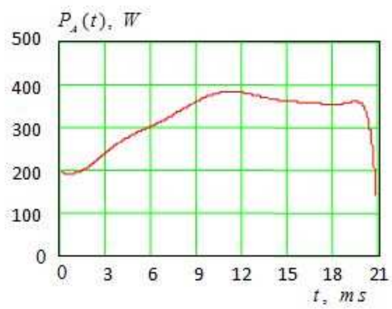

Figure 7: Dynamics of the arc power

If we identify the arc radius r_a with the initial radius of melting isotherm $\alpha(0) = b$, then the heat flux $P(t)$ can be defined by the expression $P(t) = P_A(t)/\pi r_a^2$.

The thermophysical parameters for AgCdO (Ag-90%, CdO-10%) are following [16]:
$T_m = 1233$ K, $\gamma = 10.21 \cdot 10^3$ kg·m^{-3} ,
$L = 1.06 \cdot 10^9$J·m^{-3}, $\lambda_1 = 307$·W·m^{-1}·K^{-1},
$\lambda_2 = 285$·W·m^{-1}·K^{-1}, $a_1 = 0.011$·W·m·sec$^{-1/2}$,
$a_2 = 0.008$·W·m·sec$^{-1/2}$,

The initial temperature $f(r)$ can be determined from the expression for the temperature at the pre-melting stage at the time t_m, when its value on the boundary $r = b$ reaches the melting point [16]:

$$f(r) = \frac{a_2 b}{\lambda_2 r}\int_0^{t_m}\left[\frac{\exp\left(-\frac{(r-b)^2}{4a_2^2(t_m-\tau)}\right)}{\sqrt{\pi(t_m-\tau)}} - \frac{a_2}{b}\exp\left(\frac{1}{b}(r-b)+\frac{a_2^2}{b^2}(t_m-\tau)\right)\right]\operatorname{erfc}\left(\frac{(r-b)}{2a_2\sqrt{(t_m-\tau)}}+\frac{a_2}{b}\sqrt{(t_m-\tau)}\right)P(\tau)d\tau$$

where t_m should be defined from the equation:

$$T_m = \frac{a_2}{\lambda_2}\int_0^{t_m}\left[\frac{1}{\sqrt{\pi(t_m-\tau)}} - \frac{a_2}{b}\exp\left(\frac{a_2^2}{b^2}(t_m-\tau)\right)\right]\operatorname{erfc}\left(\frac{a_2}{b}\sqrt{(t_m-\tau)}\right)P(\tau)d\tau.$$

The results of calculation of the temperature distribution on the contact spot are $t_m = 0.4$ μsec, $t_b = 2.4$ μsec, $t_a = 7.3$ μsec, where t_m is the time of the beginning of melting and t_b is the boiling start.

One can see that the duration of the contact erosion due to boiling and evaporation is $t_0 = t_a - t_b = 4.9\mu$sec. The mass of the evaporated sphere whose volume is $V = \frac{4}{3}\pi\alpha^3(t_0)$ is 42.6 μg. According to the experimental data the measured erosion is 38.4 μg. This discrepancy can be explained by the fact that the presented model does not take into account the portion of the heat flux consumed for the phase transformation at boiling and operates with an overestimated flux.

6. Conclusion

The method of the integral error functions and the heat polynomials enables us to find the analytical and approximate solutions of the Stefan problem. The elaborated mathematical model of the heat and mass transfer in electrical contacts during arcing is verified experimentally and the discrepancy can be explained by the fact that the portion of the heat flux consumed for the phase transformation is not taken into account and the operated heat flux was overestimated.

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