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A Two-Step Iterative Method Based on Diagonal and Off-Diagonal Splitting for Solving Linear Systems

Mehdi Dehghan[a], Marzieh Dehghani-Madiseh[b], Masoud Hajarian[c]

[a] *Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No. 424, Hafez Ave.,15914, Tehran, Iran*
[b] *Department of Mathematics, Faculty of Mathematical Sciences and Computer, Shahid Chamran University of Ahvaz, Iran*
[c] *Department of Mathematics, Faculty of Mathematical Sciences, Shahid Beheshti University, General Campus, Evin, Tehran 19839, Iran*

Abstract. Solving linear systems is a classical problem of engineering and numerical analysis which has various applications in many sciences and engineering. In this paper, we study efficient iterative methods, based on the diagonal and off-diagonal splitting of the coefficient matrix A for solving linear system $Ax = b$, where $A \in \mathbb{C}^{n\times n}$ is nonsingular and $x, b \in \mathbb{C}^{n\times m}$. The new method is a two-parameter two-step method that has some iterative methods as its special cases. Numerical examples are presented to illustrate the effectiveness of the new method.

1. Introduction

The linear systems play important roles in engineering, scientific computations and various other fields. Therefore a large number of papers have presented several methods for solving linear systems [8–13]. We consider numerical solution of the linear system of the form

$$Ax = b, \qquad A \in \mathbb{C}^{n\times n}, \quad x, b \in \mathbb{C}^{n\times m}, \tag{1.1}$$

where A is a nonsingular matrix with nonvanishing diagonal entries. Iterative methods for the system of linear equations (1.1) require efficient splittings of the coefficient matrix A. For example, the Jacobi, the Gauss-Seidel and the Successive Overrelaxation (SOR) methods [16, 18], split the matrix A into its diagonal and strictly lower and upper triangular parts, and as is said in [5], the generalized conjugate gradient (CG) method [6] and the generalized Lanczos method [23] split the matrix A into its Hermitian and skew-Hermitian parts; see also [2, 3, 15, 17, 22].

We consider the following splitting of A

$$A \equiv D + L + U, \tag{1.2}$$

where $D = diag(A)$ is a diagonal matrix, L is a strictly lower triangular matrix, and U is a general matrix. In this paper we will present efficient iterative methods based on this particular matrix splitting for solving

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Email address: mdehghan@aut.ac.ir (Mehdi Dehghan)

the system of linear equations (1.1). The new iteration method will be referred to as the Diagonal and off-Diagonal splitting (DOS) iteration method or, in brief, "the DOS iteration method".

We rewrite the linear system (1.1) into the system of fixed-point equations

$$Dx = \Big[\omega_1 D + (\omega_1 - 1)L + (\omega_1 - 1)U\Big]x + (1 - \omega_1)b, \tag{1.3}$$

and

$$(D + \omega_2 L)x = \Big[(1 - \omega_2)D - \omega_2 U\Big]x + \omega_2 b, \tag{1.4}$$

where ω_1 and ω_2 are prescribed parameters. Now by alternately iterating between the two systems of fixed point equations (1.3) and (1.4), we can establish the following iteration method for solving the linear system (1.1).

The DOS iteration method: Let $x^{(0)} \in \mathbb{C}^{n\times m}$ be an arbitrary initial guess, for $k = 0, 1, 2, \ldots$, until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} Dx^{(k+\frac{1}{2})} = \Big[\omega_1 D + (\omega_1 - 1)L + (\omega_1 - 1)U\Big]x^{(k)} + (1 - \omega_1)b, \\ (D + \omega_2 L)x^{(k+1)} = \Big[(1 - \omega_2)D - \omega_2 U\Big]x^{(k+\frac{1}{2})} + \omega_2 b, \end{cases} \tag{1.5}$$

where ω_1 and ω_2 are given constants.

Evidently, each iterate of the DOS iteration alternates between the diagonal part D and a lower triangular part $D + \omega_2 L$ of the matrix A, analogously to the classical alternating direction implicit (ADI) iteration method for solving partial differential equations; we refer the interested reader to [14, 20]. We can show that under some conditions, the DOS iteration (1.5) converges unconditionally to the unique solution of the system of linear equations (1.1).

Note that in the above DOS iteration method, we may first solve the system of linear equations with coefficient matrix $D + \omega_2 L$ and then solve the system of linear equations with coefficient matrix D.

The two half-steps at each DOS iterate require exact solutions with the matrices D and $D + \omega_2 L$. Because of the simple construction of these coefficient matrices, we can solve two linear sub-systems exactly, in fact, the simplicity of solving the first linear sub-system is obvious, and for the second linear sub-system we can employ the substitution methods for solving linear systems with triangular coefficient [7, 16]. Note that it is an important advantage for a method that can be inexpensive for performing.

The new method has some iterative methods as its special case. We observe that if L and U is strictly lower and upper triangular matrices, respectively and for specific values of the parameters ω_1 and ω_2, the DOS method reduces to the well-known methods. Let us mention some of them:

DOS method with $\omega_1 = 0$ and $\omega_2 = 0$ is the Jacobi method,
DOS method with $\omega_1 = 1$ and $\omega_2 = 1$ is the Gauss-Seidel method,
DOS method with $\omega_1 = 1 - \omega_1$ and $\omega_2 = 0$ is the Simultaneous Overrelaxation method,
DOS method with $\omega_1 = 1$ and free ω_2 is the Successive Overrelaxation (SOR) method.

Elimination of $x^{(k+\frac{1}{2})}$ from the second step of (1.5) yields

$$x^{(k+1)} = M(\omega_1, \omega_2)x^{(k)} + G(\omega_1, \omega_2)b, \qquad k = 0, 1, 2, \ldots,$$

where

$$M(\omega_1,\omega_2) = (D+\omega_2 L)^{-1}\Big[(1-\omega_2)D-\omega_2 U\Big]D^{-1}\Big[\omega_1 D+(\omega_1-1)L+(\omega_1-1)U\Big],$$

and

$$G(\omega_1,\omega_2) = (D+\omega_2 L)^{-1}\Big[(1-\omega_1)[(1-\omega_2)D-\omega_2 U]D^{-1}+\omega_2 I\Big].$$

Furthermore, we have $A = B(\omega_1,\omega_2) - C(\omega_1,\omega_2)$ and $M(\omega_1,\omega_2) = B(\omega_1,\omega_2)^{-1}C(\omega_1,\omega_2)$, where

$$B(\omega_1,\omega_2) = \Big[(1-\omega_1)[(1-\omega_2)D-\omega_2 U]D^{-1}+\omega_2 I\Big]^{-1}(D+\omega_2 L),$$

and

$$C(\omega_1,\omega_2) = \Big[(1-\omega_1)[(1-\omega_2)D-\omega_2 U]D^{-1}+\omega_2 I\Big]^{-1}\Big[(1-\omega_2)D-\omega_2 U\Big]D^{-1}\Big[\omega_1 D+(\omega_1-1)L+(\omega_1-1)U\Big].$$

Here $A = B(\omega_1,\omega_2) - C(\omega_1,\omega_2)$ is the splitting induced by the DOS iteration, and $B(\omega_1,\omega_2)$ can be used as a preconditioning matrix for the matrix $A \in \mathbb{C}^{n\times n}$. Matrix $B(\omega_1,\omega_2)$ will be referred to as the DOS preconditioner. Note that $M(\omega_1,\omega_2)$ is the iteration matrix of the DOS iterative method.

The rest of the paper is organized as follows: in Section 2, we study the convergence properties of the DOS iteration. Some implementation aspects are briefly discussed in Section 3. Numerical experiments are presented in Section 4. Finally, in Section 5, we end the paper with brief concluding remarks.

2. Convergence Analysis of the DOS Iteration

In this section, we study the convergence analysis of the DOS iteration. First note that the DOS iteration method can be considered as a two-step splitting iteration framework, and the following lemma describes a general convergence criterion for a two-step splitting iteration.

Lemma 2.1. *[5]. Let $A \in \mathbb{C}^{n\times n}$, $A = M_i - N_i$ (i=1, 2) be two splittings of the matrix A, and let $x^{(0)} \in \mathbb{C}^n$ be a given initial vector. If $\{x^{(k)}\}$ is a two-step iteration sequence defined by*

$$\begin{cases} M_1 x^{(k+\frac{1}{2})} = N_1 x^{(k)} + b, \\ M_2 x^{(k+1)} = N_2 x^{(k+\frac{1}{2})} + b, \end{cases} \tag{2.1}$$

$k = 0, 1, 2, \ldots$, then

$$x^{(k+1)} = M_2^{-1}N_2M_1^{-1}N_1x^{(k)} + M_2^{-1}(I + N_2M_1^{-1})b, \qquad k = 0, 1, 2, \ldots.$$

Moreover, if the spectral radius $\rho(M_2^{-1}N_2M_1^{-1}N_1)$ of the iteration matrix $M_2^{-1}N_2M_1^{-1}N_1$ is less than 1, then the iterative sequence $\{x^{(k)}\}$ converges to the unique solution $x^* \in \mathbb{C}^n$ of the system of linear equations (1.1) for all initial vectors $x^{(0)} \in \mathbb{C}^n$. Note that $A = M - N$ is called a splitting of the matrix A if M is a nonsingular matrix.

For the convergence property of the DOS iteration, we apply the above lemma and a part of a theorem (Theorem 6.33) in [1] to obtain the following theorem.

Theorem 2.1. *Let $A_{n\times n} = (a_{ij}) \in \mathbb{C}^{n\times n}$ be diagonally dominant and*

$$\sum_{j=2}^{n} a_{1j} < |a_{11}|,$$

If $L = (l_{ij})$ and $U = (u_{ij})$, and $l_{ij}u_{ij} \geq 0$, $0 \leq \omega_1 \leq 1$ and $0 < \omega_2 \leq 1$, then the DOS iteration converges to the unique solution $x^* \in \mathbb{C}^n$ of the linear system (1.1) for any initial guess.

Proof. First suppose $L_{\omega_1} = D^{-1}\Big[\omega_1 D + (\omega_1 - 1)L + (\omega_1 - 1)U\Big]$. An easy computation establishes the identity

$$\|L_{\omega_1}\|_\infty = \max_{1\leq i\leq n}\{\omega_1 + (1-\omega_1)\sum_{j\neq i}|\frac{a_{ij}}{a_{ii}}|\}.$$

Because of the diagonally dominance property of A we conclude that $\|L_{\omega_1}\|_\infty \leq 1$.

Now let $L_{\omega_2} = -(D+\omega_2 L)^{-1}\Big[(\omega_2 - 1)D + \omega_2 U\Big]$. Suppose then x is an arbitrary vector with $\max_{1\leq i\leq n}|x_i| = 1$, and let $y = L_{\omega_2}x$. Then $(D+\omega_2 L)y = \Big[(1-\omega_2)D - \omega_2 U\Big]x$, and we have

$$y_1 = (1-\omega_2)x_1 - \omega_2(\sum_{j\geq 2}\frac{a_{1j}}{a_{11}}x_j),$$

thus

$$|y_1| \leq (1-\omega_2)|x_1| + \omega_2\sum_{j\geq 2}|\frac{a_{1j}}{a_{11}}||x_j| < 1.$$

Further by induction we can write

$$y_i = (1-\omega_2)x_i - \omega_2\sum_{j<i}\frac{l_{ij}}{a_{ii}}y_j - \omega_2\sum_{j<i}\frac{u_{ij}}{a_{ii}}x_j - \omega_2\sum_{j>i}\frac{a_{ij}}{a_{ii}}x_j.$$

Now because $l_{ij}u_{ij} \geq 0$, we have

$$|y_i| \leq (1-\omega_2)|x_i| + \omega_2\sum_{j<i}|\frac{a_{ij}}{a_{ii}}||y_j| + \omega_2\sum_{j>i}|\frac{a_{ij}}{a_{ii}}||x_j| < 1,$$

for $i = 2, ..., n$. Hence, $\|L_{\omega_2}\|_\infty = \max_{1\leq i\leq n}|y_i| < 1$.

Now, the iteration matrix $M(\omega_1,\omega_2)$ is given by

$$M(\omega_1,\omega_2) = (D+\omega_2 L)^{-1}\Big[(1-\omega_2)D - \omega_2 U\Big]D^{-1}\Big[\omega_1 D + (\omega_1 - 1)L + (\omega_1 - 1)U\Big].$$

Hence the following bound for the spectral radius of $M(\omega_1,\omega_2)$ holds

$$\rho(M(\omega_1,\omega_2)) \leq \Big\|(D+\omega_2 L)^{-1}\Big[(1-\omega_2)D - \omega_2 U\Big]\Big\|_\infty \Big\|D^{-1}\Big[\omega_1 D + (\omega_1 - 1)L + (\omega_1 - 1)U\Big]\Big\|_\infty,$$

that yields

$$\rho(M(\omega_1,\omega_2)) \leq \|L_{\omega_2}\|_\infty \|L_{\omega_1}\|_\infty < 1,$$

showing that the method converges unconditionally for $0 \leq \omega_1 \leq 1$ and $0 < \omega_2 \leq 1$. □

It is to be noted that constructing a strictly lower triangular matrix L and a general matrix U that satisfy in the conditions of Theorem 2.1 is easy. It is sufficient to set $l_{ij} = 0$ and $u_{ij} = a_{ij}$ for $i \le j$. For $i > j$, we should have $l_{ij} + u_{ij} = a_{ij}$ and also $l_{ij}u_{ij} \ge 0$. One of the simplest choices is $l_{ij} = u_{ij} = a_{ij}/2$. It is obvious that for $i > j$, there are many other choices for l_{ij} and u_{ij}.

Corollary 2.1. *If $A \in \mathbb{C}^{n\times n}$ is strictly diagonally dominant, then the DOS iteration for suitable choices of matrices L and U that satisfy in conditions of Theorem 2.1, converges for all* $0 \le \omega_1 \le 1$ *and* $0 < \omega_2 \le 1$.

Before continuing, note to the following definitions and preliminaries.

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called an M-matrix if $a_{ij} \le 0$ for $i \ne j$ and $A^{-1} \ge 0$. The comparison matrix $\langle A\rangle = (\alpha_{ij})$ of a matrix $A = (a_{ij})$ is defined by

$$\alpha_{ij} = \begin{cases} |a_{ij}| & \text{if } i = j, \\ -|a_{ij}| & \text{if } i \ne j. \end{cases}$$

A matrix A is called an H-matrix if $\langle A\rangle$ is an M-matrix. A splitting $A = M - N$ is called regular if $M^{-1} \ge 0$ and $N \ge 0$, and weak regular if $M^{-1} \ge 0$ and $M^{-1}N \ge 0$.

For a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $A \ge 0$ ($A > 0$) denotes that all components of A are nonnegative (positive). For two matrices $A, B \in \mathbb{R}^{n\times n}$, $A \ge B$ ($A > B$) means that $A - B \ge 0$ ($A - B > 0$). For a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $|A|$ denotes the matrix whose components are the absolute values of the corresponding components of A. We have $|AB| \le |A||B|$ for any two matrices A and B of compatible sizes. Varga [21] showed for any square matrices A and B, $|A| \le B$ implies $\rho(A) \le \rho(B)$. OLeary and White [19] showed $\rho(M(\omega_1, \omega_2)) < 1$ when $A^{-1} \ge 0$ and the splitting $A = B(\omega_1, \omega_2) - C(\omega_1, \omega_2)$ is weak regular.

Theorem 2.2. *Let $A \in \mathbb{R}^{n\times n}$ be an H-matrix. Suppose $A = D - L - U$, where $D = diag(A)$, L is a strictly lower triangular matrix, and U is a general matrix. Then, the DOS method converges to the exact solution of the linear system (1.1) for any initial point if* $0 \le \omega_1 \le 1$ *and* $0 < \omega_2 \le 1$.

Proof. Let $M(\omega_1, \omega_2) = B^{-1}(\omega_1, \omega_2)C(\omega_1, \omega_2)$. Then, it suffices to show that $\rho(M(\omega_1, \omega_2)) < 1$ for $0 \le \omega_1 \le 1$ and $0 < \omega_2 \le 1$. Clearly $D - \omega_2 L$ is an H-matrix. We can write $B(\omega_1, \omega_2)$ and $C(\omega_1, \omega_2)$ as follow

$$B(\omega_1, \omega_2) = \frac{1}{1 - \omega_1 + \omega_1\omega_2} D\Big[D + \frac{(1-\omega_1)\omega_2}{1 - \omega_1 + \omega_1\omega_2} U\Big]^{-1} (D - \omega_2 L),$$

$$C(\omega_1, \omega_2) = \frac{1}{1 - \omega_1 + \omega_1\omega_2} D\Big[D + \frac{(1-\omega_1)\omega_2}{1 - \omega_1 + \omega_1\omega_2} U\Big]^{-1} [(1-\omega_2)D + \omega_2 U]D^{-1}[\omega_1 D + (1-\omega_1)L + (1-\omega_1)U],$$

and let

$$\tilde{B}(\omega_1, \omega_2) = |D|\Big[|D| + \frac{(1-\omega_1)\omega_2}{1 - \omega_1 + \omega_1\omega_2}|U|\Big]^{-1}(|D| - \omega_2|L|),$$

$$\tilde{C}(\omega_1, \omega_2) = |D|\Big[|D| + \frac{(1-\omega_1)\omega_2}{1 - \omega_1 + \omega_1\omega_2}|U|\Big]^{-1}[(1-\omega_2)|D| + \omega_2|U|]|D|^{-1}[\omega_1|D| + (1-\omega_1)|L| + (1-\omega_1)|U|].$$

Since $D - \omega_2 L$ is H-matrix, we can write

$$\left|\left(D\Big[D + \frac{(1-\omega_1)\omega_2}{1 - \omega_1 + \omega_1\omega_2} U\Big]^{-1}(D - \omega_2 L)\right)^{-1}\right| \le |(D - \omega_2 L)^{-1}|\left|D + \frac{(1-\omega_1)\omega_2}{1 - \omega_1 + \omega_1\omega_2} U\right||D^{-1}|$$

$$\le \langle D - \omega_2 L\rangle^{-1}\Big(|D| + \frac{(1-\omega_1)\omega_2}{1 - \omega_1 + \omega_1\omega_2}|U|\Big)|D|^{-1}$$

$$= (|D| - \omega_2|L|)^{-1}\Big(|D| + \frac{(1-\omega_1)\omega_2}{1-\omega_1+\omega_1\omega_2}|U|\Big)|D|^{-1}$$
$$= \tilde{B}^{-1}(\omega_1, \omega_2). \tag{2.2}$$

Using (2.2) we can write

$$|M(\omega_1, \omega_2)| = |B^{-1}(\omega_1, \omega_2)C(\omega_1, \omega_2)| \leq \tilde{B}^{-1}(\omega_1, \omega_2)\tilde{C}(\omega_1, \omega_2). \tag{2.3}$$

Now we have $\tilde{B}(\omega_1, \omega_2) - \tilde{C}(\omega_1, \omega_2) = (1 - \omega_1 + \omega_1\omega_2)\langle A\rangle$. Since $\langle A\rangle = \frac{1}{1-\omega_1+\omega_1\omega_2}\tilde{B}(\omega_1, \omega_2) - \frac{1}{1-\omega_1+\omega_1\omega_2}\tilde{C}(\omega_1, \omega_2)$ is a regular splitting of $\langle A\rangle$ and $\langle A\rangle^{-1} \geq 0$, $\rho(\tilde{B}^{-1}(\omega_1, \omega_2)\tilde{C}(\omega_1, \omega_2)) < 1$. From (2.3), we conclude that $\rho(M(\omega_1, \omega_2)) < 1$. □

Corollary 2.2. *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix. Suppose $A = D - L - U$, where $D = diag(A)$, $L \geq 0$ is a strictly lower triangular matrix, and $U \geq 0$ is a general matrix. Then, the DOS method converges to the exact solution of the linear system (1.1) for any initial vector if $0 \leq \omega_1 \leq 1$ and $0 < \omega_2 \leq 1$.*

Proof. Since A is an M-matrix, A is an H-matrix and $\langle A\rangle = A = D - L - U = |D| - |L| - |U|$. By previous theorem, the corollary 2.2 follows. □

3. Implementation Aspects

It is well-known that the strictly diagonally dominant matrices are usually well conditioned and many iterative methods, such as the Jacobi, the Gauss-Seidel and the SOR methods, are convergent for strictly diagonally dominant systems. Unfortunately, in practice, most matrices of nonsingular linear systems are not diagonally dominant. Hence, many iterative methods perform badly when apply to general matrices. Some researches have been done to overcome the trouble by preconditioned techniques [24, 25].

As said in [25], for every nonsingular matrix A it follows immediately from the Singular Value Decomposition (SVD) that there exist nonsingular matrices P and Q such that PAQ is strictly diagonally dominant. A different proof of this is given in [24]. The author also showed that there exists a nonsingular matrix P such that PA is strictly diagonally dominant. A tridiagonal matrix P is constructed such that PA is strictly diagonally dominant for the 3-cyclic matrices as an example in [24].

As mentioned in previous section, the DOS iteration method converges unconditionally when A is strictly diagonally dominant, for $0 \leq \omega_1 \leq 1$ and $0 < \omega_2 \leq 1$. Now it is clear that after finding the preconditioners P and Q such that PAQ is strictly diagonally dominant, we can apply this method for solving

$$PAQy = Pb, \quad \text{and} \quad x = Qy,$$

instead of solving

$$Ax = b.$$

4. Numerical Experiments

In this section, we use some test problems to demonstrate the feasibility and effectiveness of the DOS iteration method, when it used either as a solver or as a preconditioner for solving the system of linear equations (1.1). We also compare DOS method with some other iterative methods as an iterative solver and as a preconditioner for the GMRES method.

In our implementations, the initial guess is chosen to be $x^{(0)} = 0$ and the iteration is terminated when the current iterate $x^{(k)}$ satisfies

$$||x^{(k)} - x^{(k-1)}||_2 < 10^{-5}.$$

4.1. Example descriptions

In this section we describe some numerical examples to show the performance of the DOS iteration method. These examples are taken from the literature [4].

Example 4.1. The system of linear equations (1.1) is of the form

$$(\omega C_V + C_H)x = b,$$

where C_V and C_H are the viscous and the hysteretic damping matrices, respectively, and ω is the driving circular frequency. We take $C_V = 10I$ and $C_H = \mu K$ with μ a damping coefficient, and $K \in \mathbb{R}^{n\times n}$ is the five-point centered difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0,1]\times[0,1]$ with the mesh-size $h = \frac{1}{m+1}$. The matrix K possesses the tensor-product form $K = I \otimes V_m + V_m \otimes I$, with $V_m = h^{-2} tridiag(-1,2,-1) \in \mathbb{R}^{m\times m}$. Hence, K is an $n \times n$ block-tridiagonal matrix, with $n = m^2$. In addition we set $\omega = \pi$, $\mu = 0.02$, and the right-hand vector b to be $b = (-\omega^2 I + K + \omega C_V + C_H)B$, with B being the vector of all entries equal to 1, [4].

Example 4.2. The system of linear equations (1.1) is of the form

$$(I \otimes V + V \otimes I)x = b,$$

where $V = tridiag(-1,2,-1) \in \mathbb{R}^{m\times m}$. We take the right-hand side vector to be

$$b = \left[10(I \otimes V_C + V_C \otimes I) + 9(e_1 e_m^T + e_m e_1^T) \otimes I - (I \otimes V + V \otimes I)\right]B,$$

where $V_C = V - e_1 e_m^T - e_m e_1^T \in \mathbb{R}^{m\times m}$, and e_1 and e_m are the first and the last unit vectors in $\mathbb{R}^m$, respectively, and B is the vector of all entries equal to 1, [4].

Example 4.3. The system of linear equations (1.1) is of the form

$$(K + \frac{3-\sqrt{3}}{\tau} I)x = b,$$

where τ is the time step-size and K is the five-point centered difference matrix approximating the negative Laplacian operator $L = -\Delta$ with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0,1]\times[0,1]$ with the mesh-size $h = \frac{1}{m+1}$. The matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form $K = I \otimes V_m + V_m \otimes I$, with $V_m = h^{-2} tridiag(-1,2,-1) \in \mathbb{R}^{m\times m}$. Hence, K is an $n \times n$ block-tridiagonal matrix with $n = m^2$, [4]. In our tests, we take $\tau = h$. The right-hand side vector b with its jth entry b_j is given by

$$b_j = \frac{j}{\tau(j+1)^2}, \quad j = 1, 2, ..., n.$$

4.2. Experimental results

For the tests reported in this section, L and U are strictly lower and upper triangular matrices, respectively and we use different values of ω_1 and ω_2 for DOS iteration method.

Let $\rho(J)$, $\rho(GS)$, $\rho(SOR_{0.2})$, and $\rho(DOS)$ denote the spectral radius of the iteration matrices of the Jacobi, Gauss-Seidel, $SOR_{0.2}$, and DOS iteration methods, respectively. In Table 1, we give a comparison between the spectral radius of these methods for Example 1, and for different grids.

As it can be seen the spectral radius for DOS method for all mesh-sizes is less than one for the Jacobi, Gauss-Seidel, and $SOR_{0.2}$ methods. Note that for all cases the spectral radius increases when the size of problem increases.

Now we will solve Example 1, by the Jacobi, Gauss-Seidel, $SOR_{0.2}$, and DOS iteration methods. Let iter(J), iter(GS), iter($SOR_{0.2}$), and iter(DOS) denote the iteration numbers of Jacobi, Gauss-Seidel, $SOR_{0.2}$, and DOS methods, respectively. The number of required iterations for these methods is given in Table 2.

Now we solve Examples 1,2 and 3 with DOS method while $\omega_2 = 1$ is fixed and ω_1 is variable. Let $error(k) = \|b - Ax^{(k)}\|_2$, where k denotes the iteration number. The numerical results are drawn in Figure 1.

Figure 2 shows the results obtained from solving Examples 1,2 and 3 by DOS method while $\omega_1 = 0$ is fixed and ω_2 is variable.

In Figure 3, we solve Example 1 with Jacobi, Gauss-Seidel, DOS and GMRES methods while $\omega_2 = 1$ is fixed and ω_1 is variable.

In Figure 4, we solve Example 1 with Jacobi, Gauss-Seidel, DOS and GMRES methods while $\omega_1 = 0$ is fixed and ω_2 is variable.

Now we want to use the DOS preconditioner for solving Example 1. The numerical results by applying GMRES-preconditioned Jacobi, Gauss- Seidel and DOS methods and GMRES method while $\omega_2 = 1$ is fixed and ω_1 is variable, are given in Figure 5. Also the numerical results by applying GMRES-preconditioned Jacobi, Gauss-Seidel and DOS methods and GMRES method for $\omega_1 = 0$ and for various values of ω_2 are shown in Figure 6.

Table 1 The comparison of spectral radius

n	10×10	20×20	30×30	40×40	50×50
$\rho(J)$	0.2260	0.5231	0.7063	0.8083	0.8672
$\rho(GS)$	0.0511	0.2736	0.4988	0.6533	0.7520
$\rho(SOR_{0.2})$	0.8000	0.8000	0.8000	0.8000	0.8000
$\rho(DOS)$	0.0211	0.1632	0.3665	0.5360	0.6566

Table 2 The comparison of iteration number

n	10×10	20×20	30×30	40×40	50×50
iter(J)	8	14	22	33	45
iter(GS)	6	10	15	20	27
iter($SOR_{0.2}$)	46	65	92	126	163
iter(DOS)	4	6	9	13	18

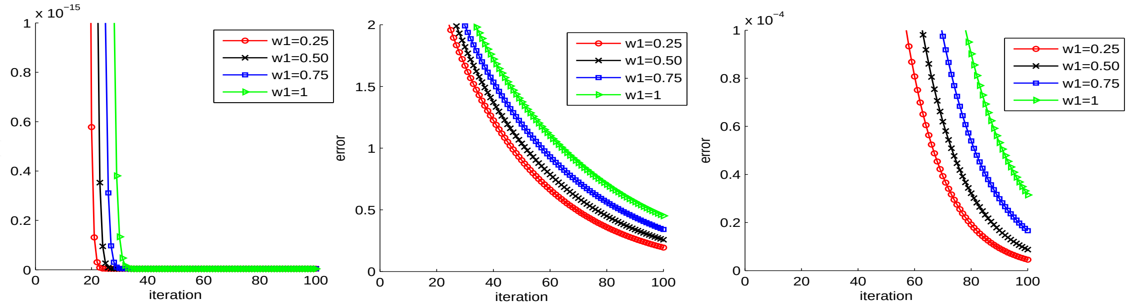

Figure 1: DOS method for solving Examples, left:Example 1, middle:Example 2, right:Example 3

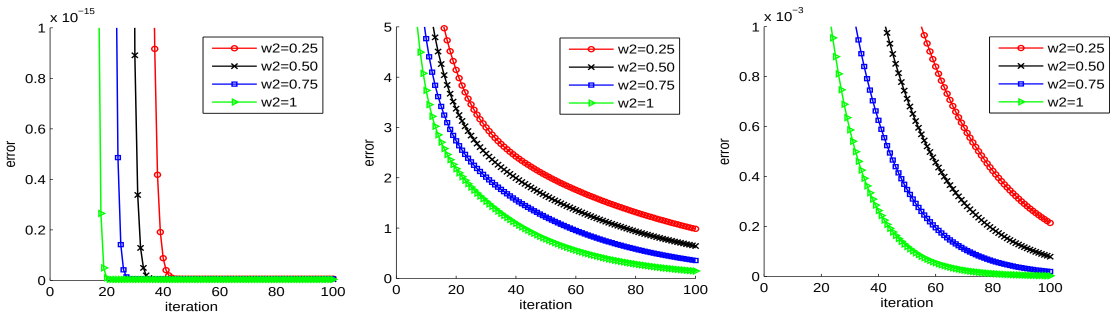

Figure 2: DOS method for solving Examples, left:Example 1, middle:Example 2, right:Example 3

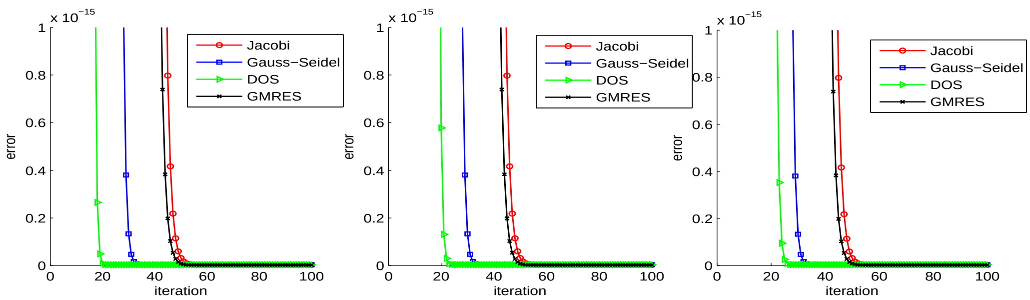

Figure 3: Some methods for solving Example 1, left: $\omega_1 = 0$, middle: $\omega_1 = 0.25$, right: $\omega_1 = 0.50$

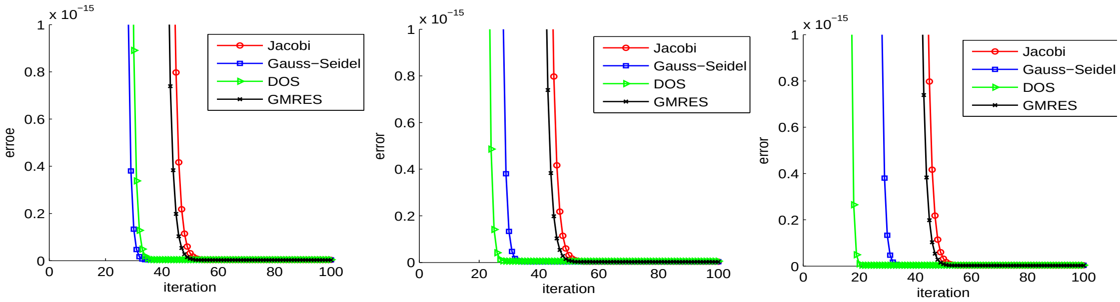

Figure 4: Some methods for solving Example 1, left: $\omega_2 = 0.50$, middle: $\omega_2 = 0.75$, right: $\omega_2 = 1$

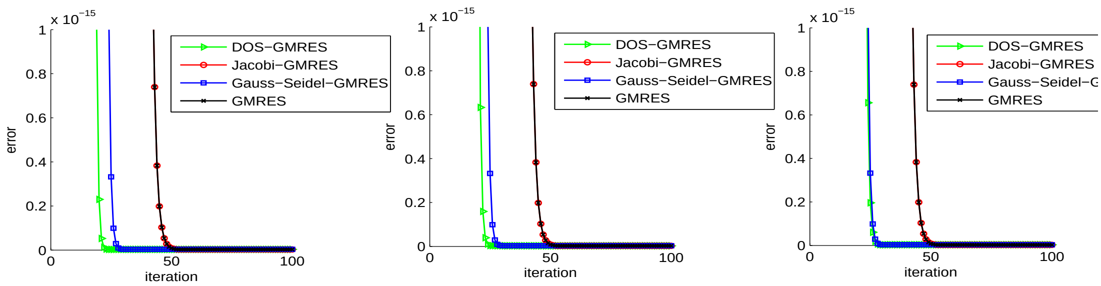

Figure 5: Some methods for solving Example 1, left: $\omega_1 = 0$, middle: $\omega_1 = 0.25$, right: $\omega_1 = 0.50$

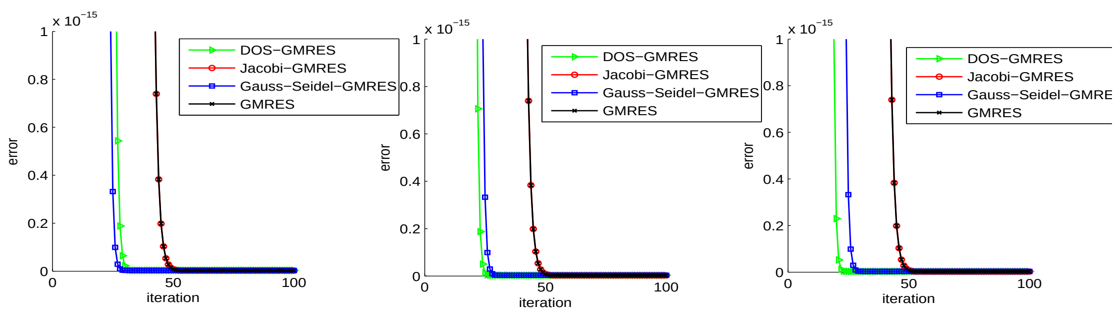

Figure 6: Some methods for solving Example 1, left: $\omega_2 = 0.50$, middle: $\omega_2 = 0.75$, right: $\omega_2 = 1$

In Table 2, we have reported the iteration numbers for Jacobi, Gauss-Seidel, $SOR_{0.2}$ and DOS iteration methods, for solving Example 1. One can see that for all mesh-sizes the number of iterations for DOS method is less than one obtained by other three methods. Also for all methods, the number of iterations grows with the problem size. However this growth is slower for the DOS than for the other three methods.

From Figure 1, for all examples with $\omega_2 = 1$ fixed, when ω_1 increases the error in a fixed iteration increases, also. In other words, with smaller ω_1 we have less error. Also, from Figure 2, for all examples with $\omega_1 = 0$ fixed, when ω_2 increases the error in a fixed iteration decreases. In other words, with larger ω_1, we have less error.

From Figure 3, we see that when $\omega_2 = 1$ is fixed and ω_1 is variable, in all cases, DOS iteration method for solving Example 1 converges faster than other methods, but with less ω_1, we have better improvement.

From Figure 4, where $\omega_1 = 0$ is fixed and ω_2 is variable, we can say, except for $\omega_2 = 0.50$, in two other cases the DOS method for solving Example 1, can perform better than the other methods, in fact by increasing in the values of the ω_2, we have better results.

From Figure 5, we find that when $\omega_2 = 1$ is fixed and ω_1 is variable, in all cases, DOS-preconditioned GMRES method for solving Example 1 converges faster than other preconditioned GMRES methods, of course with less values of ω_1, we have better results.

From Figure 6, where $\omega_1 = 0$ is fixed and ω_2 is variable, we can say except for $\omega_2 = 0.50$, in two other cases the DOS-preconditioned GMRES method for solving Example 1, has less error respect to the other preconditioned GMRES methods, and for larger values of ω_2 we have less error for DOS-preconditioned GMRES method.

5. Concluding Remarks

In this paper an iterative method for solving a system of linear equations is proposed. The powerfulness of the new method that has some other iterative methods as its special cases, compared with some other methods and numerical experiments showed its feasibility and effectiveness.

References

[1] O. Axelsson, Iterative Solution Methods, Cambridge University Press, 1996.

[2] O. Axelsson, Z.-Z. Bai, S.-X. Qiu, A class of nested iteration schemes for linear systems with a coefficient matrix with a dominant positive definite symmetric part, Numer. Algorithms, 35 (2004) 351-372.

[3] Z.-Z. Bai, Sharp error bounds of some Krylov subspace methods for non-Hermitian linear systems, Appl. Math. Comput., 109 (2000) 273-285.

[4] Z.-Z. Bai, M. Benzi, F. Chen, Modified HSS iteration methods for a class of complex symmetric linear systems, Computing. 87 (2010) 93-111.

[5] Z.-Z. Bai, G.H. Golub, M.K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, SIAM J. Matrix Anal. Appl. 24 (2003) 603-626.

[6] P. Concusand, G.H. Golub, A generalized conjugate gradient method for non-symmetric systems of linear equations, in Computing Methods in Applied Sciences and Engineering, Lecture Notes in Econom. and Math. Systems 134, R. Glowinski and J.R. Lions, eds., Springer-Verlag, Berlin, (1976) 56-65; also available online from http://wwwsccm. stanford.edu.

[7] B.N. Datta, Numerical Linear Algebra and Applications, Brooks/Cole Publishing Company, United States of America, 1995.

[8] M. Dehghan, M. Hajarian, Finite iterative algorithms for the reflexive and anti-reflexive solutions of the matrix equation $A_1X_1B_1 + A_2X_2B_2 = C$, Math. Comput. Model. 49 (2009) 1937-1959.

[9] M. Dehghan, M. Hajarian, Efficient iterative method for solving the second-order Sylvester matrix equation $EVF^2 - AVF - CV = BW$, IET Control Theory Appl. 3 (2009) 1401-1408.

[10] M. Dehghan, M. Hajarian, The general coupled matrix equations over generalized bisymmetric matrices, Linear Algebra Appl. 432 (2010) 1531-1552.

[11] F. Ding, T. Chen, Iterative least squares solutions of coupled Sylvester matrix equations, Systems Control Lett. 54 (2005) 95-107.

[12] F. Ding, T. Chen, On iterative solutions of general coupled matrix equations, SIAM J. Control Optim. 44 (2006) 2269-2284.

[13] F. Ding, P.X. Liu, J. Ding, Iterative solutions of the generalized Sylvester matrix equations by using the hierarchical identification principle, Appl. Math. Comput. 197 (2008) 41-50.
[14] J. Douglas, Jr., and H.H. Rachford, Jr., Alternating direction methods for three space variables, Numer. Math., 4 (1956) 41-63.
[15] M. Eiermann, W. Niethammer, R.S. Varga, Acceleration of relaxation methods for non-Hermitian linear systems, SIAM J. Matrix Anal. Appl., 13 (1992) 979-991.
[16] G.H. Golub, C.F. Van Loan, Matrix Computations, 3rd edn. The Johns Hopkins University Press, Baltimore, 1996.
[17] G.H. Golub, A.J. Wathen, An iteration for indefinite systems and its application to the NavierStokes equations, SIAM J. Sci. Comput., 19 (1998) 530-539.
[18] A. Hadjidimos, Successive overrelaxation (SOR) and related methods, J. Comput. Appl. Math., 123 (2000) 177-191.
[19] D.P. OLeary, R.E. White, Multisplittings of matrices and parallel solution of linear systems, SIAM J. Algebraic Discrete Methods, 6 (1985) 630-640.
[20] D.W. Peaceman, H.H. Rachford, Jr., The numerical solution of parabolic and elliptic differential equations, J. Soc. Indust. Appl. Math., 3 (1955) 28-41.
[21] R.S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, 1962.
[22] C.-L. Wang, Z.-Z. Bai, Sufficient conditions for the convergent splittings of non-Hermitian positive definite matrices, Linear Algebra Appl., 330 (2001) 215-218.
[23] O. Widlund, A Lanczos method for a class of nonsymmetric systems of linear equations, SIAM J. Numer. Anal., 15 (1978) 801-812.
[24] J.Y. Yuan, Preconditioned diagonal dominant matrices, Appl. Math. Comput., 114 (2000) 255-262.
[25] J.Y. Yuan, P. Yalamov, A method for constructing diagonally dominant preconditioners based on jacobi rotations, Appl. Math. Comput., 174 (2006) 74-80.