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A Class of Non-Holonomic Projective Connections on Sub-Riemannian Manifolds

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Abstract. The authors define a semi-symmetric non-holonomic (SSNH)-projective connection on sub-Riemannian manifolds and find an invariant of the SSNH-projective transformation. The authors further derive that a sub-Riemannian manifold is of projective flat if and only if the Schouten curvature tensor of a special SSNH-connection is zero.

1. Introduction

Since A. Friedmann and J. A. Schouten [8], in the early days of 1924, firstly introduced the concept of semi-symmetric linear connections, the research related to the semi-symmetric connection was unusually brilliant, and made a series of fruitful research results.

K. Yano [21] introduced and studied the semi-symmetric metric connection of Riemannian manifolds. N. S. Agashe and M. R. Chafle [1] introduced a semi-symmetric non-metric connection on a Riemannian manifold and this was further studied by U. C. De etc. [4–7], T. Imai [15], P. B. Zhao and H. Z. Song [27], and so on.

D. K. Sen and J. R. Vastane [19] studied the Weyl manifold by using the idea of semi-symmetric connections. Later the Weyl structure was extended in the semi-Riemannian distribution framework by O. Constantinescu and M. Crasmareanu, see [3]. I. Hinterleitner and J. Mikeš [13] studied the geodesic mappings onto Weyl manifolds. M. Tripathi and N. Nakkar [20] studied the semi-symmetric non-metric connection in a Kenmotsu manifold. B. Barua and A. K. Ray [2] studied the curvature properties of semi-symmetric metric connections and derived a sufficient and necessary condition for the Ricci tensor being of symmetric.

H. B. Yilmaz, F. O. Zengin and S. Aynur Uysal [22] considered a manifold equipped with a semi-symmetric metric connection whose torsion tensor satisfied a special condition and proved that if a manifold

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mentioned as above was conformally flat, then it was a mixed quasi-Einstein manifold. F. Y. Fu, X. P. Yang and P. B. Zhao [9] considered the geometric and physical properties of conformal mappings for the semi-Riemannian manifolds.

J. Mikeš [17, 18] studied the projective and geodesic mappings of special Riemannian spaces. F. Y. Fu and P. B. Zhao [10] studied the geodesic mapping of pseudo-symmetric Riemannian Manifolds. In particular, the second author [23] recently studied the conformal and projective characteristics of sub-Riemannian manifolds by using the so-called non-holonomic sub-Riemannian connection. I. Hinterleitner [11] studied geodesic mappings on compact Riemannian manifolds. I. Hinterleitner and J. Mikeš [12, 13, 16] studied projective and affine connections. M. Zlatanović and etc. [24–26] studied geodesic mappings and similar problems.

However, to the author's knowledge, the study of geometric and analysis in sub-Riemannian manifolds on view of the semi-symmetric metric connection in sub-Riemannian manifolds is still a gap.

In this paper, we will, based on the setting of [14], investigate a class of semi-symmetric non-holonomic connections, find the sub-Weyl projective invariant, and study the projective flatness of sub-Riemannian manifolds.

2. The SSNH-Projective Transformation

Let $(M, \triangle, g_\triangle)$ be a n-dimensional sub-Riemannian manifold, where $\triangle$ is a ℓ-dimensional sub-bundle of tangent bundles, and is called a horizontal bundle, and $g_\triangle$ is a Riemannian metric defined on $\triangle$. In particular, when $\triangle = TM$, $(M, \triangle, g_\triangle)$ will be degenerated into a Riemanian manifold. Without loss of generality, we assume $\triangle \neq TM$. In this subsection, we will define a semi-symmetric non-holonomic(SSNH) metric connection and discuss the SSNH-projective transformation following the work in [14].

We use unless otherwise noted the following ranges for indices: $i, j, k, h, \cdots \in \{1, \cdots, \ell\}$. The repeated indices with one upper index and one lower index indicates the summation over their range. The projection of X on the horizontal bundle is denoted by X_h.

Definition 2.1. *A non-holonomic connection on sub-bundle $Q \subset TM$ is a mapping $\nabla : \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$ satisfying the following*

$$\nabla_{X_h}(Y_h + Z_h) = \nabla_{X_h} Y_h + \nabla_{X_h} Z_h, \nabla_{X_h}(fY_h) = X_h(f)Y_h + f\nabla_{X_h} Y_h, \nabla_{fX_h+gY_h} Z_h = f\nabla_{X_h} Z_h + g\nabla_{Y_h} Z_h,$$

where $X, Y, Z \in \Gamma(TM)$, $f, g \in C^\infty(M)$.

Definition 2.2. *A non-holonomic connection is said to be metric and symmetric if it satisfies respectively,*

$$(\nabla_{Z_h} g_\Delta)(X_h, Y_h) = Z_h(g_\Delta(X_h, Y_h)) - g_\Delta(\nabla_{Z_h} X_h, Y_h) - g_\Delta(X_h, \nabla_{Z_h} Y_h) = 0,$$
$$T(X_h, Y_h)) = \nabla_{X_h} Y_h - \nabla_{Y_h} X_h - [X_h, Y_h]_h = 0.$$

Definition 2.3. *A non-holonomic connection is said to be a sub-Riemannian connection if it is both metric and symmetric.*

Definition 2.4. *A horizontal curve $\gamma(t) : [0, 1] \to M$ (i.e. $\dot\gamma(t) \in \Delta_{\gamma(t)}$) is said to be a sub-Riemannian parallel (in briefly, SR-parallel) curve if it satisfies*

$$\nabla_{\dot\gamma}\dot\gamma = 0, \tag{1}$$

where ∇ is the sub-Riemannian connection.

Let $\gamma : x^a = x^a(t)$, the corresponding equation (1) is

$$\frac{d^2x^k}{dt^2} + \left\{ {k \atop ij} \right\} \frac{dx^i}{dt}\frac{dx^j}{dt} = 0, \tag{2}$$

where t is an affine parameter.

Definition 2.5. *Let D_1, D_2 be two classes of non-holonomic connections. If a SR-parallel curve corresponding to D_1 coincides always with one corresponding to D_2, then we say that D_1 is a projective correspondence to D_2.*

The second author [23] proved that a non-holonomic symmetric connection D is a projective correspondence if and only if there exists a smooth horizontal 1-form φ (i.e. a 1-form defined on $\triangle$), such that, for any two horizontal vector fields X_h, Y_h, there holds

$$D_{X_h}Y_h = \nabla_{X_h}Y_h + \varphi(X_h)Y_h + \varphi(Y_h)X_h. \tag{3}$$

If D is a non-holonomic connection with torsion, then we have the following

Proposition 2.6. *A non-holonomic connection D with torsion is a projective correspondence to ∇ if and only if there exists 1-form λ such that the symmetric part of tensor $A(X_h, Y_h) = D_{X_h}Y_h - \nabla_{X_h}Y_h$ is of the form*

$$(A(X_h, Y_h) + A(Y_h, X_h))/2 = \lambda(X_h)Y_h + \lambda(Y_h)X_h, \textit{ for } X, Y \in \Gamma(TM). \tag{4}$$

Proof. The necessity is obvious. We only prove the sufficiency. If (4) holds, we denote by $(D_{X_h}Y_h+D_{Y_h}X_h)/2 = \tilde{D}_{X_h}Y_h, (\nabla_{X_h}Y_h + \nabla_{Y_h}X_h)/2 = \tilde{\nabla}_{X_h}Y_h$, then (4) is equivalent to $\tilde{D}_{X_h}Y_h - \tilde{\nabla}_{X_h}Y_h = \lambda(X_h)Y_h + \lambda(Y_h)X_h$. Hence $\tilde{D}$ and $\tilde{\nabla}$ have the same SR-parallel curves by (3).

On the other hand, if $\gamma(t)$ is a SR-parallel curve of D, then $\gamma(t)$ is also a SR-parallel curve of $\tilde{D}$ by a simple computation. Hence $\tilde{D}$ and D also have the same SR-parallel curves, so do $\tilde{\nabla}$ and ∇. Therefore, D and ∇ have the same SR-parallel curves, namely, D is a projective correspondence of ∇. □

Definition 2.7. *If $\bar{\nabla}$ is a projective correspondence to ∇ with torsion,*

$$\bar{T}(X_h, Y_h) = \pi(Y_h)X_h - \pi(X_h)Y_h, \tag{5}$$

where π is a given 1-form, then we say that $\bar{\nabla}$ is a semi-symmetric non-holonomic projective connection, in briefly, a SSNH-projective connection.

Theorem 2.8. *$\bar{\nabla}$ is a SSNH-projective connection if and only if there exist two 1-form p, q such that*

$$\bar{\nabla}_{X_h}Y_h = \nabla_{X_h}Y_h + p(X_h)Y_h + q(Y_h)X_h, \tag{6}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Let $A(X_h, Y_h) = \bar{\nabla}_{X_h}Y_h - \nabla_{X_h}Y_h$. Since $\bar{\nabla}$ is a SSNH-projective connection, from Proposition 2.6, there exists a smooth 1-form φ such that

$$(A(X_h, Y_h) + A(Y_h, X_h))/2 = \varphi(X_h)Y_h + \varphi(Y_h)X_h, for X, Y \in TM \tag{7}$$

and 1-form π such that the torsion of $\bar{\nabla}$ is of the form $\bar{T}(X_h, Y_h) = \pi(Y_h)X_h - \pi(X_h)Y_h$, we can deduce from the above equation

$$A(X_h, Y_h) - A(Y_h, X_h) = \pi(Y_h)X_h - \pi(X_h)Y_h. \tag{8}$$

By (6) and (8), we arrive at $A(X_h, Y_h) = (\varphi - \pi/2)(X_h)Y_h + (\varphi + \pi/2)(Y_h)X_h$ for $p = \varphi - \pi/2, q = \varphi + \pi/2$.

Conversely, we assume $\bar{\nabla}_{X_h}Y_h = \nabla_{X_h}Y_h + p(Y_h)X_h + q(X_h)Y_h$, then

$$(A(X_h, Y_h) + A(Y_h, X_h))/2 = \frac{p+q}{2}(Y_h)X_h + \frac{p+q}{2}(X_h)Y_h,$$

$$A(X_h, Y_h) - A(Y_h, X_h) = (p - q)(Y_h)X_h - (p - q)(X_h)Y_h.$$

By virtue of Proposition 2.6 again, we know $\bar{\nabla}$ is a projective correspondence to ∇, and we get

$$\bar{T}(X_h, Y_h) = \bar{\nabla}_{X_h}Y_h - \bar{\nabla}_{Y_h}X_h - [X_h, Y_h]_h = (p - q)(Y_h)X_h - (p - q)(X_h)Y_h.$$

Let $\pi = p - q$, then $\bar{T}(X_h, Y_h) = \pi(Y_h)X_h - \pi(X_h)Y_h$.

This completes the proof of Theorem 2.8. □

In a basis $\{e_i\}$, (6) can be rewritten as

$$\bar{\Gamma}^k_{ij} = \{^{\,k}_{ij}\} + p_i\delta^k_j + q_j\delta^k_i = \{^{\,k}_{ij}\} + \varphi_i\delta^k_j + \varphi_j\delta^k_i + \rho_j\delta^k_i - \rho_i\delta^k_j,$$

where $\rho_i = \pi_i/2, p_i = \varphi_i - \rho_i, q_i = \varphi_i + \rho_i$. The Schouten curvature tensor, Ricci tensor and sub-Weyl projective curvature tensor are given, respectively, as

$$\begin{cases} \bar{R}^h_{ijk} = R^h_{ijk} + \beta_{ij}\delta^h_k + \alpha_{ik}\delta^h_j - \alpha_{jk}\delta^h_i - \delta^h_k\Omega^s_{ij}p_s - \Omega^h_{ij}p_k, \\ \bar{R}_{jk} = R_{jk} + \beta_{kj} - (\ell-1)\alpha_{jk} - \Omega^s_{kj}p_s - \Omega^\varepsilon_{\varepsilon j}p_k, \\ W^h_{ijk} = R^h_{ijk} + \frac{1}{\ell-1}(\delta^h_j R_{ik} - \delta^h_i R_{jk}), \end{cases} \tag{9}$$

where

$$\begin{cases} \beta_{ij} = (\nabla_i p)(e_j) - (\nabla_j p)(e_i) = \varphi_{ij} - \varphi_{ji} + \rho_{ji} - \rho_{ij}, \\ \alpha_{ij} = (\nabla_i q)(e_j) - q(e_i)q(e_j) = \varphi_{ij} + \rho_{ij} - \varphi_i\rho_j - \varphi_j\rho_i, \\ \varphi_{ij} = e_i(\varphi_j) - \Gamma^e_{ij}\varphi_e - \varphi_i\varphi_j = \nabla_i\varphi_j - \varphi_i\varphi_j, \\ \rho_{ij} = e_i(\rho_j) - \Gamma^e_{ij}\rho_e - \rho_i\rho_j = \nabla_i\rho_j - \rho_i\rho_j, \\ R_{jk} = R^\varepsilon_{\varepsilon jk},\ \bar{R}_{jk} = \bar{R}^\varepsilon_{\varepsilon jk}. \end{cases} \tag{10}$$

Theorem 2.9. *The tensor S^h_{ijk} is an invariant under a SSNH-projective transformation, where*

$$\begin{aligned} S^h_{ijk} \quad = \quad & R^h_{ijk} + \frac{1}{\ell-1}(\delta^h_j R_{ik} - \delta^h_i R_{jk}) \\ & + \frac{1}{(\ell-1)(\ell^2-\ell-2)}\{\delta^h_j(R_{ik} - R_{ki}) - \delta^h_i(R_{jk} - R_{kj}) - (\ell-1)\delta^h_k(R_{ij} - R_{ji})\} \\ & + \frac{1}{\ell^2-\ell-2}\{\delta^h_j A_{ik} - \delta^h_i A_{jk} - (\ell-1)\delta^h_k A_{ij}\}, \end{aligned} \tag{11}$$

and $A_{ij} = R^s_{ijs}$.

Proof. For simplicity, we choose $\{e_i\}$ as a local frame field such that $[e_i, e_j] \in VM$, and hence we have $\Omega^h_{ij} = 0$. Then the Schouten curvature tensors and Ricci curvature tensors can be written simply as

$$\bar{R}^h_{ijk} = R^h_{ijk} + \beta_{ij}\delta^h_k + \alpha_{ik}\delta^h_j - \alpha_{jk}\delta^h_i \text{ and } \bar{R}_{jk} = R_{jk} + \beta_{kj} - (\ell-1)\alpha_{jk},$$

Let $k = h = \varepsilon$, and denote by $A_{ij} = R^\varepsilon_{ij\varepsilon}, \bar{A}_{ij} = \bar{R}^\varepsilon_{ij\varepsilon}$, one obtains

$$\bar{A}_{ij} = A_{ij} + \ell\beta_{ij} + \alpha_{ij} - \alpha_{ji},$$

hence one arrives at

$$\begin{aligned} \beta_{jk} \quad &= \quad \frac{1}{\ell^2-\ell-2}[(\bar{R}_{jk} - \bar{R}_{kj}) - (R_{jk} - R_{kj}) + (\ell-1)(\bar{A}_{jk} - A_{jk})], \\ \alpha_{jk} \quad &= \quad \frac{1}{\ell-1}(R_{jk} - \bar{R}_{jk}) - \frac{1}{(\ell-1)(\ell^2-\ell-2)}[(\bar{R}_{jk} - \bar{R}_{kj}) - (R_{jk} - R_{kj})] \\ &- \quad \frac{1}{(\ell-1)(\ell^2-\ell-2)}(\bar{A}_{jk} - A_{jk}). \end{aligned}$$

moreover one has

$$\begin{aligned}
\bar{R}^h_{ijk} &= R^h_{ijk} + \frac{\delta^h_k}{\ell^2-\ell-2}[(\bar{R}_{ij}-\bar{R}_{ji})-(R_{ij}-R_{ji})+(\ell-1)(\bar{A}_{ij}-A_{ij})] \\
&+ \frac{\delta^h_j}{\ell-1}(R_{ik}-\bar{R}_{ik}) - \frac{\delta^h_j}{(\ell-1)(\ell^2-\ell-2)}[(\bar{R}_{ik}-\bar{R}_{ki})-(R_{ik}-R_{ki})] \\
&- \frac{\delta^h_j}{(\ell-1)(\ell^2-\ell-2)}(\bar{A}_{ik}-A_{ik}) - \frac{\delta^h_i}{\ell-1}(R_{jk}-\bar{R}_{jk}) \\
&+ \frac{\delta^h_i}{(\ell-1)(\ell^2-\ell-2)}[(\bar{R}_{jk}-\bar{R}_{kj})-(R_{jk}-R_{kj})] \\
&+ \frac{\delta^h_i}{(\ell-1)(\ell^2-\ell-2)}(\bar{A}_{jk}-A_{jk}).
\end{aligned}$$

Rewriting the above equation by

$$\begin{aligned}
&\bar{R}^h_{ijk} - \frac{\delta^h_k}{\ell^2-\ell-2}(\bar{R}_{ij}-\bar{R}_{ji}) + \frac{\ell-1}{\ell^2-\ell-2}\delta^h_k\bar{A}_{ij} + \frac{1}{\ell-1}\delta^h_j\bar{R}_{ik} \\
&+\frac{1}{(\ell-1)(\ell^2-\ell-2)}\delta^h_j(\bar{R}_{ik}-\bar{R}_{ki}) + \frac{1}{(\ell-1)(\ell^2-\ell-2)}\delta^h_j\bar{A}_{ik} \\
&-\frac{1}{\ell-1}\delta^h_i\bar{R}_{jk} - \frac{1}{(\ell-1)(\ell^2-\ell-2)}\delta^h_i(\bar{R}_{jk}-\bar{R}_{kj}) - \frac{1}{(\ell-1)(\ell^2-\ell-2)}\delta^h_i\bar{A}_{jk} \\
= \; & R^h_{ijk} - \frac{\delta^h_k}{\ell^2-\ell-2}(R_{ij}-R_{ji}) + \frac{\ell-1}{\ell^2-\ell-2}\delta^h_kA_{ij} + \frac{1}{\ell-1}\delta^h_jR_{ik} \\
&+\frac{1}{(\ell-1)(\ell^2-\ell-2)}\delta^h_j(R_{ik}-R_{ki}) + \frac{1}{(\ell-1)(\ell^2-\ell-2)}\delta^h_jA_{ik} \\
&-\frac{1}{\ell-1}\delta^h_iR_{jk} - \frac{1}{(\ell-1)(\ell^2-\ell-2)}\delta^h_i(R_{jk}-R_{kj}) - \frac{1}{(\ell-1)(\ell^2-\ell-2)}\delta^h_iA_{jk},
\end{aligned}$$

that is

$$\begin{aligned}
&\bar{R}^h_{ijk} + \frac{1}{\ell-1}(\delta^h_j\bar{R}_{ik}-\delta^h_i\bar{R}_{jk}) + \frac{1}{(\ell-1)(\ell^2-\ell-2)}\{\delta^h_j(\bar{R}_{ik}-\bar{R}_{ki})-\delta^h_i(\bar{R}_{jk}-\bar{R}_{kj}) \\
&-(\ell-1)\delta^h_k(\bar{R}_{ij}-\bar{R}_{ji})\} + \frac{1}{\ell^2-\ell-2}\{\delta^h_j\bar{A}_{ik}-\delta^h_i\bar{A}_{jk}-(\ell-1)\delta^h_k\bar{A}_{ij}\} \\
= \; & R^h_{ijk} + \frac{1}{\ell-1}(\delta^h_jR_{ik}-\delta^h_iR_{jk}) + \frac{1}{(\ell-1)(\ell^2-\ell-2)}\{\delta^h_j(R_{ik}-R_{ki})-\delta^h_i(R_{jk}-R_{kj}) \\
&-(\ell-1)\delta^h_k(R_{ij}-R_{ji})\} + \frac{1}{\ell^2-\ell-2}\{\delta^h_jA_{ik}-\delta^h_iA_{jk}-(\ell-1)\delta^h_kA_{ij}\}.
\end{aligned}$$

Denote by

$$\begin{aligned}
\bar{S}^h_{ijk} &= \bar{R}^h_{ijk} + \frac{1}{\ell-1}(\delta^h_j\bar{R}_{ik}-\delta^h_i\bar{R}_{jk}) + \frac{1}{(\ell-1)(\ell^2-\ell-2)}\{\delta^h_j(\bar{R}_{ik}-\bar{R}_{ki})-\delta^h_i(\bar{R}_{jk}-\bar{R}_{kj}) \\
&\quad -(\ell-1)\delta^h_k(\bar{R}_{ij}-\bar{R}_{ji})\} + \frac{1}{\ell^2-\ell-2}\{\delta^h_j\bar{A}_{ik}-\delta^h_i\bar{A}_{jk}-(\ell-1)\delta^h_k\bar{A}_{ij}\},
\end{aligned}$$

and

$$\begin{aligned}
S^h_{ijk} &= R^h_{ijk} + \frac{1}{\ell-1}(\delta^h_jR_{ik}-\delta^h_iR_{jk}) + \frac{1}{(\ell-1)(\ell^2-\ell-2)}\{\delta^h_j(R_{ik}-R_{ki})-\delta^h_i(R_{jk}-R_{kj}) \\
&\quad -(\ell-1)\delta^h_k(R_{ij}-R_{ji})\} + \frac{1}{\ell^2-\ell-2}\{\delta^h_jA_{ik}-\delta^h_iA_{jk}-(\ell-1)\delta^h_kA_{ij}\}.
\end{aligned}$$

then one obtains $S^h_{ijk} = \bar{S}^h_{ijk}$. This ends the proof of Theorem 2.9. □

We now similarly define the sub-Weyl projective curvature tensor of the SSNH-projective connection by

$$\bar{W}^h_{ijk} = \bar{R}^h_{ijk} + \frac{1}{\ell - 1}(\delta^h_j \bar{R}_{ik} - \delta^h_i \bar{R}_{jk}), \tag{12}$$

then we have

$$\begin{aligned} \bar{W}^h_{ijk} &= W^h_{ijk} + \delta^h_k(\beta_{ij} - \Omega^s_{ij} p_s) + \frac{1}{\ell - 1}(\delta^h_j \beta_{ik} - \delta^h_i \beta_{jk}) \\ &- \frac{1}{\ell - 1}(\delta^h_j \Omega^s_{ki} - \delta^h_i \Omega^s_{kj}) p_s - [\Omega^h_{ij} - \frac{1}{\ell - 1}(\delta^h_j \Omega^\varepsilon_{i\varepsilon} - \delta^h_i \Omega^\varepsilon_{j\varepsilon})] p_k. \end{aligned}$$

We denote by

$$\begin{aligned} B^h_{ijk} &= \delta^h_k(\beta_{ij} - \Omega^s_{ij} p_s) + \frac{1}{\ell - 1}(\delta^h_j \beta_{ik} - \delta^h_i \beta_{jk}) \\ &- \frac{1}{\ell - 1}(\delta^h_j \Omega^s_{ki} - \delta^h_i \Omega^s_{kj}) p_s - [\Omega^h_{ij} - \frac{1}{\ell - 1}(\delta^h_j \Omega^\varepsilon_{i\varepsilon} - \delta^h_i \Omega^\varepsilon_{j\varepsilon})] p_k, \end{aligned}$$

then it is obvious that $\bar{W}^h_{ijk} = W^h_{ijk} + B^h_{ijk}$.

Definition 2.10. *If the 1-form p and q in (6) are horizontally closed, that is,*

$$\begin{aligned} dp(X_h, Y_h) &= X_h(p(Y_h)) - Y_h(p(X_h)) - p([X_h, Y_h]_h) = 0, \\ dq(X_h, Y_h) &= X_h(q(Y_h)) - Y_h(q(X_h)) - q([X_h, Y_h]_h) = 0, \end{aligned}$$

then we call a SSNH-projective connection $\tilde{\nabla}$ the special SSNH-projective connection.

Theorem 2.11. *The sub-Weyl projective curvature tensor is an invariant under a special SSNH-projective transformation.*

Proof. If $\tilde{\nabla}$ is a special SSNH-projective connection, then the 1-form p and q in (9) are all horizontally closed. Therefore there holds

$$\begin{aligned} 0 &= dp(e_i, e_j) = e_i(p_j) - e_i(p_j) - p([e_i, e_j]_h) = \varphi_{ij} - \varphi_{ji} + \rho_{ij} - \rho_{ji}, \\ 0 &= dq(e_i, e_j) = e_i(q_j) - e_i(q_j) - q([e_i, e_j]_h) = \varphi_{ij} - \varphi_{ji} + \rho_{ji} - \rho_{ij}. \end{aligned}$$

By adding above two equations one gets $\varphi_{ij} = \varphi_{ji}$, and $\rho_{ij} = \rho_{ji}$ by subtracting these equations. Then one obtains $\beta_{ij} = 0$ and

$$\tilde{R}^h_{ijk} = R^h_{ijk} + \alpha_{ik}\delta^h_j - \alpha_{jk}\delta^h_i,$$

Contracting by i and h, one gets

$$\tilde{R}_{jk} = R_{jk} - (\ell - 1)\alpha_{jk},$$

Therefore, one obtains

$$\begin{aligned} \tilde{W}^h_{ijk} &= \tilde{R}^h_{ijk} + \frac{1}{\ell - 1}(\delta^h_j \tilde{R}_{ik} - \delta^h_i \tilde{R}_{jk}) \\ &= R^h_{ijk} + \alpha_{ik}\delta^h_j - \alpha_{jk}\delta^h_i + \frac{1}{\ell - 1}\delta^h_j(R_{ik} - (\ell - 1)\alpha_{ik}) - \frac{1}{\ell - 1}\delta^h_i(R_{jk} - (\ell - 1)\alpha_{jk}) \\ &= W^h_{ijk}. \end{aligned}$$

The proof is finished. □

Remark 2.12. *It is obvious that a projectively flat sub-Riemannina manifold is transformed to a projectively flat sub-Riemannina manifold by a SSNH-projective transformation.*

Theorem 2.13. *A sub-Riemannian manifold $(M, \triangle, g_\triangle)$ is projective flat if and only if the Schouten curvature tensor $\tilde{R}$ of the special SSNH-projective connection $\tilde{D}$ is vanished.*

Proof. If $\tilde{\nabla}$ is a special SSNH-projective connection and

$$\tilde{R}^h_{ijk} = R^h_{ijk} + \beta_{ij}\delta^h_k + \alpha_{ik}\delta^h_j - \alpha_{jk}\delta^h_i = 0, \tag{13}$$

then by contracting (13) with i, h, we have $\tilde{R}_{jk} = R_{jk} + \beta_{kj} - (\ell - 1)\alpha_{jk} = 0$. Since $\tilde{\nabla}$ is special, then the 1-form p is horizontally closed. Hence we get $\beta_{ij} = 0$, and

$$R^h_{ijk} = \alpha_{jk}\delta^h_i - \alpha_{ik}\delta^h_j \,,\, R_{ik} = (\ell - 1)\alpha_{ik}, \tag{14}$$

By substituting (14) into the following equation

$$W^h_{ijk} = R^h_{ijk} + \frac{1}{\ell - 1}(\delta^h_j R_{ik} - \delta^h_i R_{jk}),$$

we obtain $W^h_{ijk} = 0$, that is, M is projectively flat.

Conversely, if M is projectively flat, then $W^h_{ijk} = 0$, and $R^h_{ijk} = \frac{1}{\ell-1}(\delta^h_i R_{jk} - \delta^h_j R_{ik})$, namely, $R_{ijkh} = \frac{1}{\ell-1}(g_{ih}R_{jk} - g_{jh}R_{ik})$. Since $R_{ijhh} = 0$, we get $R_{ik} = \frac{R}{\ell} g_{ik}$. If the 1-form p is horizontally closed, then the equation $\tilde{R}_{ij} = R_{ij} + \beta_{ij} - (\ell - 1)\alpha_{ij} = 0$ is equivalent to

$$(\nabla_i q)(e_j) - q_i q_j = \frac{R}{\ell(\ell - 1)} g_{ij}, \tag{15}$$

where $(\nabla_i q)(e_j) - q_i q_j = \alpha_{ij}$.

Now taking a covariant derivative of Equation (15), we get

$$\begin{aligned}
&(\nabla_i\nabla_j q)(e_k) + (\nabla_j q)(\nabla_i e_k) - (\nabla_i q)(e_j)q(e_k) - q(\nabla_i e_j)q(e_k) - q(e_j)(\nabla_i q)(e_k) - q(\nabla_i e_k)q(e_j)\\
&= \frac{K}{\ell(\ell - 1)}(g(\nabla_i e_j, e_k) + g(e_j, \nabla_i e_k))\\
&= (\nabla_{\nabla_i e_j} q)(e_k) - q(\nabla_i e_j)q(e_k) + (\nabla_j q)(\nabla_i e_k) - q(\nabla_i e_k)q(e_j),
\end{aligned}$$

where the last equality follows from Equation (15). Namely,

$$(\nabla_i\nabla_j q)(e_k) - (\nabla_i q)(e_j)q(e_k) - q(e_j)(\nabla_i q)(e_k) = (\nabla_{\nabla_i e_j} q)(e_k). \tag{16}$$

Since the horizontal 1-form p is closed, then by (15), (16) and $W^h_{ijk} = 0$, we obtain

$$(\nabla_i\nabla_j q - \nabla_j\nabla_i q - \nabla_{[e_i,e_j]_h} q)(e_k) = -R^h_{ijk} q_h, \tag{17}$$

therefore there exists a solution q to Equation (15), let

$$\tilde{\Gamma}^k_{ij} = \{^{\,k}_{ij}\} + p_i\delta^k_j + q_j\delta^k_i, \tag{18}$$

where p is a closed horizontal 1-form.

By Theorem 2.8, we know $\tilde{\nabla}$ whose connection coefficients are defined by (18) is a SSNH-projective connection. On the other hand, α_{ij} is proportional to g_{ij} by (15), so it is symmetric and $dq(e_i, e_j) = \alpha_{ij} - \alpha_{ji} = 0$, which implies that the 1-form q is horizontally closed.

This completes the proof of Theorem 2.13. □

3. Example

Example 3.1. *(Almost contact metric sub-Riemannian manifold)*

Let (M, Δ, g_Δ) be a $(2n+1)$-dimensional sub-Riemannian manifold, an almost contact structure is denoted by (φ, ξ, η), where φ is a horizontal $(1,1)$-tensor field(i.e. $\varphi(X_h) \in \Delta$), ξ is a vector field and η is a 1-form such that

$$\varphi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1,\ g(\varphi X_h, \varphi Y_h) = g(X_h, Y_h) - \eta(X_h)\eta(Y_h).$$

then $(M, \Delta, g, \varphi, \xi, \eta)$ is called an almost contact metric sub-Riemannian manifold. In virtue of this 1-form η, one defines a metric connection,

$$\tilde{\nabla}_{X_h} Y_h = \nabla_{X_h} Y_h + \eta(X_h) Y_h + \eta(Y_h) X_h, \tag{19}$$

in local coordinate, that is,

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \eta_i \delta^k_j + \eta_j \delta^k_i, \tag{20}$$

where ∇ is the sub-Riemannian connection, then $\tilde{\nabla}$ is actually a SSNH-projective connection.

In fact, if $\gamma : x^a = x^a(t)$ is a SR-parallel curve with respect to sub-Riemannian connection, then it satisfies Equations (2), substituting (20) into the above Equations, one obtains,

$$\frac{d^2x^k}{dt^2} + \tilde{\Gamma}^k_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt} = -2\eta_i \frac{dx^i}{dt}\frac{dx^k}{dt},$$

Now we introduce a new parameter s by the equation

$$s = \int e^{\int -2\eta_i dx^i} dt,$$

and obtain the following relations by straight-forward calculation,

$$\begin{aligned} &\frac{ds}{dt} = e^{\int -2\eta_j dx^j}, \frac{d^2s}{dt^2} = e^{\int -2\eta_j dx^j}(-2\eta_j \frac{dx^j}{dt}), \\ &\frac{dx^i}{dt} = e^{\int -2\eta_j dx^j} \frac{dx^i}{ds}, \frac{d^2x^i}{dt^2} = e^{2\int -2\eta_j dx^j}(\frac{d^2x^i}{ds^2} - 2\eta_j \frac{dx^j}{ds}\frac{dx^i}{ds}). \end{aligned}$$

hence, we have

$$\frac{d^2x^k}{ds^2} + \tilde{\Gamma}^k_{ij} \frac{dx^i}{ds}\frac{dx^j}{ds} = 0.$$

that is $\gamma : x^a = x^a(t)$ is also a SR-parallel curve associated with the connection (19). On the other hand, one can prove the converse statement is also true by the same method. Therefore, the metric connection (19) is a SSNH-projective connection.

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