Fundamental Serrin Type Regularity Criteria for 3D MHD Fluid Passing Through the Porous Medium

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Abstract. In this paper, we consider the Cauchy problem for the 3D MHD fluid passing through the porous medium, and provide some fundamental Serrin type regularity criteria involving the velocity or its gradient, the pressure or its gradient. This extends and improves [S. Rahman, Regularity criterion for 3D MHD fluid passing through the porous medium in terms of gradient pressure, *J. Comput. Appl. Math.*, **270** (2014), 88–99].

1. Introduction

This paper concerns about the 3D flows of an incompressible fluid passing through the porous medium. Let $\boldsymbol{u} = (u_1, u_2, u_3)$ be the fluid velocity, and ϕ and K are the porosity and permeability of the medium respectively. Then there exists a pressure drop

$$\boldsymbol{R} = -m\boldsymbol{u}, \quad m = \frac{\phi}{K} > 0$$

by using Darcy's law (see [4]), when the fluid passes through the porous medium. Consequently, the fundamental equations governing the dynamics of an incompressible, unsteady MHD fluid passing through the porous medium are (see also [12])

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{b} - \mu \triangle \boldsymbol{u} + \nabla \pi = -m\boldsymbol{u}, \\ \partial_t \boldsymbol{b} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{b} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} - \nu \triangle \boldsymbol{b} = \boldsymbol{0}, \\ \nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{b} = 0, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{b}(0) = \boldsymbol{b}_0, \end{cases} \tag{1}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ is the fluid velocity, $\boldsymbol{b} = (b_1, b_2, b_3)$ is the magnetic field, π is a pressure scalar, $\mu > 0$ and $\nu > 0$ are the kinematic viscosity and magnetic diffusivity respectively. For simplicity, we set $\mu = \nu = 1$.

When $m = 0$, (1) reduces to the classical MHD equations, which have drawn a lot of mathematicians' attention. Duvaut and Lions [6] constructed a class of global weak solution with finite energy. Sermange

and Temam [13] considered the strong and classical solutions, their results show that the MHD system has a global unique classical solution in 2D case, while has only local unique strong solution in 3D case. Thus whether or not this local strong solution being global is an outstanding problem. In [11] and [15], the authors first established the following regularity criteria in terms of velocity or velocity gradient only:

$$u \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 1, \quad 3 < q \le \infty, \tag{2}$$

and

$$\nabla u \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} < q \le \infty; \tag{3}$$

that is, if (2) or (3) holds, then the solution (u, b) of the MHD system is regular on $(0, T)$. These two criteria have been improved in larger spaces and other related models, see [2, 7–10, 18–20] for example. Subsequently, we consider the regularity criteria involving the pressure or the pressure gradient. Zhou [16] first showed that each of the following conditions on the pressure (or its gradient) and the magnetic field

$$\pi \in L^p(0, T; L^q(\mathbb{R}^3)), \quad b \in L^{2p}(0, T; L^{2q}(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} < q \le \infty; \tag{4}$$

or

$$\nabla \pi \in L^p(0, T; L^q(\mathbb{R}^3)), \quad b \in L^{3p}(0, T; L^{3q}(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 3, \quad 1 < q \le \infty \tag{5}$$

implies the smoothness of the solution. Then by converting the MHD system into a mathematically symmetric form

$$\begin{aligned}
&\partial_t \omega^+ + (\omega^- \cdot \nabla)\omega^+ - \triangle\omega^+ + \nabla\pi = \mathbf{0}, \\
&\partial_t \omega^- + (\omega^+ \cdot \nabla)\omega^- - \triangle\omega^- + \nabla\pi = \mathbf{0}, \\
&\nabla \cdot \omega^+ = \nabla \cdot \omega^- = 0, \\
&\omega^+(0) = \omega_0^+, \quad \omega^-(0) = \omega_0^-
\end{aligned} \tag{6}$$

with

$$\omega^+ = u + b, \quad \omega^- = u - b, \quad \omega_0^+ = u_0 + b_0, \quad \omega_0^- = u_0 - b_0,$$

Duan [5] was able to remove the conditions of the magnetic field, and established the following regularity conditions:

$$\pi \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} < q \le \infty; \tag{7}$$

and

$$\nabla \pi \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 3, \quad 1 < q \le \infty. \tag{8}$$

For the system (1), Rahman [12] showed that (4) or (5) again entails the regularity of the solution. In this paper, the authors first extend the regularity criteria in terms of velocity or velocity gradient, (2) and (3), to the system (1), and then improve [12] by removing the conditions of the magnetic field.

Before stating the precise result, let us first recall the definition of weak solutions to (1) (see also [12]).

Definition 1.1. *Let $u_0, b_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u = \nabla \cdot b = 0$. A measurable pair (u, b) is called a weak solution to the system* (1) *provided the following conditions holds:*
(1) $u, b \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$;

(2) $\boldsymbol{u}, \boldsymbol{b}$ verify $(1)_{1,2,3}$ in the distributional sense;

(3) the energy inequality is valid for almost every $t \geq 0$,

$$\|(\boldsymbol{u}, \boldsymbol{b})\|_{L^2}^2 (t) + 2 \int_0^t \|\nabla(\boldsymbol{u}, \boldsymbol{b})\|_{L^2}^2 (s) \, ds + 2m \int_0^t \|\boldsymbol{u}\|_{L^2}^2 (s) \, ds \leq \|(\boldsymbol{u}_0, \boldsymbol{b}_0)\|_{L^2}^2 . \tag{9}$$

Here and in what follows, we shall use the notation

$$\|(\boldsymbol{u}, \boldsymbol{b})\|_{L^2}^2 = \|\boldsymbol{u}\|_{L^2}^2 + \|\boldsymbol{b}\|_{L^2}^2 , \quad \|\nabla(\boldsymbol{u}, \boldsymbol{b})\|_{L^2}^2 = \|\nabla\boldsymbol{u}\|_{L^2}^2 + \|\nabla\boldsymbol{b}\|_{L^2}^2 .$$

The existence of a weak solution to (1) can be established similarly as in the proof of Theorem 6.1 in [6]. In fact, compared with the MHD system, the system (1) has only one more damping term (see [3, 14, 17]) $m\boldsymbol{u}$, which is linear in $\boldsymbol{u}$. By a strong solution, we mean a weak solution such that

$$\boldsymbol{u}, \boldsymbol{b} \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)).$$

It is well known (see [13]) that the strong solutions exists locally, and are regular and unique in the class of weak solutions.

Now, our first result is the following regularity criteria via the velocity or its gradient.

Theorem 1.2. *Let $\boldsymbol{u}_0, \boldsymbol{b}_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{b} = 0$. Assume $(\boldsymbol{u}, \boldsymbol{b})$ is a weak solution pair to (1) on $(0, T)$. If (2) or (3) holds, then the solution is smooth on $(0, T)$.*

Next, by converting the system (1) into the following mathematically symmetric form

$$\begin{cases} \partial_t \omega^+ + (\omega^- \cdot \nabla)\omega^+ - \triangle\omega^+ + \nabla\pi + \frac{1}{2}(\omega^+ + \omega^-) = \boldsymbol{0}, \\ \partial_t \omega^- + (\omega^+ \cdot \nabla)\omega^- - \triangle\omega^- + \nabla\pi + \frac{1}{2}(\omega^+ + \omega^-) = \boldsymbol{0}, \\ \nabla \cdot \omega^+ = \nabla \cdot \omega^- = 0, \\ \omega^+(0) = \omega_0^+, \quad \omega^-(0) = \omega_0^- \end{cases} \tag{10}$$

with

$$\omega^+ = \boldsymbol{u} + \boldsymbol{b}, \quad \omega^- = \boldsymbol{u} - \boldsymbol{b}, \quad \omega_0^+ = \boldsymbol{u}_0 + \boldsymbol{b}_0, \quad \omega_0^- = \boldsymbol{u}_0 - \boldsymbol{b}_0,$$

we show

Theorem 1.3. *Let $\boldsymbol{u}_0, \boldsymbol{b}_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{b} = 0$. Assume $(\boldsymbol{u}, \boldsymbol{b})$ is a weak solution pair to (1) on $(0, T)$. If (7) or (8) holds, then the solution is smooth on $(0, T)$.*

The proof of Theorem 1.2 is given in Section 2, and Section 3 is devoted to proving Theorem 1.3.

2. Proof of Theorem 1.2

In this section, we shall provide the proof of Theorem 1.2.

Step I Preliminary Reductions.

For any $\varepsilon \in (0, T)$, due to the fact that $\boldsymbol{u}, \boldsymbol{b} \in L^2(0, T; H^1(\mathbb{R}^3))$, we may find a $\delta \in (0, \varepsilon)$, such that $\nabla\boldsymbol{u}(\delta), \nabla\boldsymbol{b}(\delta) \in L^2(\mathbb{R}^3)$. Take this $\boldsymbol{u}(\delta), \boldsymbol{b}(\delta)$ as initial data, there exists a strong solution pair $\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{b}} \in C([\delta, \Gamma^*), H^1(\mathbb{R}^3)) \cap L^2(0, \Gamma^*; H^2(\mathbb{R}^3))$, where $[\delta, \Gamma^*)$ is the life span of the unique strong solution, see [13]. Moreover, $\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{b}} \in C^\infty(\mathbb{R}^3 \times (\delta, \Gamma^*))$. According to the uniqueness result, $\tilde{\boldsymbol{u}} = \boldsymbol{u}, \tilde{\boldsymbol{b}} = \boldsymbol{b}$ on $[\delta, \Gamma^*)$. If $\Gamma^* \geq T$, we have already that $\boldsymbol{u}, \boldsymbol{b} \in C^\infty(\mathbb{R}^3 \times (0, T))$, due to the arbitrariness of $\varepsilon \in (0, T)$. In case $\Gamma^* < T$, our strategy is to show that $\|\nabla(\boldsymbol{u}, \boldsymbol{b})\|_2 (t)$ is uniformly bounded for $t \in [\delta, \Gamma^*)$. The standard continuation argument then yields that $[\delta, \Gamma^*)$ can not be the maximal interval of existence of the solution pair $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{b}})$, and consequently $\Gamma^* \geq T$. This concludes the proof.

Step II H^1-estimates.

Taking the inner product of $(1)_1$ with $-\triangle u$, $(1)_2$ with $-\triangle b$ in $L^2(\mathbb{R}^3)$ respectively, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(u,b)\|_{L^2}^2 + \|\triangle(u,b)\|_{L^2}^2 + m\|\nabla u\|_{L^2}^2 \\
&= \int_{\mathbb{R}^3}[(u\cdot\nabla)u]\cdot\triangle u\,\mathrm{d}x - \int_{\mathbb{R}^3}[(b\cdot\nabla)b]\cdot\triangle u\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}[(u\cdot\nabla)b]\cdot\triangle b\,\mathrm{d}x - \int_{\mathbb{R}^3}[(b\cdot\nabla)u]\cdot\triangle b\,\mathrm{d}x \\
&= -\sum_{k=1}^{3}\int_{\mathbb{R}^3}[(\partial_k u\cdot\nabla)u]\cdot\partial_k u\,\mathrm{d}x + \sum_{k=1}^{3}\int_{\mathbb{R}^3}[(\partial_k b\cdot\nabla)b]\cdot\partial_k u\,\mathrm{d}x \\
&\quad - \sum_{k=1}^{3}\int_{\mathbb{R}^3}[(\partial_k u\cdot\nabla)b]\cdot\partial_k b\,\mathrm{d}x + \sum_{k=1}^{3}\int_{\mathbb{R}^3}[(\partial_k b\cdot\nabla)u]\cdot\partial_k b\,\mathrm{d}x \\
&\equiv I.
\end{aligned}
\tag{11}
$$

Here, we use the following facts (by integrating by parts):

$$
\nabla\cdot u = 0 \Rightarrow \sum_{k=1}^{3}\int_{\mathbb{R}^3}[u\cdot\nabla(\partial_k u)]\cdot\partial_k u\,\mathrm{d}x = 0,
$$

$$
\nabla\cdot u = 0 \Rightarrow \sum_{k=1}^{3}\int_{\mathbb{R}^3}[u\cdot\nabla(\partial_k b)]\cdot\partial_k b\,\mathrm{d}x = 0,
$$

$$
\nabla\cdot b = 0 \Rightarrow \sum_{k=1}^{3}\int_{\mathbb{R}^3}[b\cdot\nabla(\partial_k b)]\cdot\partial_k u\,\mathrm{d}x + \sum_{k=1}^{3}\int_{\mathbb{R}^3}[b\cdot\nabla(\partial_k u)]\cdot\partial_k b\,\mathrm{d}x = 0.
$$

Suppose (2) holds, we may apply integrating by parts formula to each term in I to get

$$
\begin{aligned}
I &\leq C\int_{\mathbb{R}^3}|u|\cdot|\nabla(u,b)|\cdot|\nabla^2(u,b)|\,\mathrm{d}x \\
&\leq C\|u\|_{L^q}\|\nabla(u,b)\|_{L^{\frac{2q}{q-2}}}\left\|\nabla^2(u,b)\right\|_{L^2}\quad\text{(Hölder inequality)} \\
&\leq C\|u\|_{L^q}\|\nabla(u,b)\|_{L^2}^{1-\frac{3}{q}}\left\|\nabla^2(u,b)\right\|_{L^2}^{1+\frac{3}{q}},
\end{aligned}
$$

where, in the last inequality, we have used the interpolation inequality

$$
\|f\|_{L^s} \leq \|f\|_{L^2}^{\frac{3}{s}-\frac{1}{2}}\|f\|_{L^2}^{\frac{3}{2}-\frac{3}{s}},\quad 2\leq s\leq 6,
$$

the Sobolev inequality

$$
\|f\|_{L^6} \leq C\|\nabla f\|_{L^2},
$$

and the boundedness of the Riesz transformation $\mathcal{R}_i$ ($1\leq i\leq 3$) in $L^s(\mathbb{R}^3)$ ($1<s<\infty$) spaces,

$$
\left\|\partial_i\partial_j f\right\|_{L^s} = \left\|\frac{\partial_i}{\sqrt{-\triangle}}\frac{\partial_j}{\sqrt{-\triangle}}(-\triangle)f\right\|_{L^s} = \left\|\mathcal{R}_i\mathcal{R}_j(-\triangle)f\right\|_{L^s} \leq C\left\|\triangle f\right\|_{L^s}.
$$

By Young inequality, we may further dominate I as

$$
I \leq C\|u\|_{L^q}^{\frac{2q}{q-3}}\|\nabla(u,b)\|_{L^2}^2 + \frac{1}{2}\|\triangle(u,b)\|_{L^2}^2.
$$

Substituting the above inequality into (11), and absorbing $\frac{1}{2}\|\triangle(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2$ into the left hand side, we may invoke the Gronwall inequality to deduce that

$$
\begin{aligned}
\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2(t) &\leq \|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2(\delta) \cdot \exp\left[C\int_\delta^t \|\boldsymbol{u}\|_{L^q}^{\frac{2q}{q-3}}(s)\,\mathrm{d}s\right] \\
&\leq \|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2(\delta) \cdot \exp\left[C\int_0^T \|\boldsymbol{u}\|_{L^q}^{\frac{2q}{q-3}}(s)\,\mathrm{d}s\right] < \infty, \ \delta \leq t < \Gamma^*,
\end{aligned}
\tag{12}
$$

as desired. The proof of Theorem 1.2 in case (2) is completed.

Suppose now (3) is valid, we may estimate I as

$$
\begin{aligned}
I &\leq C\int_{\mathbb{R}^3} |\nabla\boldsymbol{u}| \cdot |\nabla(\boldsymbol{u},\boldsymbol{b})|^2\,\mathrm{d}x \\
&\leq C\|\nabla\boldsymbol{u}\|_{L^q}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^{\frac{2q}{q-1}}}^2 \\
&\leq C\|\nabla\boldsymbol{u}\|_{L^q}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{2-\frac{3}{q}}\|\triangle(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{\frac{3}{q}} \\
&\leq C\|\nabla\boldsymbol{u}\|_{L^q}^{\frac{2q}{2q-3}}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 + \frac{1}{2}\|\triangle(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.
\end{aligned}
$$

Substituting the above inequality into (11), and absorbing $\frac{1}{2}\|\triangle(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2$ into the left hand side, we may invoke the Gronwall inequality to deduce that

$$
\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2(t) \leq \|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2(\delta) \cdot \exp\left[C\int_0^T \|\nabla\boldsymbol{u}\|_{L^q}^{\frac{2q}{2q-3}}(s)\,\mathrm{d}s\right] < \infty, \delta \leq t < \Gamma^*,
$$

as desired. The proof of Theorem 1.2 in case (2) is completed.

3. Proof of Theorem 1.3

In this section, we shall prove Theorem 1.3.

As stated in Section 3, we need only to bound $\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2(t)$ uniformly in $t \in [\delta,\Gamma^*)$, where $\delta > 0$ can be chosen arbitrarily small such that $\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}(\delta)$ is finite. Due to (12), we have to dominate $\int_\delta^{\Gamma^*} \|\boldsymbol{u}\|_{L^q}^{\frac{2q}{q-3}}(s)\,\mathrm{d}s$ for some $3 < q \leq \infty$. In the following, we shall in fact prove

$$
\boldsymbol{u},\boldsymbol{b} \in L^\infty(\delta,\Gamma^*;L^4(\mathbb{R}^3)) \subset L^8(\delta,\Gamma^*;L^4(\mathbb{R}^3)),
$$

and thus complete the proof.

To this end, taking the inner product of $(1)_1$ with $|\omega^+|^2\omega^+$, $(1)_2$ with $|\omega^-|^2\omega^-$ in $L^2(\mathbb{R}^3)$ respectively, and adding the resulting equations together, we find

$$
\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(|\omega^+|^4 + |\omega^-|^4)\,\mathrm{d}x \\
&+ \frac{1}{2}\int_{\mathbb{R}^3} |\nabla|\omega^+|^2|^2 + |\nabla|\omega^-|^2|^2\,\mathrm{d}x + \int_{\mathbb{R}^3} |\nabla\omega^+|^2 \cdot |\omega^+|^2 + |\nabla\omega^-|^2 \cdot |\omega^-|^2\,\mathrm{d}x \\
&= -\frac{1}{2}\int_{\mathbb{R}^3}(\omega^+ + \omega^-)(|\omega^+|^2\omega^+ + |\omega^-|^2\omega^-)\,\mathrm{d}x - \int_{\mathbb{R}^3}\nabla\pi \cdot (|\omega^+|^2\omega^+ + |\omega^-|^2\omega^-)\,\mathrm{d}x.
\end{aligned}
$$

Noticing that

$$-\frac{1}{2}\int_{\mathbb{R}^3}(\omega^+ + \omega^-)(|\omega^+|^2\omega^+ + |\omega^-|^2\omega^-)\,dx$$

$$= -\frac{1}{2}\int_{\mathbb{R}^3}(|\omega^+|^4 + |\omega^-|^2\omega^+\cdot\omega^- + |\omega^+|^2\omega^+\cdot\omega^- + |\omega^-|^4)\,dx$$

$$\leq -\frac{1}{2}\int_{\mathbb{R}^3}(|\omega^+|^4 - |\omega^+|\cdot|\omega^-|^3 - |\omega^+|^3\cdot|\omega^-| + |\omega^-|^4)\,dx$$

$$= -\frac{1}{2}\int_{\mathbb{R}^3}(|\omega^+| - |\omega^-|)(|\omega^+|^3 - |\omega^-|^3)\,dx$$

$$= -\frac{1}{2}\int_{\mathbb{R}^3}(|\omega^+| - |\omega^-|)^2(|\omega^+|^2 + |\omega^+|\cdot|\omega^-| + |\omega^-|^2)\,dx$$

$$\leq 0,$$

we have

$$\frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}^3}(|\omega^+|^4 + |\omega^-|^4)\,dx$$

$$+ \frac{1}{2}\int_{\mathbb{R}^3}|\nabla|\omega^+|^2|^2 + |\nabla|\omega^-|^2|^2\,dx + \int_{\mathbb{R}^3}|\nabla\omega^+|^2\cdot|\omega^+|^2 + |\nabla\omega^-|^2\cdot|\omega^-|^2\,dx \qquad (13)$$

$$\leq -\int_{\mathbb{R}^3}\nabla\pi\cdot(|\omega^+|^2\omega^+ + |\omega^-|^2\omega^-)\,dx$$

$$\equiv J.$$

Up to now, we could just estimate J as in [5] to get

$$\frac{d}{dt}\left\||\omega^+|^2 + |\omega^-|^2\right\|_{L^2}^2 \leq C\|\pi\|_{L^q}^{\frac{2q}{2q-3}}\left\||\omega^+|^2 + |\omega^-|^2\right\|_{L^2}^2,$$

as well as

$$\frac{d}{dt}\left\||\omega^+|^2 + |\omega^-|^2\right\|_{L^2}^2 \leq C\|\nabla\pi\|_{L^q}^{\frac{2q}{3(q-1)}}\left\||\omega^+|^2 + |\omega^-|^2\right\|_{L^2}^2.$$

Applying Gronwall inequality to the last two inequalities, we deduce

$$\left\||\omega^+|^2 + |\omega^-|^2\right\|_{L^2}^2(t) \leq \left\||\omega^+|^2 + |\omega^-|^2\right\|_{L^2}^2(\delta)\cdot\exp\left[C\int_\delta^t\|\pi\|_{L^q}^{\frac{2q}{2q-3}}(s)\,ds\right]$$

$$\leq \|(u,b)\|_{L^4}^4(\delta)\cdot\exp\left[C\int_\delta^t\|\pi\|_{L^q}^{\frac{2q}{2q-3}}(s)\,ds\right]$$

$$\leq \|(u,b)\|_{L^2}^2(\delta)\cdot\|\nabla(u,b)\|_{L^2}^2(\delta)\cdot\exp\left[C\int_\delta^t\|\pi\|_{L^q}^{\frac{2q}{2q-3}}(s)\,ds\right].$$

and

$$\left\||\omega^+|^2 + |\omega^-|^2\right\|_{L^2}^2(t) \leq \|(u,b)\|_{L^2}^2(\delta)\cdot\|\nabla(u,b)\|_{L^2}^2(\delta)\cdot\exp\left[C\int_\delta^t\|\nabla\pi\|_{L^q}^{\frac{2q}{3(q-1)}}(s)\,ds\right].$$

Hence, under the assumption (7) or (8), we could deduce

$$\omega^+,\ \omega^- \in L^\infty(0,T;L^4(\mathbb{R}^3)) \Rightarrow u = \frac{1}{2}(\omega^+ + \omega^-) \in L^\infty(0,T;L^4(\mathbb{R}^3))$$

as desired. The proof of Theorem 1.3 is completed.

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