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Comparability of Lower Attouch–Wets Topologies

Marco Rosa[a], **Paolo Vitolo**[a]

[a]*Dipartimento di Matematica, Informatica ed Economia, Università degli studi della Basilicata, Via dell'Ateneo Lucano 10, 85100 Potenza (Italy)*

Abstract. Beer and Di Concilio [4] have given necessary and sufficient conditions for a two-sided Attouch–Wets topology to contain another on the hyperspace of non-empty closed subsets of a metrizable space as determined by metrics compatible with the topology. In the present paper, we characterize comparability of lower Attouch–Wets topologies as determined by compatible metrics.

1. Introduction

In the literature there are many topologies that can be defined on the hyperspace $\mathrm{CL}(X)$ of a metrizable space X, i.e. the collection of all non-empty closed subsets of X. We refer to the reader to [3] for comprehensive discussion of these. One of the best known is undoubtedly the Hausdorff metric topology τ_{H_d}. Even if this topology works well for bounded sets, it is in general too strong for applications to unbounded sets. For instance, if E_n denotes the line $y = x/n$ in $\mathbb{R}^2$ and E is the x-axis, then the sequence $(E_n)_{n\in\mathbb{N}}$ does not converge to E with respect to τ_{H_d}.

A weakening of the Hausdorff metric topology that is more useful in applications is the Attouch–Wets topology (which is also called "bounded-Hausdorff topology"). It appears for the first time as a convergence in Mosco's paper [7], and was later deeply studied by Attouch and Wets [1, 2].

Given a metric space (X, d), the Attouch–Wets topology τ_{AW_d} on $\mathrm{CL}(X)$ is defined as the topology that $\mathrm{CL}(X)$ inherits from the space $C(X, \mathbb{R})$ of all continuous real-valued functions on X, equipped with the topology of uniform convergence on bounded subsets of X, under the identification $E \leftrightarrow d(\cdot, E)$, where $d(\cdot, E)\colon x \mapsto d(x, E) = \inf_{y\in E} d(x, y)$.

For our purposes, we split this topology in two halves: the upper and the lower Attouch–Wets topologies, respectively denoted by $\tau^+_{AW_d}$ and $\tau^-_{AW_d}$.

On a metrizable space X, let $\mathfrak{M}(X)$ be the set of all compatible metrics. For any pair $d, \rho \in \mathfrak{M}(X)$ we consider the topologies $\tau^-_{AW_d}, \tau^+_{AW_d}, \tau_{AW_d}$ and $\tau^-_{AW_\rho}, \tau^+_{AW_\rho}, \tau_{AW_\rho}$. In [4], the following theorem is established by Beer and Di Concilio to characterize $\tau_{AW_d} \subseteq \tau_{AW_\rho}$. Even if the comparison of upper Attouch–Wets topologies was never explicitly characterized, it can be easily proved in a similar way that the same condition characterizes also $\tau^+_{AW_d} \subseteq \tau^+_{AW_\rho}$.

Theorem 1.1. ([4, Theorem 3.1]) *Let X be a metrizable space and let $d, \rho \in \mathfrak{M}(X)$. The following are equivalent:*

(1) $\tau_{AW_d} \subseteq \tau_{AW_\rho}$ *on* $\mathrm{CL}(X)$.

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Email addresses: marco.rosa@unibas.it (Marco Rosa), paolo.vitolo@unibas.it (Paolo Vitolo)

(2) $\tau^+_{AW_d} \subseteq \tau^+_{AW_\rho}$ *on* CL(X).
(3) $\mathcal{B}_d \subseteq \mathcal{B}_\rho$ *and* $\iota\colon (X,\rho) \to (X,d)$ *is strongly uniformly continuous on* $\mathcal{B}_d$.

Here, in condition (3), the symbol ι denotes the identity function and $\mathcal{B}_d$ is the collection of d-bounded subsets of X. Strong uniform continuity is a stronger (as the name suggests) concept than uniform continuity. We will see its definition later on.

Of course the previous result gives also a condition to characterize $\tau_{AW_d} = \tau_{AW_\rho}$ and $\tau^+_{AW_d} = \tau^+_{AW_\rho}$. Another condition to characterize the equality $\tau_{AW_d} = \tau_{AW_\rho}$ was found in [3], and again it can be easily shown, by a slight modification of the proof, that the same condition also characterizes $\tau^+_{AW_d} = \tau^+_{AW_\rho}$.

Theorem 1.2. ([3, Theorem 3.3.3]) *Let X be a metrizable space and let $d, \rho \in \mathfrak{M}(X)$. The following are equivalent:*

(1) $\tau_{AW_d} = \tau_{AW_\rho}$ *on* CL(X).
(2) $\tau^+_{AW_d} = \tau^+_{AW_\rho}$ *on* CL(X).
(3) $\mathcal{B}_d = \mathcal{B}_\rho$ *and* $\iota\colon (X,\rho) \to (X,d)$ *is bi-uniformly continuous on* $\mathcal{B}_d$.

Note that condition (3) of Theorem 1.2 is seemingly weaker than what one expects from Theorem 1.1.

Our aim is to characterize the inclusion $\tau^-_{AW_d} \subseteq \tau^-_{AW_\rho}$, using again a condition on the collections of d-bounded and ρ-bounded sets, and on strong uniform continuity of the identity map on a certain collection of sets. We also show that our condition is strictly weaker than the condition for $\tau^+_{AW_d} \subseteq \tau^+_{AW_\rho}$ or equivalently, than the condition for $\tau_{AW_d} \subseteq \tau_{AW_\rho}$.

2. Preliminaries and Notation

In a metric space (X,d), the ε-ball about a point x will be denoted, as usual, by $B^d_\varepsilon(x)$. Given a set $A \subseteq X$ we denote by $B^d_\varepsilon[A]$ the ε-expansion of A, namely $B^d_\varepsilon[A] = \bigcup_{x\in A} B^d_\varepsilon(x)$.

A local base at $E \in$ CL(X) with respect to the topologies $\tau^-_{AW_d}$ and $\tau^+_{AW_d}$, is constituted by all collections of the form

$$\{\, F \in \mathrm{CL}(X) \mid E \cap B \subseteq B^d_\varepsilon[F] \,\}$$

and, respectively,

$$\{\, F \in \mathrm{CL}(X) \mid F \cap B \subseteq B^d_\varepsilon[E] \,\}$$

where B runs over the d-bounded subsets of X and $\varepsilon > 0$. Now the Attouch–Wets topology can be defined as $\tau_{AW_d} = \tau^-_{AW_d} \vee \tau^+_{AW_d}$.

Consider the filter Σ_d on CL$(X) \times$ CL(X) having as a base all sets of the form

$$U_d[x_0, n] = \left\{ (E,F) \mid E \cap B^d_n(x_0) \subseteq B^d_{\frac{1}{n}}[F] \text{ and } F \cap B^d_n(x_0) \subseteq B^d_{\frac{1}{n}}[E] \right\}$$

when n runs over $\mathbb{N}$ and x_0 is an arbitrary fixed point of X. It has been shown in [3, Prop. 3.1.6], that Σ_d is a uniformity and it is compatible with τ_{AW_d}.

Given $x_0 \in X$, arbitrarily fixed point, consider the filters Σ^-_d and Σ^+_d having as a base respectively all sets of the form

$$U^-_d[x_0, n] = \left\{ (E,F) \mid E \cap B^d_n(x_0) \subseteq B^d_{\frac{1}{n}}[F] \right\}$$

and

$$U^+_d[x_0, n] = \left\{ (E,F) \mid F \cap B^d_n(x_0) \subseteq B^d_{\frac{1}{n}}[E] \right\}$$

when n runs over $\mathbb{N}$. It can be easily shown using the same technique of [3, Prop. 3.1.6], that both Σ^-_d and Σ^+_d are quasi-uniformities and that they are compatible respectively with $\tau^-_{AW_d}$ and $\tau^+_{AW_d}$.

Definition 2.1. Let (X,ρ) and (Y,d) be metric spaces. A function $f\colon (X,\rho)\to(Y,d)$ is strongly uniformly continuous on a set $A\subseteq X$ if

$$\forall\varepsilon>0\quad\exists\delta>0\quad\forall x\in A\quad\forall y\in X\qquad[\rho(x,y)<\delta\implies d(f(x),f(y))<\varepsilon].$$

We say that f is strongly uniformly continuous on a collection $\mathcal{A}$ if it is strongly uniformly continuous on each $A\in\mathcal{A}$.

This notion of continuity was investigated and explicitly defined in [5]. A basic reference for strong uniform continuity is [6]. Note that strong uniform continuity on a set A is stronger than uniform continuity on A.

Let X be a metrizable space and let $d,\rho\in\mathfrak{M}(X)$. It follows immediately from the definition that the identity $\iota\colon(X,\rho)\to(X,d)$ is strongly uniformly continuous on a set A if, and only if,

$$\forall\varepsilon>0\quad\exists\delta>0\quad\forall x\in A\qquad B^{\rho}_{\delta}(x)\subseteq B^{d}_{\varepsilon}(x).$$

For every compatible metric d on X, we denote by D_d the gap between two non-empty closed sets $E,F\in\mathrm{CL}(X)$, defined as:

$$D_d(E,F)=\inf_{x\in E}\inf_{y\in F}d(x,y).$$

3. When are Lower Attouch–Wets Topologies Comparable?

In this section we give necessary and sufficient conditions for $\tau^-_{AW_d}\subseteq\tau^-_{AW_\rho}$ on $\mathrm{CL}(X)$.

Recall that $A\subseteq X$ is d-uniformly discrete with respect to $\varepsilon>0$ if $d(x,y)\geq\varepsilon$ for every distinct $x,y\in A$. A set A is d-uniformly discrete if there exists $\varepsilon>0$ such that A is d-uniformly discrete with respect to ε.

A set $A\subseteq X$ is d-totally bounded if for every $\varepsilon>0$ there exists $F\subseteq X$ finite such that $A\subseteq B^d_\varepsilon[F]$. When we say that A is not d-totally bounded with respect to $\sigma>0$ we mean of course that $A\nsubseteq B^d_\sigma[F]$ for every finite $F\subseteq X$.

We use the following notations:

$$\mathcal{B}_d=\{\,E\subseteq X\mid E\text{ is }d\text{-bounded}\,\},$$
$$\mathcal{D}_d=\{\,E\subseteq X\mid E\text{ is }d\text{-uniformly discrete}\,\}.$$

We first need some preliminary results. The following two lemmas deal with the comparison of the lower Attouch–Wets topology $\tau^-_{AW_d}$ and the lower Vietoris topology τ^-_V, and will be useful to prove our main theorem. Recall that a subbase for τ^-_V is constituted by all collections of the form $V^-=\{\,E\in\mathrm{CL}(X)\mid E\cap V\neq\varnothing\,\}$, where V runs over all open subsets of X.

Lemma 3.1. *Let (X,d) be a metric space. Then $\tau^-_V\subseteq\tau^-_{AW_d}$.*

Proof. Let $W\subseteq X$ be open and let $W^-=\{\,E\in\mathrm{CL}(X)\mid E\cap W\neq\varnothing\,\}$ be a subbasic open set of τ^-_V. We show that W^- is open with respect to $\tau^-_{AW_d}$.

Let $E\in W^-$. Given $x\in E\cap W$, there exists $n_0\in\mathbb{N}$ such that $x\in B^d_{n_0}(x_0)$ and there exists $n_1\in\mathbb{N}$ such that $B^d_{\frac{1}{n_1}}(x)\subseteq W$. Set $n=\max\{\,n_0,n_1\,\}$. We prove $U^-_d[x_0,n](E)\subseteq W^-$. Indeed, if $F\in U^-_d[x_0,n](E)$, then $E\cap B^d_n(x_0)\subseteq B^d_{\frac{1}{n}}[F]$. Since $x\in E\cap B^d_n(x_0)$, there exists $y\in F$ such that $d(x,y)<\frac{1}{n}$. Therefore $y\in B^d_{\frac{1}{n}}(x)\subseteq W$ and then $y\in F\cap W\neq\varnothing$, that is $F\in W^-$. $\square$

Lemma 3.2. *Let (X,d) be a metric space and let $E\in\mathrm{CL}(X)$. If E is d-totally bounded, then every neighbourhood of E with respect to $\tau^-_{AW_d}$ contains a neighbourhood of E with respect to τ^-_V. As a consequence, if (X,d) is totally bounded, then $\tau^-_{AW_d}=\tau^-_V$.*

Proof. Let $U_d^-[x_0, n](E)$ be a basic neighbourhood of E with respect to $\tau_{AW_d}^-$. Since E is d-totally bounded, there exists $F \subseteq E$ finite such that $E \subseteq B_{\frac{1}{2n}}^d[F]$.

Set $W_z = B_{\frac{1}{2n}}^d(z)$ for every $z \in F$ and set $\mathcal{G} = \bigcap_{z \in F} W_z^-$. Then $\mathcal{G}$ is a neighbourhood of E in τ_V^-, because each W_z is open in X and $z \in E \cap W_z \neq \emptyset$ for every $z \in F$.

We want to prove that $\mathcal{G} \subseteq U_d^-[x_0, n](E)$, that is $E \cap B_n^d(x_0) \subseteq B_{\frac{1}{n}}^d[G]$ for every $G \in \mathcal{G}$. Let $G \in \mathcal{G}$ and $x \in E \cap B_n^d(x_0)$. There exists $z \in F$ such that $x \in B_{\frac{1}{2n}}^d(z)$. Since $G \in W_z^-$, there exists $y \in G \cap W_z$, and then $d(x, y) \leq d(x, z) + d(z, y) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$. It follows that $x \in B_{\frac{1}{n}}^d[G]$ and therefore $E \cap B_n^d(x_0) \subseteq B_{\frac{1}{n}}^d[G]$. $\square$

We now prove our main result.

Theorem 3.3. *Let X be a metrizable space and let $d, \rho \in \mathfrak{M}(X)$. The following are equivalent:*

(1) $\tau_{AW_d}^- \subseteq \tau_{AW_\rho}^-$;
(2) $\mathcal{B}_d \cap \mathcal{D}_d \subseteq \mathcal{B}_\rho$, *and* $\iota\colon (X, \rho) \to (X, d)$ *is strongly uniformly continuous on* $\mathcal{B}_d \cap \mathcal{D}_d$;
(3) $\mathcal{B}_d \cap \mathcal{D}_d \subseteq \mathcal{B}_\rho$, *and for every pair of sequences* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *in* X, *where either* $\{ x_n \mid n \in \mathbb{N} \}$ *or* $\{ y_n \mid n \in \mathbb{N} \}$ *is d-bounded and d-uniformly discrete,*

$$\lim_{n\to\infty} \rho(x_n, y_n) = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} d(x_n, y_n) = 0;$$

(4) $\mathcal{B}_d \cap \mathcal{D}_d \subseteq \mathcal{B}_\rho$, *and for every* $E, F \in \mathrm{CL}(X)$, *where at least one of them is d-bounded and d-uniformly discrete,*

$$D_d(E, F) > 0 \quad \Longrightarrow \quad D_\rho(E, F) > 0.$$

Proof. (1) ⇒ (2) We prove the first condition. On the contrary suppose that there exists $E \in \mathrm{CL}(X)$ which is d-bounded and d-uniformly discrete but not ρ-bounded.

Let $x_1 \in E$. Since E is not ρ-bounded, for every $n \in \mathbb{N}$ we can choose a point $x_{n+1} \in E \setminus B_n^\rho[\{ x_1, \dots, x_n \}]$. Then $E' = \{ x_n \mid n \in \mathbb{N} \} \subseteq E$ is ρ-uniformly discrete.

For every $n \in \mathbb{N}$ set $E_n = \{ x_1, \dots, x_n \}$. We first prove that $(E_n)_{n\in\mathbb{N}}$ converges to E' with respect to $\tau_{AW_\rho}^-$.

Let $\varepsilon > 0$ and let B be ρ-bounded. There exists $q \in \mathbb{N}$ such that $B \subseteq B_q^\rho(x_1)$. Then $E' \cap B \subseteq \{ x_i \mid \rho(x_i, x_1) < q \} \subseteq \{ x_1, \dots, x_q \}$. Therefore $E' \cap B \subseteq B_\varepsilon^\rho[E_n]$ for every $n \geq q$. Hence $E_n \to E'$ with respect to $\tau_{AW_\rho}^-$.

By (1) we have $E_n \to E'$ with respect to $\tau_{AW_d}^-$. Consider the d-bounded set E: for every $\varepsilon > 0$, eventually $E' = E' \cap E \subseteq B_\varepsilon^d[E_n]$. Since each E_n is finite, this would imply that E' is d-totally bounded, a contradiction because E' is infinite and d-uniformly discrete being contained in E.

We now prove strong uniform continuity of ι. Let E be d-uniformly discrete with respect to some $\sigma > 0$ and d-bounded. There exists $h \in \mathbb{N}$ such that $E \subseteq B_h^d(x_0)$. Let $\varepsilon > 0$, $k \in \mathbb{N}$ such that $k > \max\{ h, \frac{1}{\sigma}, \frac{1}{\varepsilon} \}$ and consider $U_d^-[x_0, k](E)$. Since $\tau_{AW_d}^- \subseteq \tau_{AW_\rho}^-$, there exists $m \in \mathbb{N}$ such that $U_\rho^-[x_0, m](E) \subseteq U_d^-[x_0, k](E)$. Let $x \in E$ and $y \in X$ such that $\rho(x, y) < \frac{1}{m}$. We want to prove that $d(x, y) < \varepsilon$. Set $F = (E \setminus \{ x \}) \cup \{ y \}$. Then $E \cap B_m^\rho(x_0) \subseteq E \subseteq B_{\frac{1}{m}}^\rho[F]$, that is $F \in U_\rho^-[x_0, m](E)$. Thus $F \in U_d^-[x_0, k](E)$, that is $E \cap B_k^d(x_0) \subseteq B_{\frac{1}{k}}^d[F]$. Since $x \in E = E \cap B_h^d(x_0) \subseteq E \cap B_k^d(x_0)$, there exists $z \in F$ such that $d(z, x) < \frac{1}{k}$. Since E is uniformly discrete with respect to $\sigma > \frac{1}{k}$, it must be $z = y$ and therefore $d(x, y) < \frac{1}{k} < \varepsilon$.

(2) ⇒ (1) Let $E \in \mathrm{CL}(X)$ and let $n \in \mathbb{N}$. Consider the neighbourhood $U_d^-[x_0, n](E)$ of E with respect to $\tau_{AW_d}^-$. We want to prove that it is also a neighbourhood of E with respect to $\tau_{AW_\rho}^-$.

If $E \cap B_n^d(x_0)$ is d-totally bounded, then by Lemma 3.2, the neighbourhood $U_d^-[x_0, n](E) = U_d^-[x_0, n](E \cap B_n^d(x_0))$ of $E \cap B_n^d(x_0)$ contains a neighbourhood $\mathcal{G}$ of $E \cap B_n^d(x_0)$ with respect to τ_V^-. Then $\mathcal{G}$ is also a neighbourhood of E with respect to τ_V^- and finally, by Lemma 3.1, $\mathcal{G}$ is a neighbourhood of E with respect to $\tau_{AW_\rho}^-$.

Otherwise, if $E \cap B^d_n(x_0)$ is not d-totally bounded with respect to some $\sigma > 0$ (and in particular is infinite), then it is not d-totally bounded with respect to $\sigma' = \min\{\sigma, \frac{1}{2n}\}$. Therefore, by Zorn's lemma, we can construct a maximal set $E' \subseteq E \cap B^d_n(x_0)$, which is (infinite and) d-uniformly discrete, such that

$$E' \subseteq E \cap B^d_n(x_0) \subseteq B^d_{\sigma'}[E'] \subseteq B^d_{\frac{1}{2n}}[E'].$$

The set $E' = E' \cap B^d_n(x_0)$ is d-bounded and d-uniformly discrete, hence there exists $m_0 \in \mathbb{N}$ such that $E' \subseteq B^\rho_{m_0}(x_0)$. Moreover, since ι is strongly uniformly continuous on E',

$$\exists m_1 \in \mathbb{N} \quad \forall x \in E' \qquad B^\rho_{\frac{1}{m_1}}(x) \subseteq B^d_{\frac{1}{2n}}(x).$$

Set $m = \max\{m_0, m_1\}$, we prove that $U^-_\rho[x_0, m](E) \subseteq U^-_d[x_0, n](E)$. Let $F \in U^-_\rho[x_0, m](E)$, that is $E \cap B^\rho_m(x_0) \subseteq B^\rho_{\frac{1}{m}}[F]$. Given $z \in E \cap B^d_n(x_0)$, we claim that $z \in B^d_{\frac{1}{n}}[F]$.

Since $z \in E \cap B^d_n(x_0)$, there exists $x \in E'$ such that $d(z, x) < \frac{1}{2n}$. Thus $B^d_{\frac{1}{2n}}(x) \subseteq B^d_{\frac{1}{n}}(z)$. Moreover $x \in E' \subseteq E \cap B^\rho_m(x_0) \subseteq B^\rho_{\frac{1}{m}}[F]$ and hence there exists $y \in F$ such that $\rho(x, y) < \frac{1}{m}$. Then $y \in B^\rho_{\frac{1}{m}}(x) \subseteq B^d_{\frac{1}{2n}}(x) \subseteq B^d_{\frac{1}{n}}(z)$. Consequently, $z \in B^d_{\frac{1}{n}}[F]$ as claimed

(2) ⇔ (3) This follows from [8, Proposition 4.3].

(2) ⇔ (4) This follows from [5, Theorem 3.1]. □

Now we show by a counterexample that the condition for $\tau^-_{AW_d} \subseteq \tau^-_{AW_\rho}$ is strictly weaker that the condition for $\tau^+_{AW_d} \subseteq \tau^+_{AW_\rho}$ (or equivalently than the condition for $\tau_{AW_d} \subseteq \tau_{AW_\rho}$).

Counterexample 3.4. *Let $X = \mathbb{N}$ and, for every $n, m \in \mathbb{N}$, define $\rho(n, m) = |n - m|$ and $d(n, m) = \left|\frac{1}{n} - \frac{1}{m}\right|$. Then $\tau^-_{AW_d} \subseteq \tau^-_{AW_\rho}$ but $\tau^+_{AW_d} \nsubseteq \tau^+_{AW_\rho}$ and $\tau_{AW_d} \nsubseteq \tau_{AW_\rho}$.*

Proof. Observe first that both d and ρ induce on $\mathbb{N}$ the discrete topology.

Now, as $(\mathbb{N}, d)$ is totally bounded, Lemma 3.2 gives $\tau^-_{AW_d} = \tau^-_V$. Therefore $\tau^-_{AW_d} \subseteq \tau^-_{AW_\rho}$ by Lemma 3.1. Since $\mathbb{N}$ is not ρ-bounded but is d-bounded, we have $\mathcal{B}_d \nsubseteq \mathcal{B}_\rho$. Applying Theorem 1.1 we get both $\tau^+_{AW_d} \nsubseteq \tau^+_{AW_\rho}$ and $\tau_{AW_d} \nsubseteq \tau_{AW_\rho}$. □

4. Comparison of Topologies on Ideals

In [5] Beer and Levi gave conditions on two compatible metrics ρ and d which ensure that the Hausdorff metric topology τ_{H_ρ} induced by ρ restricted to an ideal $\mathcal{I} \subseteq \mathrm{CL}(X)$ is stronger than τ_{H_d} so restricted. This is of course equivalent to continuity of the identity function $\hat{\iota}\colon (\mathrm{CL}(X), \tau^-_{H_\rho}) \to (\mathrm{CL}(X), \tau^-_{H_d})$ on $\mathcal{I}$.

In the same spirit, a similar condition was given in [8] which ensures continuity of the identity function $\hat{\iota}\colon (\mathrm{CL}(X), \mathscr{H}^-_{\mathcal{V}}) \to (\mathrm{CL}(X), \mathscr{H}^-_{\mathcal{U}})$ on a particular collection $\mathcal{I}$ of $\mathrm{CL}(X)$, where $\mathscr{H}^-_{\mathcal{U}}$ and $\mathscr{H}^-_{\mathcal{V}}$ are the lower Hausdorff quasi-uniform hypertopologies generated by two uniformities $\mathcal{U}$ and $\mathcal{V}$.

We want to give a similar result for the lower Attouch–Wets topologies generated by two compatible metrics d and ρ. We omit the proof of the following theorem because it can be easily obtained by a slight modification of the proof of Theorem 3.3.

Theorem 4.1. *Let $\mathcal{A} \subseteq \mathrm{CL}(X)$ be stable under taking nonempty closed subsets and let $d, \rho \in \mathfrak{M}(X)$. The following conditions are equivalent:*

1. *$\hat{\iota}\colon (\mathcal{A}, \tau^-_{AW_\rho}) \to (\mathcal{A}, \tau^-_{AW_d})$ is continuous;*
2. *$\mathcal{B}_d \cap \mathcal{D}_d \cap \mathcal{A} \subseteq \mathcal{B}_\rho$ and $\iota\colon (X, \rho) \to (X, d)$ is strongly uniformly continuous on $\mathcal{B}_d \cap \mathcal{D}_d \cap \mathcal{A}$;*
3. *$\mathcal{B}_d \cap \mathcal{D}_d \cap \mathcal{A} \subseteq \mathcal{B}_\rho$ and for every pair of sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X, where either $\{x_n \mid n \in \mathbb{N}\}$ or $\{y_n \mid n \in \mathbb{N}\}$ belongs to $\mathcal{B}_d \cap \mathcal{D}_d \cap \mathcal{A}$,*

$$\lim_{n\to\infty} \rho(x_n, y_n) = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} d(x_n, y_n) = 0;$$

(4) $\mathcal{B}_d \cap \mathcal{D}_d \cap \mathcal{A} \subseteq \mathcal{B}_\rho$ *and for every* $E, F \in \mathrm{CL}(X)$*, where at least one of them belongs to* $\mathcal{B}_d \cap \mathcal{D}_d \cap \mathcal{A}$*,*

$$D_d(E, F) > 0 \quad \Longrightarrow \quad D_\rho(E, F) > 0.$$

Note that for $\mathcal{A} = \mathrm{CL}(X)$ we obtain again Theorem 3.3.

As a consequence of Theorem 4.1, we can characterize continuity of $\hat{\iota}$ on an ideal $\mathcal{I}$. As observed for example in [8], $\mathcal{D}_d$ is not an ideal since the union of two uniformly discrete sets need not be uniformly discrete.

Given a collection C of subsets of X, denote by $\Im(C)$ be the ideal generated by C. Note that $\Im(\mathcal{D}_d) = \left\{ \bigcup \mathcal{A} \mid \mathcal{A} \text{ finite}, \mathcal{A} \subseteq \mathcal{D}_d \right\}$.

Corollary 4.2. *Let* $\mathcal{I} \subseteq \mathrm{CL}(X)$ *be an ideal and let* $d, \rho \in \mathfrak{M}(X)$*. The following conditions are equivalent:*

(1) $\hat{\iota} \colon (\mathcal{I}, \tau^-_{AW_\rho}) \to (\mathcal{I}, \tau^-_{AW_d})$ *is continuous;*
(2) $\mathcal{B}_d \cap \Im(\mathcal{D}_d) \cap \mathcal{I} \subseteq \mathcal{B}_\rho$ *and* $\iota \colon (X, \rho) \to (X, d)$ *is strongly uniformly continuous on* $\mathcal{B}_d \cap \Im(\mathcal{D}_d) \cap \mathcal{I}$*.*

Remark 4.3. As remarked in [5], a function $f \colon (X, \rho) \to (X, d)$ is continuous at $x \in X$ if, and only if, it is strongly uniformly continuous on $\{x\}$. Hence $\iota \colon (X, \rho) \to (X, d)$ is continuous if, and only if, it is strongly uniformly continuous on $\mathcal{F}(X) = \{ A \subseteq X \mid A \text{ is finite} \}$.

If we consider as $\mathcal{I}$ the ideal of all d-totally bounded subsets of X, then $\Im(\mathcal{D}_d) \cap \mathcal{I} = \mathcal{F}(X)$. Therefore $\mathcal{B}_d \cap \Im(\mathcal{D}_d) \cap \mathcal{I} = \mathcal{F}(X) \subseteq \mathcal{B}_\rho$. for every $\rho \in \mathfrak{M}(X)$.

By Remark 4.3, ι is strongly uniformly continuous on $\mathcal{B}_d \cap \Im(\mathcal{D}_d) \cap \mathcal{I}$ and this implies that $\hat{\iota} \colon (\mathcal{I}, \tau^-_{AW_\rho}) \to (\mathcal{I}, \tau^-_{AW_d})$ is continuous for every $\rho \in \mathfrak{M}(X)$.

Note that indeed $\tau^-_{AW_d} = \tau^-_V$ on $\mathcal{I}$ by Lemma 3.2, and hence continuity of $\hat{\iota}$ on $\mathcal{I}$ could be also seen as a consequence of Lemma 3.1.

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