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Three-step Iterations for Total Asymptotically Nonexpansive Mappings in CAT(0) Spaces

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Abstract. In this paper, we establish strong and Δ-convergence theorems of modified three-step iterations for total asymptotically nonexpansive mapping which is wider than the class asymptotically nonexpansive mappings in the framework of CAT(0) spaces. Our results extend and generalize the corresponding results of Chang *et al*. [Demiclosed principle and Δ-convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces, Appl. Math. Comput. 219(5) (2012) 2611-2617], Nanjaras and Panyanak [Demiclosed principle for asymptotically nonexpansive mappings in CAT(0) spaces, Fixed Point Theory Appl. Vol. 2010, Art. ID 268780], and many others.

1. Introduction

As we know, iteration methods are numerical procedures which compute a sequence of gradually accurate iterates to approximate the solution of a class of problems. Such methods are useful tools of applied mathematics for solving real life problems ranging from economics and finance or biology to transportation, network analysis or optimization. When we design iteration methods, we have to study their qualitative properties such as: convergence, stability, error propagation, stopping criteria. This is an active area of research, several well known scientists in the world paid and still pay attention to the qualitative study of iteration methods; please, see: Ishikawa [23], Mann [29], Noor *et al*. [33–35], Ćiric *et al*. [9–11, 14], Kirk and Shahzad [28], Ofoedu *et al*. [36, 37], Shahzad and Zegeye [42], Wan [45], Yao *et al*. [48, 49]. Special emphasis is given to studies on CAT(0) spaces; Abbas *et al*. [2], Saluja [38–40], Shahzad [41], Chang *et al*. [6], Panyanak *et al*. [17, 31, 32], Wu *et al* [46].

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Fixed point theory in CAT(0) space has been first studied by Kirk (see [25, 26]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(m) space for every $m \geq k$ (see [4]).

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Nanjaras and Panyanak [31] proved the demiclosed principle for asymptotically nonexpansive mappings and gave the Δ-convergence theorem of the modified Mann iteration process for above mentioned mappings in a CAT(0) space. In 2010, Y. Niwongsa and B. Panyanak [32] studied the Noor iteration scheme in CAT(0) spaces and they proved some Δ and strong convergence theorems for asymptotically nonexpansive mappings which extend and improve some recent results from the literature. In 2012, Chang *et al*. [6] introduced the concept of total asymptotically nonexpansive mappings and proved the demiclosed principle for said mapping in a CAT(0) space. Also, they established the Δ-convergence theorem of the modified Mann iteration process for total asymptotically nonexpansive mappings in a CAT(0) space. Recently, Başarir and Şahin [3] studied the modified S-iteration process, modified two-step iteration process and established strong and Δ-convergence theorems for total asymptotically nonexpansive mappings in the framework of CAT(0) spaces.

Algorithm 1. The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$\begin{aligned} y_n &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \\ x_{n+1} &= (1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, \quad n \geq 1, \end{aligned} \tag{1}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in (0,1) is called modified S-iterative sequence [1].

If $T^n = T$ for all $n \geq 1$, then Algorithm 1 reduces to the following.

Algorithm 2. The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$\begin{aligned} y_n &= (1-\beta_n)x_n \oplus \beta_n T x_n, \\ x_{n+1} &= (1-\alpha_n)T x_n \oplus \alpha_n T y_n, \quad n \geq 1, \end{aligned} \tag{2}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in (0,1) is called S-iterative sequence [1].

Algorithm 3. The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$\begin{aligned} y_n &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \\ x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n T^n y_n, \quad n \geq 1, \end{aligned} \tag{3}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in [0,1] is called an Ishikawa iterative sequence [23]; please, see also Ćirić [7], and Ćirić and Ume [8].

If $\beta_n = 0$ for all $n \geq 1$, then Algorithm 3 reduces to the following.

Algorithm 4. The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n, \quad n \geq 1, \tag{4}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in (0,1) is called a Mann iterative sequence [29]; please, see also [12, 13].

Motivated and inspired by [1] and some others, we modify iteration scheme (1) as follows.

Algorithm 5. The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$\begin{aligned} z_n &= (1-\gamma_n)x_n \oplus \gamma_n T^n x_n \\ y_n &= (1-\beta_n)x_n \oplus \beta_n T^n z_n, \\ x_{n+1} &= (1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, \quad n \geq 1, \end{aligned} \tag{5}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ are appropriate sequences in (0,1) is called modified three-step iterative sequence. Iteration scheme (5) is independent of modified Noor iteration [30], modified Ishikawa iteration and modified Mann iteration schemes.

If $\gamma_n = 0$ for all $n \geq 1$, then Algorithm 5 reduces to the Algorithm 1.

Iteration procedures in fixed point theory are lead by the considerations in summability theory. For example, if a given sequence converges, then we don't look for the convergence of the sequence of its arithmetic means. Similarly, if the sequence of Picard iterates of any mapping T converges, then we don't look for the convergence of other iteration procedures.

The three-step iterative approximation problems were studied extensively by Noor [33, 34], Glowinsky and Le Tallec [20], and Haubruge et al [21]. It has been shown [20] that three-step iterative scheme gives better numerical results than the two step and one step approximate iterations. Thus we conclude that three step scheme plays an important and significant role in solving various problems, which arise in pure and applied sciences.

Motivated by Chang *et al.* [6], Başarir and Şahin [3] and some others, in this paper, we establish strong and Δ-convergence theorems of modified three-step iteration process for total asymptotically nonexpansive mappings in the framework of CAT(0) spaces. Our results extend and generalize the corresponding results of [3, 6, 31, 32] many others.

2. Preliminaries and Lemmas

Let (X, d) be a metric space and K a nonempty subset of X. Let $T\colon K \to K$ be a mapping. A point $x \in K$ is called a fixed point of T if $Tx = x$ and we denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$.

Definition 2.1. Let (X, d) be a metric space and K a nonempty subset of X. Then $T\colon K \to K$ is said to be

(1) *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$;

(2) *asymptotically nonexpansive* [18] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $d(T^n x, T^n y) \leq k_n\, d(x, y)$ for all $x, y \in K$ and $n \geq 1$;

(3) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq L\, d(x, y)$ for all $x, y \in K$ and $n \geq 1$;

(4) *semi-compact* if for a sequence $\{x_n\}$ in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.

(5) a sequence $\{x_n\}$ in K is called *approximate fixed point sequence* for T (AFPS in short) if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Chang et al. [6] defined the concept of total asymptotically nonexpansive mapping as follows.

Definition 2.2 ([6] Definition 2.1). Let (X, d) be a metric space, K be its nonempty subset and let $T\colon K \to K$ be a mapping. T is said to be a *total asymptotically nonexpansive mapping* if there exist non-negative real sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\zeta\colon [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x, y) + \nu_n \zeta(d(x, y)) + \mu_n$$

for all $x, y \in K$ and $n \geq 1$.

Remark 2.3. From the definitions given above, it follows that each nonexpansive mapping is an asymptotically nonexpansive mapping with the constant sequence $\{k_n\} = \{1\}$, $\forall\, n \geq 1$, each asymptotically nonexpansive mapping is a total asymptotically nonexpansive mapping with $\mu_n = 0$, $\nu_n = k_n - 1$ for all $n \geq 1$, $\zeta(t) = t$, $t \geq 0$ and each asymptotically nonexpansive mapping is a uniformly L-Lipschitzian mapping with $L = \sup_{n\geq 1}\{k_n\}$.

We now give the definition and some basic properties of CAT(0) space. For this purpose, we consider (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say that X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denote by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [4]).

CAT(0) space. A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \quad \leq \quad d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{6}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [24]). If x, y_1, y_2 are points of a $CAT(0)$ space and y_0 is the midpoint of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \quad \leq \quad \frac{1}{2} d^2(x, y_1) + \frac{1}{2} d^2(x, y_2) - \frac{1}{4} d^2(y_1, y_2). \tag{7}$$

Inequality (7) is the (CN) inequality of Bruhat and Tits [5]. The above inequality was extended in [17] as

$$d^2(z, \alpha x \oplus (1-\alpha)y) \quad \leq \quad \alpha d^2(z, x) + (1-\alpha) d^2(z, y) - \alpha(1-\alpha) d^2(x, y) \tag{8}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [2, p.163]). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \quad \leq \quad \alpha d(z, x) + (1-\alpha) d(z, y), \tag{9}$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0, 1]\}$.

A subset K of a $CAT(0)$ space X is convex if for any $x, y \in K$, we have $[x, y] \subset K$.

In the sequel, we need the following definitions and useful lemmas to prove our main results of this paper.

Lemma 2.4 ([32]). *Let X be a $CAT(0)$ space.*
(i) For $x,\ y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x,\ y]$ such that

$$d(x,\ z) = t\, d(x,\ y) \quad \text{and} \quad d(y,\ z) = (1-t)\, d(x,\ y). \quad (A)$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (A).
(ii) For $x,\ y \in X$ and $t \in [0, 1]$, we have

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z).$$

Let $\{x_n\}$ be a bounded sequence in a closed convex subset K of a CAT(0) space X. For $x \in X$, set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \Big\{x \in X : r(\{x_n\}) = r(x, \{x_n\})\Big\}.$$

It is known that, in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point; please, see [15], Proposition 7.

We now recall the definition of Δ-convergence and weak convergence ($\rightharpoonup$) in CAT(0) space.

Definition 2.5 ([27]). A sequence $\{x_n\}$ in a CAT(0) space X is said to *Δ-converge to $x \in X$* if x is the unique asymptotic center of $\{x_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

In this case we write Δ-$\lim_n x_n = x$ and call x is the Δ-limit of $\{x_n\}$.

Recall that a bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In the Banach space it is known that, every bounded sequence has a regular subsequence; [19], Lemma 15.2.

Since in a CAT(0) space every regular sequence Δ-converges, we see that every bounded sequence in X has a Δ-convergent subsequence; noticed in [27], p. 3690.

Lemma 2.6 ([2]). *Given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$, then*

$$\limsup_n d(x_n, x) < \limsup_n d(x_n, y).$$

In a Banach space the above condition is known as the Opial property.

Now, recall the definition of weak convergence in a CAT(0) space.

Definition 2.7 ([22]). Let K be a closed convex subset of a CAT(0) space X. A bounded sequence $\{x_n\}$ in K is said to *converge weakly to $q \in K$* if and only if $\Phi(q) = \inf_{x\in K} \Phi(x)$, where $\Phi(x) = \limsup_{n\to\infty} d(x_n, x)$.

Note that $\{x_n\} \rightharpoonup q$ if and only if $A_K\{x_n\} = \{q\}$.

Nanjaras and Panyanak [31] established the following relation between Δ-convergence and weak convergence in a CAT(0) space:

Lemma 2.8 ([31], Proposition 3.12). *Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X and let K be a closed convex subset of X which contains $\{x_n\}$. Then*

(i) Δ-$\lim_{x_n} = x$ implies $x_n \rightharpoonup x$.

(ii) The converse of (i) is true if $\{x_n\}$ is regular.

Lemma 2.9 ([17], Lemma 2.8). *If $\{x_n\}$ is a bounded sequence in a CAT(0) space X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.10 ([16], Proposition 2.1). *If K is a closed convex subset of a CAT(0) space X and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

Lemma 2.11 ([6], Theorem 3.8). *Let K be closed convex subset of a complete CAT(0) space X and let $T\colon K \to K$ be a total asymptotically nonexpansive and uniformly L-Lipschitzian mapping. Let $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = p$. Then $Tp = p$.*

Lemma 2.12 ([44]). *Suppose that $\{a_n\}$, $\{b_n\}$ and $\{r_n\}$ are sequences of nonnegative real numbers such that $a_{n+1} \leq (1 + b_n)a_n + r_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

3. The Main Results

Now, we prove the following lemma using modified three-step iteration scheme (5) needed in the sequel.

Lemma 3.1. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon K \to K$ be a total asymptotically nonexpansive and uniformly L-Lipschitzian mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (5).*

If the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) there exists a constant $M_1 > 0$ such that $\zeta(r) \leq M_1 r$, $r \geq 0$.

Then $\lim_{n\to\infty} d(x_n, p)$ and $\lim_{n\to\infty} d(x_n, F(T))$ exist for all $p \in F(T)$.

Proof. Let $p \in F(T)$. From (5) and Lemma 2.1(ii), we have

$$\begin{aligned} d(z_n, p) &= d((1-\gamma_n)x_n \oplus \gamma_n T^n x_n, p) \\ &\leq \gamma_n\, d(T^n x_n, p) + (1-\gamma_n)d(x_n, p) \\ &\leq \gamma_n[d(x_n, p) + \nu_n(d(x_n, p)) + \mu_n] + (1-\gamma_n)d(x_n, p) \\ &\leq \gamma_n[d(x_n, p) + \nu_n M_1 d(x_n, p) + \mu_n] + (1-\gamma_n)d(x_n, p) \\ &= \gamma_n(1 + M_1\nu_n)d(x_n, p) + \gamma_n\mu_n + (1-\gamma_n)d(x_n, p) \\ &\leq \gamma_n(1 + M_1\nu_n)d(x_n, p) + \gamma_n\mu_n \\ &\quad +(1-\gamma_n)(1 + M_1\nu_n)d(x_n, p) \\ &\leq (1 + M_1\nu_n)d(x_n, p) + \mu_n. \end{aligned} \tag{10}$$

Again using (5), (10) and Lemma 2.1(ii), we have

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)x_n \oplus \beta_n T^n z_n, p) \\ &\leq \beta_n\, d(T^n z_n, p) + (1-\beta_n)d(x_n, p) \\ &\leq \beta_n[d(z_n, p) + \nu_n(d(z_n, p)) + \mu_n] + (1-\beta_n)d(x_n, p) \\ &\leq \beta_n[d(z_n, p) + \nu_n M_1 d(z_n, p) + \mu_n] + (1-\beta_n)d(x_n, p) \\ &= \beta_n(1 + M_1\nu_n)d(z_n, p) + \beta_n\mu_n + (1-\beta_n)d(x_n, p) \\ &\leq \beta_n(1 + M_1\nu_n)[(1 + M_1\nu_n)d(x_n, p) + \mu_n] + \beta_n\mu_n \\ &\quad +(1-\beta_n)d(x_n, p) \\ &= \beta_n(1 + M_1\nu_n)^2 d(x_n, p) + \beta_n(1 + M_1\nu_n)\mu_n + \beta_n\mu_n \\ &\quad +(1-\beta_n)d(x_n, p) \\ &\leq \beta_n(1 + M_1\nu_n)^2 d(x_n, p) + \beta_n(1 + M_1\nu_n)\mu_n + \beta_n\mu_n \\ &\quad +(1-\beta_n)(1 + M_1\nu_n)^2 d(x_n, p) \\ &\leq (1 + M_1\nu_n)^2 d(x_n, p) + (2 + M_1\nu_n)\mu_n. \end{aligned} \tag{11}$$

Now using (5), (11) and Lemma 2.1(ii), we get

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, p) \\ &\leq \alpha_n\, d(T^n y_n, p) + (1-\alpha_n)d(T^n x_n, p) \\ &\leq \alpha_n[d(y_n, p) + \nu_n(d(y_n, p)) + \mu_n] + (1-\alpha_n)[d(x_n, p) \\ &\quad +\nu_n(d(x_n, p)) + \mu_n] \\ &\leq \alpha_n[d(y_n, p) + \nu_n M_1 d(y_n, p) + \mu_n] + (1-\alpha_n)[d(x_n, p) \\ &\quad +\nu_n M_1 d(x_n, p) + \mu_n] \\ &= \alpha_n(1 + M_1\nu_n)d(y_n, p) + (1-\alpha_n)(1 + M_1\nu_n)d(x_n, p) + \mu_n \\ &\leq \alpha_n(1 + M_1\nu_n)[(1 + M_1\nu_n)^2 d(x_n, p) + (2 + M_1\nu_n)\mu_n] \\ &\quad +(1-\alpha_n)(1 + M_1\nu_n)d(x_n, p) + \mu_n \\ &\leq \alpha_n(1 + M_1\nu_n)^3 d(x_n, p) + \alpha_n(1 + M_1\nu_n)(2 + M_1\nu_n)\mu_n \\ &\quad +(1-\alpha_n)(1 + M_1\nu_n)^3 d(x_n, p) + \mu_n \\ &\leq (1 + M_1\nu_n)^3 d(x_n, p) + 3\mu_n(1 + M_1\nu_n) \\ &= (1 + A_n)d(x_n, p) + B_n, \end{aligned} \tag{12}$$

where $A_n = 3M_1\nu_n + 3M_1^2\nu_n^2 + M_1^3\nu_n^3$ and $B_n = 3\mu_n(1 + M_1\nu_n)$. Since by the assumption of the theorem $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\sum_{n=1}^{\infty}\nu_n < \infty$, it follows that $\sum_{n=1}^{\infty}A_n < \infty$ and $\sum_{n=1}^{\infty}B_n < \infty$. Equation (12) implies that

$$d(x_{n+1}, F(T)) \leq (1 + A_n)d(x_n, F(T)) + B_n. \tag{13}$$

Hence from Lemma 2.12, (12) and (13), we get $\lim_{n\to\infty} d(x_n, p)$ and $\lim_{n\to\infty} d(x_n, F(T))$ both exist. □

Theorem 3.2. *Let X, K, T, $\{x_n\}$ satisfy the hypothesis of Lemma* 3.1. *Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$*, where* $d(x, F(T)) = \inf\{d(x, p) : p \in F(T)\}$.

Proof. The necessity is obvious.

To prove the converse, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. As proved in Lemma 3.1, for all $p \in F(T)$, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Thus by the hypothesis $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in K. With the help of inequality $1 + x \le e^x$, $x \ge 0$. For any integer $m \ge 1$, therefore from (9), we have

$$\begin{aligned}
d(x_{n+m}, p) &\le (1 + A_{n+m-1})d(x_{n+m-1}, p) + B_{n+m-1} \\
&\le e^{A_{n+m-1}} d(x_{n+m-1}, p) + B_{n+m-1} \\
&\le e^{A_{n+m-1}}[e^{A_{n+m-2}} d(x_{n+m-2}, p) + B_{n+m-2}] + B_{n+m-1} \\
&\le e^{(A_{n+m-1}+A_{n+m-2})} d(x_{n+m-2}, p) + e^{A_{n+m-1}}[B_{n+m-2} + B_{n+m-1}] \\
&\le \dots \\
&\le \left(e^{\sum_{k=n}^{n+m-1} A_k}\right) d(x_n, p) + \left(e^{\sum_{k=n}^{n+m-1} A_k}\right) \sum_{k=n}^{n+m-1} B_k \\
&\le \left(e^{\sum_{n=1}^{\infty} A_n}\right) d(x_n, p) + \left(e^{\sum_{n=1}^{\infty} A_n}\right) \sum_{k=n}^{n+m-1} B_k \\
&= W\, d(x_n, p) + W \sum_{k=n}^{n+m-1} B_k, \qquad (14)
\end{aligned}$$

where $W = e^{\sum_{n=1}^{\infty} A_n}$.

Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, without loss of generality, we may assume that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_{n_k}\} \subset F(T)$ such that $d(x_{n_k}, p_{n_k}) \to 0$ as $k \to \infty$. Then for any $\varepsilon > 0$, there exists $k_\varepsilon > 0$ such that

$$d(x_{n_k}, p_{n_k}) < \frac{\varepsilon}{4W} \quad \text{and} \quad \sum_{k=n_{k_\varepsilon}}^{\infty} B_k < \frac{\varepsilon}{4W}, \qquad (15)$$

for all $k \ge k_\varepsilon$.

For any $m \ge 1$ and for all $n \ge n_{k_\varepsilon}$, by (14), we have

$$\begin{aligned}
d(x_{n+m}, x_n) &\le d(x_{n+m}, p_{n_k}) + d(x_n, p_{n_k}) \\
&\le W\, d(x_n, p_{n_k}) + W \sum_{k=n_{k_\varepsilon}}^{\infty} B_k \\
&\quad + W\, d(x_n, p_{n_k}) + W \sum_{k=n_{k_\varepsilon}}^{\infty} B_k \\
&= 2W\, d(x_n, p_{n_k}) + 2W \sum_{k=n_{k_\varepsilon}}^{\infty} B_k \\
&< 2W.\frac{\varepsilon}{4W} + 2W.\frac{\varepsilon}{4W} = \varepsilon. \qquad (16)
\end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in K. Thus, the completeness of X implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = z$. Since K is closed, therefore $z \in K$. Next, we show that $z \in F(T)$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ we get $d(z, F(T)) = 0$, closedness of $F(T)$ gives that $z \in F(T)$. Thus $\{x_n\}$ converges strongly to a point in $F(T)$. This completes the proof. □

Lemma 3.3. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon K \to K$ be a uniformly continuous and total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (5). *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in* $(0, 1)$ *such that* $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$ *and* $\liminf_{n\to\infty} \gamma_n(1 - \gamma_n) > 0$.

If the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) there exists a constant $M_1 > 0$ such that $\zeta(r) \leq M_1 r$, $r \geq 0$.

Then $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Proof. Let $p \in F(T)$. Then by Lemma 3.1, we have $\lim_{n\to\infty} d(x_n, p)$ exists, so we can assume that $\lim_{n\to\infty} d(x_n, p) = a$, where $a > 0$. We claim that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Since $\{x_n\}$ is bounded, there exists $R > 0$ such that $\{x_n\}, \{y_n\}, \{z_n\} \subset B_R(p)$ for all $n \geq 1$. Using (1) and (8), we have

$$\begin{aligned}
d^2(z_n, p) &= d^2((1-\gamma_n)x_n \oplus \gamma_n T^n x_n, p) \\
&\leq \gamma_n d^2(T^n x_n, p) + (1-\gamma_n) d^2(x_n, p) \\
&\quad -\gamma_n(1-\gamma_n) d(T^n x_n, x_n) \\
&\leq \gamma_n [d(x_n, p) + \nu_n(d(x_n, p)) + \mu_n]^2 + (1-\gamma_n) d^2(x_n, p) \\
&\quad -\gamma_n(1-\gamma_n) d(T^n x_n, x_n) \\
&\leq \gamma_n [d(x_n, p) + \nu_n M_1 d(x_n, p) + \mu_n]^2 + (1-\gamma_n) d^2(x_n, p) \\
&\quad -\gamma_n(1-\gamma_n) d(T^n x_n, x_n) \\
&= \gamma_n [(1+\nu_n M_1) d(x_n, p) + \mu_n]^2 + (1-\gamma_n) d^2(x_n, p) \\
&\quad -\gamma_n(1-\gamma_n) d(T^n x_n, x_n) \\
&\leq (1+\nu_n M_1)^2 \gamma_n d^2(x_n, p) + (1+\nu_n M_1)^2 (1-\gamma_n) d^2(x_n, p) \\
&\quad +\gamma_n [2(1+\nu_n M_1)\mu_n d(x_n, p) + \mu_n^2] - \gamma_n(1-\gamma_n) d(T^n x_n, x_n) \\
&\leq (1+\nu_n M_1)^2 d^2(x_n, p) + \gamma_n [2(1+\nu_n M_1)\mu_n d(x_n, p) + \mu_n^2] \\
&\quad -\gamma_n(1-\gamma_n) d(T^n x_n, x_n) \\
&\leq d^2(x_n, p) + a\nu_n + b\mu_n - \gamma_n(1-\gamma_n) d(T^n x_n, x_n), \qquad (17)
\end{aligned}$$

for some a, $b > 0$. This implies that

$$d^2(z_n, p) \leq d^2(x_n, p) + a\nu_n + b\mu_n, \qquad (18)$$

Again from (1) and (8), we have

$$\begin{aligned}
d^2(y_n, p) &= d^2((1-\beta_n)x_n \oplus \beta_n T^n z_n, p) \\
&\leq \beta_n d^2(T^n z_n, p) + (1-\beta_n) d^2(x_n, p) \\
&\quad -\beta_n(1-\beta_n) d^2(T^n z_n, x_n) \\
&\leq \beta_n [d(z_n, p) + \nu_n(d(z_n, p)) + \mu_n]^2 + (1-\beta_n) d^2(x_n, p) \\
&\quad -\beta_n(1-\beta_n) d^2(T^n z_n, x_n) \\
&\leq \beta_n [d(z_n, p) + \nu_n M_1 d(z_n, p) + \mu_n]^2 + (1-\beta_n) d^2(x_n, p) \\
&\quad -\beta_n(1-\beta_n) d^2(T^n z_n, x_n) \\
&= \beta_n [(1+\nu_n M_1) d(z_n, p) + \mu_n]^2 + (1-\beta_n) d^2(x_n, p) \\
&\quad -\beta_n(1-\beta_n) d^2(T^n z_n, x_n) \\
&\leq \beta_n (1+\nu_n M_1)^2 d^2(z_n, p) + (1-\beta_n)(1+\nu_n M_1)^2 d^2(x_n, p) \\
&\quad +\beta_n [2\mu_n(1+\nu_n M_1) d(z_n, p) + \mu_n^2] \\
&\quad -\beta_n(1-\beta_n) d^2(T^n z_n, x_n). \qquad (19)
\end{aligned}$$

Substituting (18) into (19), we have

$$\begin{aligned} d^2(y_n,p) &\leq \beta_n(1+\nu_n M_1)^2[d^2(x_n,p)+a\nu_n+b\mu_n] \\ &\quad +(1-\beta_n)(1+\nu_n M_1)^2 d^2(x_n,p) \\ &\quad +\beta_n[2\mu_n(1+\nu_n M_1)d(z_n,p)+\mu_n^2] \\ &\quad -\beta_n(1-\beta_n)d^2(T^n z_n,x_n) \\ &\leq d^2(x_n,p)+f\nu_n+g\mu_n \\ &\quad -\beta_n(1-\beta_n)d^2(T^n z_n,x_n), \end{aligned} \tag{20}$$

for some f, $g > 0$.

This implies that

$$d^2(y_n,p) \leq d^2(x_n,p)+f\nu_n+g\mu_n. \tag{21}$$

Finally, from (1) and (8), we have

$$\begin{aligned} d^2(x_{n+1},p) &= d^2((1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n,p) \\ &\leq \alpha_n d^2(T^n y_n,p)+(1-\alpha_n)d^2(T^n x_n,p) \\ &\quad -\alpha_n(1-\alpha_n)d^2(T^n x_n,T^n y_n) \\ &\leq \alpha_n[d(y_n,p)+\nu_n(d(y_n,p))+\mu_n]^2+(1-\alpha_n)[d(x_n,p) \\ &\quad +\nu_n(d(x_n,p))+\mu_n]^2-\alpha_n(1-\alpha_n)d^2(T^n x_n,T^n y_n) \\ &\leq \alpha_n[d(y_n,p)+\nu_n M_1 d(y_n,p)+\mu_n]^2+(1-\alpha_n)[d(x_n,p) \\ &\quad +\nu_n M_1 d(x_n,p)+\mu_n]^2-\alpha_n(1-\alpha_n)d^2(T^n x_n,T^n y_n) \\ &= \alpha_n[(1+\nu_n M_1)d(y_n,p)+\mu_n]^2+(1-\alpha_n)[(1+\nu_n M_1)d(x_n,p) \\ &\quad +\mu_n]^2-\alpha_n(1-\alpha_n)d^2(T^n x_n,T^n y_n) \\ &\leq \alpha_n(1+\nu_n M_1)^2 d^2(y_n,p)+(1+\nu_n M_1)^2(1-\alpha_n)d^2(x_n,p) \\ &\quad +2\mu_n\alpha_n(1+\nu_n M_1)d(y_n,p)+2\mu_n(1-\alpha_n)(1+\nu_n M_1)d(x_n,p) \\ &\quad +\mu_n^2-\alpha_n(1-\alpha_n)d^2(T^n x_n,T^n y_n) \\ &\leq \alpha_n(1+\nu_n M_1)^2 d^2(y_n,p)+(1+\nu_n M_1)^2(1-\alpha_n)d^2(x_n,p) \\ &\quad +2\mu_n(1+\nu_n M_1)R+\mu_n^2-\alpha_n(1-\alpha_n)d^2(T^n x_n,T^n y_n). \end{aligned} \tag{22}$$

Substituting (21) into (22), we obtain

$$\begin{aligned} d^2(x_{n+1},p) &\leq \alpha_n(1+\nu_n M_1)^2[d^2(x_n,p)+f\nu_n+g\mu_n] \\ &\quad +(1+\nu_n M_1)^2(1-\alpha_n)d^2(x_n,p)+\mu_n^2 \\ &\quad +2\mu_n(1+\nu_n M_1)R-\alpha_n(1-\alpha_n)d^2(T^n x_n,T^n y_n) \\ &\leq d^2(x_n,p)+M\nu_n+N\mu_n \\ &\quad -\alpha_n(1-\alpha_n)d^2(T^n x_n,T^n y_n), \end{aligned} \tag{23}$$

for some M, $N > 0$.

Equation (23) yields

$$\alpha_n(1-\alpha_n)d^2(T^n x_n,T^n y_n) \leq d^2(x_n,p)-d^2(x_{n+1},p)+M\nu_n+N\mu_n.$$

Since $\sum_{n=1}^{\infty}\nu_n<\infty$, $\sum_{n=1}^{\infty}\mu_n<\infty$, we have

$$\alpha_n(1-\alpha_n)d^2(T^n x_n,T^n y_n)<\infty.$$

This implies by $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$ that

$$\lim_{n\to\infty} d(T^n x_n,T^n y_n)=0.$$

Now, consider (20), we have

$$\begin{aligned} d^2(y_n, p) &\leq d^2(x_n, p) + f\nu_n + g\mu_n \\ &\quad -\beta_n(1-\beta_n)d^2(T^n z_n, x_n). \end{aligned} \tag{24}$$

Equation (24) yields

$$\beta_n(1-\beta_n)d^2(T^n z_n, x_n) \leq d^2(x_n, p) - d^2(y_n, p) + f\nu_n + g\mu_n.$$

Since $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $d(x_n, p) \leq R$ and $d(y_n, p) \leq R$ for all n, we have

$$\beta_n(1-\beta_n)d^2(T^n z_n, x_n) < \infty.$$

This implies by $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ that

$$\lim_{n\to\infty} d(T^n z_n, x_n) = 0. \tag{25}$$

Next, consider (17), we have

$$d^2(z_n, p) \leq d^2(x_n, p) + a\nu_n + b\mu_n - \gamma_n(1-\gamma_n)d(T^n x_n, x_n). \tag{26}$$

Equation (26) yields

$$\gamma_n(1-\gamma_n)d^2(T^n x_n, x_n) \leq d^2(x_n, p) - d^2(z_n, p) + a\nu_n + b\mu_n.$$

Since $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $d(x_n, p) \leq R$ and $d(z_n, p) \leq R$ for all n, we have

$$\gamma_n(1-\gamma_n)d^2(T^n x_n, x_n) < \infty.$$

This implies by $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$ that

$$\lim_{n\to\infty} d(T^n x_n, x_n) = 0. \tag{27}$$

Now, we have

$$\begin{aligned} d(T^n y_n, x_n) &\leq d(T^n y_n, T^n x_n) + d(T^n x_n, x_n) \\ &\quad \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{28}$$

Again, note that

$$d(x_n, y_n) \leq \beta_n d(x_n, T^n z_n) \to 0 \quad \text{as } n \to \infty. \tag{29}$$

By the definitions of x_{n+1} and y_n, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, T^n y_n) \\ &\leq d(x_n, T^n x_n) + d(T^n x_n, T^n y_n) \\ &\leq d(x_n, T^n x_n) + d(x_n, y_n) + \nu_n(d(x_n, y_n)) + \mu_n \\ &\leq d(x_n, T^n x_n) + d(x_n, y_n) + \nu_n M_1 d(x_n, y_n) + \mu_n \\ &\leq d(x_n, T^n x_n) + (1 + \nu_n M_1)d(x_n, y_n) + \mu_n \\ &\quad \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{30}$$

By (27), (29) and uniform continuity of T, we have

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) \\ &\quad + d(T^{n+1}x_{n+1}, T^{n+1}x_n) + d(T^{n+1}x_n, Tx_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) \\ &\quad + d(x_{n+1}, x_n) + +\nu_{n+1}(d(x_{n+1}, x_n)) + \mu_{n+1} \\ &\quad + d(T^{n+1}x_n, Tx_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) \\ &\quad + d(x_{n+1}, x_n) + \nu_{n+1}M_1 d(x_{n+1}, x_n) + \mu_{n+1} \\ &\quad + d(T^{n+1}x_n, Tx_n) \\ &= (2 + \nu_{n+1}M_1)d(x_{n+1}, x_n) + d(x_{n+1}, T^{n+1}x_{n+1}) \\ &\quad + d(T^{n+1}x_n, Tx_n) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{31}$$

This completes the proof. □

Now, we are in a position to prove the Δ-convergence and strong convergence theorems.

Theorem 3.4. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon K \to K$ be a uniformly continuous and total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (5). *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$.*

If the following conditions are satisfied:
(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;
(ii) there exists a constant $M_1 > 0$ such that $\zeta(r) \leq M_1 r$, $r \geq 0$.
Then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Proof. Let $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We can complete the proof by showing that $\omega_w(x_n) \subseteq F(T)$ and $\omega_w(x_n)$ consists of exactly one point. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.10, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in K$. Hence $v \in F(T)$ by Lemma 2.11. Since by Lemma 3.1, $\lim_{n\to\infty} d(x_n, v)$ exists, so by Lemma 2.9, $v = u$, i.e., $\omega_w(x_n) \subseteq F(T)$.

To show that $\{x_n\}$ Δ-converges to a fixed point of T, it is sufficient to show that $\omega_w(x_n)$ consists of exactly one point.

Let $\{w_n\}$ be a subsequence of $\{x_n\}$ with $A(\{w_n\}) = \{w\}$ and let $A(\{x_n\}) = \{x\}$. Since $w \in \omega_w(x_n) \subseteq F(T)$ and by Lemma 3.1, $\lim_{n\to\infty} d(x_n, w)$ exists. Again by Lemma 2.9, we have $x = w \in F(T)$. Thus $\omega_w(x_n) = \{x\}$. This shows that $\{x_n\}$ Δ-converges to a fixed point of T. This completes the proof. □

As a consequence of Theorem 3.4, we obtain the following.

Corollary 3.5. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon K \to K$ be a uniformly continuous and total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (1). *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.*

If the following conditions are satisfied:
(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;
(ii) there exists a constant $M_1 > 0$ such that $\zeta(r) \leq M_1 r$, $r \geq 0$.
Then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Proof. The proof of Corollary 3.5 immediately follows from Theorem 3.4 by taking $\gamma_n = 0$ for all $n \geq 1$. This completes the proof. □

Theorem 3.6. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $T\colon K \to K$ be a uniformly continuous and total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (5). *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in* $(0,1)$ *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ *and* $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$.

Suppose the following conditions are satisfied:
(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;
(ii) there exists a constant $M_1 > 0$ such that $\zeta(r) \leq M_1 r$, $r \geq 0$.
If T^m is semi-compact for some $m \in \mathbb{N}$, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By Lemma 3.3, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since T is uniformly continuous, we have

$$d(x_n, T^m x_n) \leq d(x_n, Tx_n) + d(Tx_n, T^2 x_n) + \cdots + d(T^{m-1}x_n, T^m x_n) \to 0$$

as $n \to \infty$. That is, $\{x_n\}$ is an AFPS for T^m. By the semi-compactness of T^m, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $p \in K$ such that $\lim_{j\to\infty} x_{n_j} = p$. Again, by the uniform continuity of T, we have

$$d(Tp, p) \leq d(Tp, Tx_{n_j}) + d(Tx_{n_j}, x_{n_j}) + d(x_{n_j}, p) \to 0 \quad \text{as } j \to \infty.$$

That is $p \in F(T)$. By Lemma 3.1, $d(x_n, p)$ exists, thus p is the strong limit of the sequence $\{x_n\}$ itself. This shows that the sequence $\{x_n\}$ converges strongly to a fixed point of T. This completes the proof. □

Senter and Dotson [43] introduced the concept of Condition (A) as follows.

Definition 3.7 ([43]). A mapping $T\colon K \to K$ is said to *satisfy Condition* (A) if there exists a non-decreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, Tx) \geq f(d(x, F(T)))$, for all $x \in K$.

As an application of Theorem 3.2, we establish another strong convergence result employing Condition (A).

Theorem 3.8. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $T\colon K \to K$ be a uniformly continuous and total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (5). *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in* $(0,1)$ *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ *and* $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$.

Suppose the following conditions are satisfied:
(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;
(ii) there exists a constant $M_1 > 0$ such that $\zeta(r) \leq M_1 r$, $r \geq 0$.
If T satisfies Condition (A), then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By similar argument as in the proof of Lemma 3.1, we have that $\lim_{n\to\infty} d(x_n, F(T))$ exists. Again by Lemma 3.3, we know that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. So Condition (A) guarantees that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since f is a non-decreasing function and $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, Theorem 3.2 implies that $\{x_n\}$ converges strongly to a fixed point of T. This completes the proof. □

Remark 3.9. Theorem 3.4 extends Theorem 5.7 of Nanjaras and Panyanak [31] and Theorem 3.5 of Niwongsa and Panyanak [32] to the case of more general class of asymptotically nonexpansive mapping and modified three-step iteration scheme considered in this paper.

Remark 3.10. Theorem 3.4 also extends Theorem 3.5 of Chang et al. [6] to the case of modified three-step iteration scheme considered in this paper.

Remark 3.11. Theorem 3.4 contains Theorem 4 of Başarir and Şahin [3] since the modified three-step iteration scheme reduces to the modified S-iteration scheme.

4. Conclusion

In this paper, we establish some Δ and strong convergence theorems using iteration scheme (5) for more general class of asymptotically nonexpansive mappings and modified three-step iteration scheme (5) which contains modified S-iteration scheme in the framework of CAT(0) spaces and also give its application. The results presented in this paper extend and generalize the previous work from the current existing literature (see, for example, [3, 31, 32, 47] and many others).

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