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Positive Solutions of Boundary Value Problems for p-Laplacian Fractional Differential Equations

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Abstract. This work is devoted to the existence of positive solutions for nonlinear fractional differential equations with p-Laplacian operator. By using five functionals fixed point theorem, the existence of at least three positive solutions are obtained. As an application, an example is presented to demonstrate our main result.

1. Introduction

In this paper, we are concerned with the existence of positive solutions for the fractional differential equations with p-Laplacian:

$$\begin{cases} D^{\beta}_{0^+}(\phi_p(D^{\alpha}_{0^+}x(t))) = f(t, x(t), x'(t)), \quad t \in (0,1), \\ x(0) = aD^{\gamma}_{0^+}x(1), \\ x'(0) = bx'(1), \\ x^{(i)}(0) = 0, \quad i = 2, 3, ..., n-1, \end{cases} \tag{1.1}$$

where $D^{\beta}_{0^+}$, $D^{\alpha}_{0^+}$, $D^{\gamma}_{0^+}$ are Caputo fractional derivatives with $0 < \beta \leq 1$, $0 < \gamma < 1$, $2 \leq n-1 < \alpha \leq n$, where n is an integer, and the constants a and b satisfy $0 < b < 1$, $a \geq 0$. $\phi_p(s)$ is a p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$ for $p > 1$, $(\phi_p)^{-1}(s) = \phi_q(s)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in C([0,1] \times [0,\infty) \times [0,\infty), [0,\infty))$ is a given nonlinear function.

Fractional differential equations have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry and engineering. There has been a significant development in the study of fractional differential equations in recent years, see the monographs of Kilbas et al. [4], Miller et al. [8], Podlubny [9] and Samko et al. [10].

Since the p-Laplacian operator and fractional calculus arises from many applied fields such as turbulant filtration in porous media, blood flow problems, rheology, modelling of viscoplasticity, material science, it is

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worth studying the fractional p-Laplacian differential equations. The research of boundary value problems for p-Laplacian equations of fractional order has just begun in recent years, see [2, 3, 5–7, 11, 12].

Very few works has been done to the existence of positive solutions to Caputo fractional boundary value problems involving p-Laplacian operator, see [5–7]. In particular, we would like to mention some results of Liu et al. [5, 6].

In [5], Liu et al. considered the existence and uniqueness of solutions of the Caputo fractional differential eqation involving the p-Laplacian operator

$$(\phi_p(D^{\alpha}x(t)))' = f(t, x(t)), \quad t \in (0, 1),$$

with the boundary value conditions

$$\begin{cases} x(0) = r_0 x(1), \\ x'(0) = r_1 x'(1), \\ x^{(i)}(0) = 0, \quad i = 2, 3, ..., [\alpha] - 1. \end{cases}$$

By using the Banach contraction mapping principle, some new results on the existence and uniqueness of a solution for the model were obtained.

In [6], Liu et al. were concerned with the multiple positive solutions of Dirichlet-Neumann boundary value problems for a type of fractional differential equation involving p-Laplacian operator

$$(\phi_p(D^{\alpha}x(t)))' = \phi_p(\lambda)f(t, x(t), x'(t)), \quad t \in (0, 1),$$

with the boundary value conditions

$$\begin{cases} k_0 x(0) - k_1 x(1) = 0, \\ m_0 x'(0) - m_1 x'(1) = 0, \\ x^{(r)}(0) = 0, \quad r = 2, 3, ..., n. \end{cases}$$

By using Avery-Peterson fixed point theorem, they obtained the existence of at least three positive solutions of the model.

Motivated by the above mentioned papers, we apply the five functionals fixed point theorem [1] to verify the existence of at least three positive solutions for the boundary value problem (BVP) (1.1).

The paper is organized as follows. In Section 2, we introduce some definitions and lemmas to prove our main result. In Section 3, we prove our main result. Finally, an example is given in Section 4 to illustrate the usefulness of our main result.

2. Preliminaries

In this section, we introduce some preliminary facts which are used throughout this article. Now we recall the following definitions, which can be found in [4, 8–10].

Definition 2.1 Let $\alpha > 0$ for a function $y : (0, \infty) \longrightarrow \mathbb{R}$. The fractional integral of order α of y is defined by

$$I^{\alpha}_{0^+} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y(s) ds$$

provided the integral exists.

Definition 2.2 Let $\alpha > 0$. The Caputo fractional derivative of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$$D^{\alpha}_{0^+} y(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n - \alpha - 1} y^{(n)}(t) ds$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Proposition 2.1. *If $y \in L^1(0,1)$, $\nu > \sigma > 0$, then*
(i) $I^{\nu}_{0^+} I^{\sigma}_{0^+} y(t) = I^{\nu+\sigma} y(t)$;
(ii) $D^{\sigma}_{0^+} I^{\nu}_{0^+} y(t) = I^{\nu-\sigma} y(t)$;
(iii) $D^{\sigma}_{0^+} I^{\sigma}_{0^+} y(t) = y(t)$.

Lemma 2.2. *[6] Let $0 < n-1 < \alpha \leq n$. If we assume $y \in C^n(0,1) \cap L[0,1]$, the fractional differential equation $D^{\alpha} y(t) = 0$ has a unique solution*

$$y(t) = y(0) + \frac{y'(0)}{1!}t + \frac{y''(0)}{2!}t^2 + ... + \frac{y^{(n-1)}(0)}{(n-1)!}t^{n-1}.$$

Lemma 2.3. *Suppose that $h \in C[0,1]$. Then the boundary value problem*

$$\begin{cases} D^{\beta}_{0^+}(\phi_p(D^{\alpha}_{0^+} x(t))) = h(t), \quad t \in (0,1), \\ x(0) = aD^{\gamma}_{0^+} x(1), \\ x'(0) = bx'(1), \\ x^{(i)}(0) = 0, \quad i = 2,3,...,n-1, \end{cases} \tag{2.1}$$

has a unique solution

$$x(t) = \frac{b}{1-b}\left[\frac{a}{\Gamma(2-\gamma)} + t\right] I^{\alpha-1}_{0^+}\phi_q(I^{\beta}_{0^+}h(1)) + I^{\alpha}_{0^+}\phi_q(I^{\beta}_{0^+}h(t)) + aI^{\alpha-\gamma}_{0^+}\phi_q(I^{\beta}_{0^+}h(1)). \tag{2.2}$$

Proof. The definition of the Caputo derivatives implies that ${}^{C}D^{\alpha}x(0) = 0$, and from Lemma 2.1 we have

$$\phi_p(D^{\alpha}_{0^+}x(t)) = \phi_p(D^{\alpha}_{0^+}x(0)) + I^{\beta}_{0^+}(h(t)).$$

So,

$$x(t) = x(0) + x'(0)t + I^{\alpha}_{0^+}\phi_q(I^{\beta}_{0^+}h(t)) \tag{2.3}$$

$$= x(0) + x'(0)t + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q(I^{\beta}_{0^+}h(s))ds. \tag{2.4}$$

Hence, we get

$$x'(t) = x'(0) + \frac{d}{dt}\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q(I^{\beta}_{0^+}h(s))ds\right],$$

i.e.,

$$x'(t) = x'(0) + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\phi_q(I^{\beta}_{0^+}h(s))ds. \tag{2.5}$$

From the boundary value condition $x'(0) = bx'(1)$ and (2.5), one has

$$x'(0) = bx'(0) + \frac{b}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi_q(I^{\beta}_{0^+}h(s))ds,$$

i.e.,

$$x'(0) = \frac{b}{(1-b)\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi_q(I^{\beta}_{0^+}h(s))ds. \tag{2.6}$$

Since $x(0) = aD^{\gamma}_{0^+}x(1)$, we get

$$x(0) = a\left[x'(0)\frac{1}{\Gamma(1-\gamma)}\int_0^1 (1-s)^{-\gamma}ds + I^{\alpha-\gamma}_{0^+}\phi_q(I^{\beta}_{0^+}h(1))\right]. \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.3), we can obtain that

$$x(t) = \frac{b}{1-b}\left[\frac{a}{\Gamma(2-\gamma)} + t\right] I_{0^+}^{\alpha-1}\phi_q(I_{0^+}^{\beta}h(1)) + I_{0^+}^{\alpha}\phi_q(I_{0^+}^{\beta}h(t)) + aI_{0^+}^{\alpha-\gamma}\phi_q(I_{0^+}^{\beta}h(1)).$$

So, the proof is complete. □

Let $E = C^1[0,1]$, $\|x\| = \max\left\{\max_{t\in[0,1]}|x(t)|, \max_{t\in[0,1]}|x'(t)|\right\}$. Then $(E, \|.\|)$ is a Banach space. Set

$$P = \{x \in E : x \text{ is a nonnegative, monotone increasing and convex function on } [0,1]\}.$$

So, P is a cone on E.

Lemma 2.4. *If $x \in P$ is a solution of BVP* (2.1)*, then*

$$\min_{t\in[0,1]} x(t) \geq \mu \max_{t\in[0,1]} x(t),$$

where

$$\mu = \frac{ab}{ab + (1 + a(1-b))\Gamma(2-\gamma)}. \tag{2.8}$$

Proof. From Lemma 2.2, we have

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left[\frac{1}{\Gamma(\beta)}\int_0^s (s-r)^{\beta-1}h(r)dr\right]ds \\ &\quad + \frac{b}{(1-b)\Gamma(\alpha-1)}\left[\frac{a}{\Gamma(2-\gamma)} + t\right]\int_0^1 (1-s)^{\alpha-2}\phi_q\left[\frac{1}{\Gamma(\beta)}\int_0^s (s-r)^{\beta-1}h(r)dr\right]ds \\ &\quad + \frac{a}{\Gamma(\alpha-\gamma)}\int_0^1 (1-s)^{\alpha-\gamma-1}\phi_q\left[\frac{1}{\Gamma(\beta)}\int_0^s (s-r)^{\beta-1}h(r)dr\right]ds. \end{aligned}$$

$$x(t) = \int_0^1 F(t,s)\phi_q(I_{0^+}^{\beta}h(s))ds,$$

where

$$F(t,s) = \begin{cases} g_1(t,s), & 0 \leq s \leq t \leq 1, \\ g_2(t,s), & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.9}$$

and

$$g_1(t,s) = \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} + \frac{b}{(1-b)\Gamma(\alpha-1)}\left[\frac{a}{\Gamma(2-\gamma)} + t\right](1-s)^{\alpha-2} + \frac{a}{\Gamma(\alpha-\gamma)}(1-s)^{\alpha-\gamma-1}, \tag{2.10}$$

$$g_2(t,s) = \frac{b}{(1-b)\Gamma(\alpha-1)}\left[\frac{a}{\Gamma(2-\gamma)} + t\right](1-s)^{\alpha-2} + \frac{a}{\Gamma(\alpha-\gamma)}(1-s)^{\alpha-\gamma-1}. \tag{2.11}$$

The proof is divided into two cases.

Case 1. If $0 \leq s \leq t \leq 1$, then by (2.10) we have

$$\frac{\partial g_1(t,s)}{\partial t} = \frac{b}{(1-b)\Gamma(\alpha-1)}(1-s)^{\alpha-2} + \frac{1}{\Gamma(\alpha-1)}(t-s)^{\alpha-2} > 0.$$

Since $g_1(t,s)$ is increasing with respect to t, we have for $s \le t$

$$g_1(s,s) \le g_1(t,s) \le g_1(1,s).$$

Case 2. If $0 \le t \le s \le 1$, then by (2.11) we have

$$\frac{\partial g_2(t,s)}{\partial t} = \frac{b}{(1-b)\Gamma(\alpha-1)}(1-s)^{\alpha-2} > 0.$$

Since $g_2(t,s)$ is increasing with respect to t, we have

$$g_2(0,s) \le g_2(t,s) \le g_2(s,s).$$

Summing up the above analysis Case 1 and Case 2, it follows that

$$g_2(0,s) \le \min_{s\in[0,1]}\{g_1(s,s), g_2(0,s)\} \le F(t,s) \le \max_{s\in[0,1]}\{g_1(1,s), g_2(s,s)\},$$

$$g_2(0,s) \le F(t,s) \le g_1(1,s) = \max_{t\in[0,1]} F(t,s).$$

It remains to show that

$$F(t,s) \ge \mu g_1(1,s),\ t,s \in [0,1],$$

and

$$\frac{F(t,s)}{g_1(t,s)} \ge \frac{g_2(0,s)}{g_1(1,s)} = \mu(s).$$

From (2.10) and (2.11), for $s \in (0,1)$ we have

$$\begin{aligned}
\mu(s) = \frac{g_2(0,s)}{g_1(1,s)} &= \frac{\dfrac{ab(1-s)^{\alpha-2}}{(1-b)\Gamma(2-\gamma)\Gamma(\alpha-1)} + \dfrac{a(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}}{\dfrac{b(1-s)^{\alpha-2}}{(1-b)\Gamma(\alpha-1)}\left[\dfrac{a}{\Gamma(2-\gamma)}+1\right] + \dfrac{a(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} + \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}} \\
&\ge \frac{\dfrac{ab}{(1-b)\Gamma(2-\gamma)\Gamma(\alpha-1)}}{\dfrac{b}{(1-b)\Gamma(\alpha-1)}\left[\dfrac{a}{\Gamma(2-\gamma)}+1\right] + \dfrac{a}{\Gamma(\alpha-\gamma)} + \dfrac{1}{\Gamma(\alpha)}} \\
&\ge \frac{ab}{ab + (1+a(1-b))\Gamma(2-\gamma)},
\end{aligned}$$

and so

$$\frac{F(t,s)}{g_1(1,s)} \ge \mu,\ s \in (0,1).$$

For $t,s \in [0,1]$,

$$F(t,s) \ge \mu g_1(1,s) = \mu \max_{t\in[0,1]} F(t,s).$$

Hence

$$\min_{t\in[0,1]} F(t,s) \ge \mu \max_{t\in[0,1]} F(t,s). \tag{2.12}$$

By (2.12), we have

$$\min_{t\in[0,1]} x(t) \ge \mu \max_{t\in[0,1]} x(t).$$

This completes the proof. □

Lemma 2.5. *If $x \in P$ is a solution of BVP* (2.1)*, then* $\|x\| \leq Mx'(1)$*, where* $M = \max\left\{\frac{a}{\Gamma(2-\gamma)}, \frac{a\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma)}\right\} + 1$.

Proof. From (2.4), it follows that

$$x(1) = x(0) + x'(0) + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q(I_{0^+}^{\beta}h(s))ds \tag{2.13}$$

and from (2.5) we get that

$$x'(1) = x'(0) + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi_q(I_{0^+}^{\beta}h(s))ds. \tag{2.14}$$

Substituting (2.7) into (2.13), it is easy to get that

$$\begin{aligned}
x(1) &= a\left[\frac{x'(0)}{\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha-\gamma)}\int_0^1 (1-s)^{\alpha-\gamma-1}\phi_q(I_{0^+}^{\beta}h(s))ds\right] \\
&\quad + x'(0) + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q(I_{0^+}^{\beta}h(s))ds \\
&< x'(0)\left[\frac{a}{\Gamma(2-\gamma)} + 1\right] + \left[\frac{a}{\Gamma(\alpha-\gamma)} + \frac{1}{\Gamma(\alpha)}\right]\int_0^1 (1-s)^{\alpha-2}\phi_q(I_{0^+}^{\beta}h(s))ds \\
&= x'(0)\left[\frac{a}{\Gamma(2-\gamma)} + 1\right] + \left[\frac{a\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma)} + \frac{1}{\alpha-1}\right]\frac{1}{\Gamma(\alpha-1}\int_0^1 (1-s)^{\alpha-2}\phi_q(I_{0^+}^{\beta}h(s))ds \\
&< \max\left\{\frac{a}{\Gamma(2-\gamma)} + 1, \frac{a\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma)} + 1\right\}\left[x'(0) + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi_q(I_{0^+}^{\beta}h(s))ds\right] \\
&= \max\left\{\frac{a}{\Gamma(2-\gamma)} + 1, \frac{a\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma)} + 1\right\}x'(1).
\end{aligned}$$

So that

$$\|x\| = \max\left\{\max_{t\in[0,1]}|x(t)|, \max_{t\in[0,1]}|x'(t)|\right\} \leq M\max_{t\in[0,1]}|x'(t)|,$$

i.e., $\|x\| \leq Mx'(1)$. The proof is complete. □

Define the operator $T : P \longrightarrow E$ by

$$\begin{aligned}
(Tx)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left[\frac{1}{\Gamma(\beta)}\int_0^s (s-r)^{\beta-1}f(r,x(r),x'(r))dr\right]ds \\
&\quad + \frac{b}{(1-b)\Gamma(\alpha-1)}\left[\frac{a}{\Gamma(2-\gamma)} + t\right]\int_0^1 (1-s)^{\alpha-2}\phi_q\left[\int_0^s \frac{(s-r)^{\beta-1}}{\Gamma(\beta)}f(r,x(r),x'(r))dr\right]ds \\
&\quad + \frac{a}{\Gamma(\alpha-\gamma)}\int_0^1 (1-s)^{\alpha-\gamma-1}\phi_q\left[\frac{1}{\Gamma(\beta)}\int_0^s (s-r)^{\beta-1}f(r,x(r),x'(r))dr\right]ds.
\end{aligned} \tag{2.15}$$

It is clear that $x \in C^n[0,1]$ is the solution of the BVP (1.1) if and only if $x \in E$ is the fixed point of the operator T.

Lemma 2.6. *$T : P \longrightarrow P$ is completely continuous.*

Proof. By the definition of the operator T, for $x \in P$ it is easy to see $(Tx)(t) \geq 0$ for any $t \in [0,1]$. Using the property of the fractional integrals and derivatives, we can get that

$$(Tx)'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \phi_q(I_{0^+}^{\beta} f(s, x(s), x'(s)))ds + \frac{b}{(1-b)\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \phi_q(I_{0^+}^{\beta} f(s, x(s), x'(s)))ds$$

$$= \frac{1}{\Gamma(\alpha-1)} \Big[\int_0^t (t-s)^{\alpha-2} \phi_q(I_{0^+}^{\beta} f(s, x(s), x'(s)))ds + \frac{b}{(1-b)} \int_0^1 (1-s)^{\alpha-2} \phi_q(I_{0^+}^{\beta} f(s, x(s), x'(s)))ds \Big] \tag{2.16}$$

$$\geq 0,$$

and

$$\begin{aligned} (Tx)''(t) &= \frac{1}{\Gamma(\alpha-1)} \frac{d}{dt} \left[\int_0^t (t-s)^{\alpha-2} \phi_q(I_{0^+}^{\beta} f(s, x(s), x'(s)))ds \right] \\ &= \frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} \phi_q(I_{0^+}^{\beta} f(s, x(s), x'(s)))ds \geq 0. \end{aligned}$$

Then Tx is nonnegative, monotone increasing and convex on $[0,1]$. Thus, $Tx \in P$. It is easy to prove that T is continuous and compact. This completes the proof. □

3. Main Results

In this section we state and prove our main result. Now we will present the five functionals fixed point theorem.

Let $\tilde{a},\ \tilde{b},\ c,\ d,\ h > 0$ be positive constants, $\vartheta,\ \psi$ be two nonnegative continuous concave functionals on the cone P, $\chi,\ \varphi,\ \theta$ be three nonnegative continuous convex functionals on the cone P. Define the convex sets as follows:

$$\begin{aligned} P(\chi, c) &= \{x \in P : \chi(x) < c\}, \\ P(\chi, \vartheta, \tilde{a}, c) &= \{x \in P : \vartheta(x) \geq \tilde{a}, \chi(x) \leq c\}, \\ P(\chi, \theta, \vartheta, \tilde{a}, \tilde{b}, c) &= \left\{x \in P : \vartheta(x) \geq \tilde{a}, \theta(x) \leq \tilde{b}, \chi(x) \leq c\right\}, \\ Q(\chi, \varphi, d, c) &= \{x \in P : \varphi(x) \leq d, \chi(x) \leq c\}, \\ Q(\chi, \varphi, \psi, h, d, c) &= \{x \in P : \psi(x) \geq h, \varphi(x) \leq d, \chi(x) \leq c\}. \end{aligned}$$

Lemma 3.1. *[1] (Five Functionals Fixed Point Theorem) Let E be a real Banach space, P be a nonempty cone in E, $\vartheta,\ \psi$ be two nonnegative continuous concave functionals on the cone P, $\chi,\ \varphi,\ \theta$ be three nonnegative continuous convex functionals on the cone P. There exist constant $M > 0$ such that*

$$\vartheta(x) \leq \varphi(x),\ \|x\| \leq M\chi(x) \text{ for all } x \in \overline{P(\chi, c)}.$$

Furthermore, suppose that $h,\ d,\ \tilde{a},\ \tilde{b},\ c > 0$ are constants with $d < a$. Let $T : \overline{P(\chi, c)} \longrightarrow \overline{P(\chi, c)}$ be a completely continuous operator. If

(C1) $\left\{x \in P(\chi, \theta, \vartheta, \tilde{a}, \tilde{b}, c) : \vartheta(x) > \tilde{a}\right\} \neq \emptyset$ *and* $\vartheta(Tx) > \tilde{a}$ *for every* $x \in P(\chi, \theta, \vartheta, \tilde{a}, \tilde{b}, c)$;

(C2) $\{x \in Q(\chi, \varphi, \psi, h, d, c) : \varphi(x) < d\} \neq \emptyset$ *and* $\varphi(Tx) < d$ *for every* $x \in Q(\chi, \varphi, \psi, h, d, c)$;

(C3) $\vartheta(Tx) > \tilde{a}$ *for* $x \in P(\chi, \vartheta, \tilde{a}, c)$ *with* $\theta(Tx) > \tilde{b}$;

(C4) $\varphi(Tx) < d$ *for each* $x \in Q(\chi, \varphi, d, c)$ *with* $\psi(Tx) < h$;

then T has at least three fixed points $x_1,\ x_2,\ x_3 \in \overline{P(\chi, c)}$ such that $\varphi(x_1) < d$, $\vartheta(x_2) > \tilde{a}$ and $\varphi(x_3) > d$ with $\vartheta(x_3) < \tilde{a}$.

Define the functionals on P by

$$\begin{aligned}\chi(x) &= \max_{t\in[0,1]} x'(t),\\ \theta(x) &= \varphi(x) = \max_{t\in[0,1]} x(t),\\ \psi(x) &= \vartheta(x) = \min_{t\in[0,1]} x(t).\end{aligned}$$

It is easy to see that ϑ, ψ are two nonnegative continuous concave functionals on the cone P, χ, φ, θ are three nonnegative continuous convex functionals on the cone P and $\vartheta(x) \le \varphi(x)$ for all $x \in P$.

Theorem 3.2. *Suppose that Q, W and Ω are given by*

$$\begin{aligned}Q &= \frac{\Gamma(\beta q-\beta+1)}{(1-b)[\Gamma(\beta+1)]^{q-1}\Gamma(\alpha+\beta q-\beta)},\\ W &= \frac{a\Gamma(\beta q-\beta+1)}{[\Gamma(\beta+1)]^{q-1}\Gamma(\alpha-\gamma+\beta q-\beta+1)},\\ \Omega &= MQ,\end{aligned}$$

and there exist positive constants e_1, e_2 and c such that

$$c \ge \frac{e_2}{\mu} > e_2 > e_1 > 0,\ W > Q.$$

If

(C5) $f(t,u,v) \le \phi_p\left(\frac{c}{Q}\right)$ *for all* $t \in [0,1]$, $u \in [0, Mc]$, $v \in [0,c]$;

(C6) $f(t,u,v) > \phi_p\left(\frac{e_2}{W}\right)$ *for all* $t \in [0,1]$, $u \in \left[e_2, \frac{e_2}{\mu}\right]$, $v \in [0,c]$;

(C7) $f(t,u,v) < \phi_p\left(\frac{e_1}{\Omega}\right)$ *for all* $t \in [0,1]$, $u \in [\mu e_1, e_1]$, $v \in [0,c]$;

then BVP (1.1) *has at least three increasing positive solutions x_1, x_2, x_3 such that*

$$x_1(1) < e_1,\ x_2(0) > e_2,\ x_3(1) > e_1 \text{ and } x_3(0) < e_2.$$

Proof. To apply Lemma 3.1, we prove that all conditions in Lemma 3.1 are satisfied. By the definitions, it is easy to see that ϑ, ψ are two nonnegative continuous concave functionals on the cone P, χ, φ, θ are three nonnegative continuous convex functionals on the cone P and $\vartheta(x) \le \varphi(x)$ for all $x \in P$. From Lemma 2.5 we have $\|x\| \le Mx'(1)$, i.e., $\|x\| \le M\chi(x)$ for all $x \in P$.

Corresponding to Lemma 3.1,

$$c = c,\ h = \mu e_1,\ d = e_1,\ \tilde{a} = e_2,\ \tilde{b} = \frac{e_2}{\mu}.$$

Now, we prove that (C1)-(C4) of Lemma 3.1 hold. One sees that $0 < e_1 < e_2$. The remainder is divided into four steps.

Step 1. Prove that $T : \overline{P(\chi,c)} \longrightarrow \overline{P(\chi,c)}$;
For $x \in \overline{P(\chi,c)}$, we have $\chi(x) \le c$. Then $\max_{t\in[0,1]} x'(t) \le c$ and $0 \le x'(t) \le c$ for all $t \in [0,1]$. So from Lemma 2.5 implies that

$$\max_{t\in[0,1]} x(t) \le M \max_{t\in[0,1]} x'(t) \le Mc.$$

So (C5) implies that

$$f(t, x(t), x'(t)) \leq \phi_p\left(\frac{c}{Q}\right).$$

We have

$$\begin{aligned}
(Tx)'(1) &= \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi_q(I_{0^+}^{\beta} f(s, x(s), x'(s)))ds \\
&\quad + \frac{b}{(1-b)\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi_q(I_{0^+}^{\beta} f(s, x(s), x'(s)))ds \\
&= \frac{1}{(1-b)\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi_q(I_{0^+}^{\beta} f(s, x(s), x'(s)))ds \\
&\leq \frac{c}{Q}\frac{1}{(1-b)\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi_q\left[\frac{1}{\Gamma(\beta)}\int_0^s (s-r)^{\beta-1}dr\right]ds \\
&= \frac{c}{Q}\frac{1}{(1-b)\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi_q\left(\frac{s^{\beta}}{\beta\Gamma(\beta)}\right)ds \\
&= \frac{c}{Q}\frac{1}{(1-b)\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\left(\frac{s^{\beta}}{\Gamma(\beta+1)}\right)^{q-1}ds \\
&= \frac{c}{Q}\frac{1}{(1-b)\Gamma(\alpha-1)}\frac{1}{[\Gamma(\beta+1)]^{q-1}}\beta(\alpha-1, \beta q-\beta+1) \\
&= \frac{c}{Q}\frac{1}{(1-b)}\frac{1}{\Gamma(\beta+1)^{q-1}}\frac{\Gamma(\beta q-\beta+1)}{\Gamma(\alpha+\beta q-\beta)} \\
&= c.
\end{aligned}$$

Then $T : \overline{P(\chi, c)} \longrightarrow \overline{P(\chi, c)}$.
This completes Step 1.

Step 2. Prove that

$$\left\{x \in P\left(\chi, \theta, \vartheta, e_2, \frac{e_2}{\mu}, c\right) : \vartheta(x) > e_2\right\} \neq \emptyset,$$

and $\vartheta(Tx) > e_2$ for every $x \in P\left(\chi, \theta, \vartheta, e_2, \frac{e_2}{\mu}, c\right)$. Choose $x(t) = \frac{e_2}{\mu}$ for all $t \in [0,1]$. It follows that $\left\{x \in P\left(\chi, \theta, \vartheta, e_2, \frac{e_2}{\mu}, c\right) : \vartheta(x) > e_2\right\} \neq \emptyset$.

For $x \in P\left(\chi, \theta, \vartheta, e_2, \frac{e_2}{\mu}, c\right)$, one has that

$$\vartheta(x) = \min_{t\in[0,1]} x(t) \geq e_2,\ \theta(x) = \max_{t\in[0,1]} x(t) \leq \frac{e_2}{\mu},\ \chi(x) = \max_{t\in[0,1]} x'(t) \leq c.$$

Then $e_2 \leq x(t) \leq \frac{e_2}{\mu}$, $t \in [0,1]$ and $0 \leq x'(t) \leq c$. Thus (C6) implies that $f(t, x(t), x'(t)) > \phi_p\left(\frac{e_2}{W}\right)$, $t \in [0,1]$. We

get

$$\begin{aligned}
\vartheta(Tx) &= (Tx)(0) \\
&> \frac{a}{\Gamma(\alpha-\gamma)}\int_0^1 (1-s)^{\alpha-\gamma-1}\phi_q\left[\frac{1}{\Gamma(\beta)}\int_0^s (s-r)^{\beta-1}\phi_p\left(\frac{e_2}{W}\right)dr\right]ds \\
&= \frac{e_2}{W}\frac{a}{\Gamma(\alpha-\gamma)}\int_0^1 (1-s)^{\alpha-\gamma-1}\phi_q\left[\frac{1}{\Gamma(\beta)}\int_0^s (s-r)^{\beta-1}dr\right]ds \\
&= \frac{e_2}{W}\frac{a}{\Gamma(\alpha-\gamma)}\int_0^1 (1-s)^{\alpha-\gamma-1}\phi_q\left(\frac{s^\beta}{\Gamma(\beta+1)}\right)ds \\
&= \frac{e_2}{W}\frac{a}{\Gamma(\alpha-\gamma)}\int_0^1 (1-s)^{\alpha-\gamma-1}\left[\frac{s^\beta}{\Gamma(\beta+1)}\right]^{q-1}ds \\
&= \frac{e_2}{W}\frac{a}{\Gamma(\alpha-\gamma)}\frac{1}{[\Gamma(\beta+1)]^{q-1}}\beta(\alpha-\gamma,\beta q-\beta+1) \\
&= \frac{e_2}{W}\frac{a}{[\Gamma(\beta+1)]^{q-1}}\frac{\Gamma(\beta q-\beta+1)}{\Gamma(\alpha-\gamma+\beta q-\beta+1)} \\
&= e_2.
\end{aligned}$$

This completes Step 2.

Step 3. Prove that

$$\{x \in Q(\chi,\varphi,\psi,\mu e_1,e_1,c) : \varphi(x) < e_1\} \neq \emptyset,$$

and $\varphi(Tx) < e_1$ for every $x \in Q(\chi,\varphi,\psi,\mu e_1,e_1,c)$. Choose $x(t) = \mu e_1$. Then $x \in P$ and $\psi(x) = \min_{t\in[0,1]} x(t) = \mu e_1$, $\varphi(x) = \theta(x) = \max_{t\in[0,1]} x(t) = \mu e_1 < e_1$, $\chi(x) = \max_{t\in[0,1]} x'(t) = 0 \le c$. It follows that

$$\{x \in Q(\chi,\varphi,\psi,\mu e_1,e_1,c) : \varphi(x) < e_1\} \neq \emptyset.$$

For $x \in Q(\chi,\varphi,\psi,\mu e_1,e_1,c)$, one has that

$$\psi(x) = \min_{t\in[0,1]} x(t) \ge \mu e_1,\ \theta(x) = \max_{t\in[0,1]} x(t) \le e_1,\ \chi(x) = \max_{t\in[0,1]} x'(t) \le c.$$

Hence we get that $\mu e_1 \le x(t) \le e_1$ and $0 \le x'(t) \le c$ for $t \in [0,1]$. Then (C7) implies that $f(t,x(t),x'(t)) < \phi_p\left(\frac{e_1}{\Omega}\right)$

for $t \in [0,1]$. From Lemma 2.5 and (2.16) we get

$$\begin{aligned}
\varphi(Tx) &= \max_{t\in[0,1]} Tx(t) \\
&\leq M \max_{t\in[0,1]} (Tx)'(t) \\
&= M(Tx)'(1) \\
&= \frac{M}{\Gamma(\alpha-1)(1-b)} \int_0^1 (1-s)^{\alpha-2} \phi_q(I_{0^+}^{\beta} h(s)) ds \\
&< \frac{M}{\Gamma(\alpha-1)(1-b)} \int_0^1 (1-s)^{\alpha-2} \phi_q \left[\frac{1}{\Gamma(\beta)} \int_0^s (s-r)^{\beta-1} \phi_p \left(\frac{e_1}{\Omega} \right) dr \right] ds \\
&= \frac{Me_1}{\Gamma(\alpha-1)(1-b)\Omega} \int_0^1 (1-s)^{\alpha-2} \phi_q \left[\frac{s^{\beta}}{\Gamma(\beta+1)} \right] ds \\
&= \frac{Me_1}{\Gamma(\alpha-1)(1-b)\Omega \left[\Gamma(\beta+1)\right]^{q-1}} \int_0^1 (1-s)^{\alpha-2} s^{\beta q-\beta} ds \\
&= \frac{Me_1}{\Gamma(\alpha-1)(1-b)\Omega \left[\Gamma(\beta+1)\right]^{q-1}} \beta\left(\alpha-1, \beta q-\beta+1\right) \\
&= \frac{Me_1}{(1-b)\Omega \left[\Gamma(\beta+1)\right]^{q-1}} \frac{\Gamma(\beta q-\beta+1)}{\Gamma(\alpha+\beta q-\beta)} \\
&= e_1.
\end{aligned}$$

This completes Step 3.

Step 4. Prove that $\vartheta(Tx) > a$ for $x \in P(\chi, \vartheta, e_2, c)$ with $\theta(Tx) > \frac{e_2}{\mu}$. We have that $\vartheta(x) = \min_{t\in[0,1]} x(t) \geq e_2$, $\chi(x) = \max_{t\in[0,1]} x'(t) \leq c$ with $\theta(Tx) = \max_{t\in[0,1]} Tx(t) > \frac{e_2}{\mu}$. From Lemma 2.4

$$\vartheta(Tx) = \min_{t\in[0,1]} (Tx)(t) = \mu \max_{t\in[0,1]} (Tx)(t) \geq \mu\varphi(Tx) > e_2.$$

This completes Step 4.

Step 5. Prove that $\varphi(Tx) < d$ for each $x \in Q(\chi, \varphi; d, c)$ with $\psi(Tx) < h$.

For $x \in Q(\chi, \varphi; e_1, c)$ with $\psi(Tx) < \mu e_1$, we have

$$\begin{aligned}
\chi(x) &= \max_{t\in[0,1]} |x'(t)| \leq c, \\
\varphi(x) &= \max_{t\in[0,1]} x(t) \leq e_1, \\
\psi(Tx) &= \min_{t\in[0,1]} (Tx)(t) < \mu e_1.
\end{aligned}$$

Then $Tx \in P$ implies from Lemma 2.4

$$\varphi(Tx) = \max_{t\in[0,1]} (Tx)(t) \leq \frac{1}{\mu} \min_{t\in[0,1]} (Tx)(t) < e_1.$$

This completes Step 5.

Then Lemma 3.1 implies that T has at least three fixed points $x_1,\ x_2,\ x_3 \in P$ such that

$$\varphi(x_1) < e_1,\ \vartheta(x_2) > e_2,\ \varphi(x_3) > e_1,\ \vartheta(x_3) < e_2.$$

Hence BVP (1.1) has three positive solutions x_1, x_2 and x_3. The proof is complete. □

4. An Example

Example 4.1 Consider the following boundary value problem,

$$\begin{cases} D_{0^+}^{\frac{1}{2}}(\phi_{\frac{6}{5}}(D_{0^+}^{\frac{7}{2}}x(t))) = f(t,x(t),x'(t)), \quad t \in (0,1), \\ x(0) = 50D_{0^+}^{\frac{1}{2}}x(1), \\ x'(0) = \dfrac{1}{3}x'(1), \\ x^{(i)}(0) = 0, \quad i = 2,3, \end{cases} \tag{4.1}$$

where

$$f(t,u,v) = \begin{cases} \dfrac{t^2}{1000} + \dfrac{2}{5} + \dfrac{v}{15000}, & (t,u,v) \in [0,1]\times[0,1]\times[0,15]; \\ \dfrac{t^2}{1000} + \dfrac{1}{75}(23u+7) + \dfrac{v}{15000}, & (t,u,v) \in [0,1]\times[1,4]\times[0,15]; \\ \dfrac{t^2}{1000} + \dfrac{1}{85730900}(48000u + 112972788) + \dfrac{v}{15000}, & (t,u,v) \in [0,1]\times[4,\infty)\times[0,15]. \end{cases}$$

Here $\beta = \dfrac{1}{2},\ p = \dfrac{6}{5},\ q = 6,\ \alpha = \dfrac{7}{2},\ \gamma = \dfrac{1}{2},\ a = 50,\ b = \dfrac{1}{3},\ n = 4.$ Then we have

$$M = \max\left\{\frac{a}{\Gamma(2-\gamma)}, \frac{a\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma)}\right\} + 1 \cong \max\left\{\frac{50}{0.8862}, \frac{66.465}{2}\right\} + 1 \cong 57.4206,\ \mu \cong 0.3539.$$

Taking $e_1 = 1,\ e_2 = 4,\ c = 15$, we get $\dfrac{e_2}{\mu} \cong 11.3026$, $\mu e_1 \cong 0.3539$, $Mc \cong 861.309$, $W \cong 1.0558$, $Q \cong 0.7599$, $\Omega = MQ \cong 43.6339$. We can obtain that

$$c \geq \frac{e_2}{\mu} > e_2 > e_1 > 0,\ W > Q.$$

It is easy to verify that (C5)-(C7) hold:

$$\begin{aligned} f(t,u,v) &\leq 1.802 \leq \phi_p\left(\frac{c}{Q}\right) \cong 1.81579,\ t \in [0,1],\ u \in [0,861.309],\ v \in [0,15]; \\ f(t,u,v) &\geq 1.32 > \phi_p\left(\frac{e_2}{W}\right) \cong 1.3052,\ t \in [0,1],\ u \in [4.11,3026],\ v \in [0,15]; \\ f(t,u,v) &\leq 0.402 < \phi_p\left(\frac{e_1}{\Omega}\right) \cong 0.4699,\ t \in [0,1],\ u \in [0.3539,1],\ v \in [0,15]. \end{aligned}$$

Then, all conditions of Theorem 3.2 hold. Hence, we get the BVP (4.1) has at least three positive solutions $x_1,\ x_2$ and x_3 such that

$$x_1(1) < 1,\ x_2(0) > 4,\ x_3(1) > 1,\ x_3(0) < 4.$$

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