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Polynomial in a Saphar Linear Relation in a Banach Space

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Abstract. In this paper, we introduce the notion of Saphar linear relation in a Banach space and we study the behaviour of such notion in polynomials.

1. Introduction and preliminaries

We adhered to the notations and terminology of the book [4]. Let E, F and G be linear spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A linear relation T from E to F, denoted by $T \in LR(E, F)$, is any mapping having domain $D(T)$ a nonempty subspace of E and taking values in the collection of nonempty subsets of F such that $T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$ for all nonzero scalars α, β and $x_1, x_2 \in D(T)$. If T maps the points of its domain to singletons then T is said to be an operator. A linear relation $T \in LR(E, F)$ is uniquely determined by its graph $G(T)$ which is defined by

$$G(T) := \{(x, y) \in E \times F : x \in D(T), y \in Tx\}.$$

For linear relations $T_1, T_2 \in LR(E, F)$ and $S \in LR(F, G)$, the linear relations $T_1^{-1}, T_1 + T_2$ and ST_1 are defined by

$$G(T_1^{-1}) := \{(y, x) : (x, y) \in G(T_1)\},$$

$$G(T_1 + T_2) := \{(x, y_1 + y_2) : (x, y_1) \in G(T_1), (x, y_2) \in G(T_2)\},$$

and

$$G(ST_1) := \{(x, z) \in E \times G : (x, y) \in G(T_1), (y, z) \in G(S) \text{ for some } y \in F\}.$$

If $G(T_1) \subset G(T_2)$ we write $T_1 \subset T_2$.

If $\lambda \in \mathbb{K}$ and T is a linear relation in E, that is, $T \in LR(E) := LR(E, E)$, then λT stands for $(\lambda I)T$ where I is the identity operator on E and $T - \lambda := T - \lambda I$.

The product of linear relations is clearly associative. Hence if $T \in LR(E)$ then T^n, $n \in \mathbb{Z}$, is defined as usual with $T^0 = I$ and $T^1 = T$.

Let $T \in LR(E, F)$. The subspaces $N(T) := T^{-1}(0), R(T) := T(D(T))$ and $T(0)$ are called the null space, the range and the multivalued part of T, respectively. We say that T is injective if $N(T) = \{0\}$, surjective if $R(T) = F$ and T is bijective if it is injective and surjective. We note that T is an operator if and only if $T(0) = \{0\}$. For $T \in LR(E)$ we shall consider the subsets

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$$\rho(T) := \{\lambda \in \mathbb{K} : T - \lambda \text{ is bijective } \} \text{ and } \sigma(T) := \mathbb{K} \backslash \rho(T),$$

called the resolvent set and the spectrum of T, respectively.

Let X and Y be normed spaces and let $T \in LR(X, Y)$. If M is a closed subspace of X, we say that M is topologically complemented in X if there exists a closed subspace M_1 of X such that $X = M \oplus M_1$. In such case, M_1 is called a topological complement of M. We denote by $T\ |_M$ the linear relation given by $G(T\ |_M) = G(T) \cap (M \times Y)$, Q_M denotes the quotient map from X onto X/M and Q_T stands for the quotient map from Y onto $Y/\overline{T(0)}$. It is easy to see that $Q_T T$ is an operator and hence we can define $\| Tx \| := \| Q_T Tx \|$, $x \in D(T)$ and $\| T \| := \| Q_T T \|$. We say that T is closed if its graph is a closed subspace of $X \times Y$, continuous if $\| T \| < \infty$ and T is called bounded if T is continuous and everywhere defined. We note that if X and Y are Banach spaces and $T \in LR(X, Y)$ is closed and everywhere defined, then T is bounded.

Bounded regular operators and bounded Saphar operators in Banach spaces were introduced and studied (under various names and notations) by several authors, see, for instance [3], [9], [12], [13] and [14] among others. Such concepts can be naturally generalized to linear relations, as follows.

Definition 1.1. *Let X and Y be Banach spaces and let $T \in LR(X, Y)$ be closed and everywhere defined. We say that T is relatively regular, denoted by $T \in RR(X, Y)$, if $N(T)$ and $R(T)$ are topologically complemented in X and Y, respectively. Assume that $X = Y$. Then we say that T is regular if $R(T)$ is closed and $N(T) \subset R(T^n)$ for all $n \in \mathbb{N}$ and T is called a Saphar relation in X if T is regular and relatively regular.*

It is evident that in Hilbert spaces the class of Saphar relations coincides with the class of regular relations which was considered in [8] with the name of the class of quasi-Fredholm relations of degree 0. On the other hand, it is not difficult to find examples of Saphar relations in Banach spaces, as we see from the next example.

Example 1.2. *Let $l_p, 1 \leq p < \infty$ be the Banach space of all complex sequences (x_n) such that $\sum_{n=1}^{\infty} | x_n |^p < \infty$ and let S_r and S_l be the bounded operators in l_p defined by*

$S_r(x_1, x_2, x_3, ...) := (0, x_1, x_2, ...)$ and $S_l(x_1, x_2, x_3, ...) := (x_2, x_3, ...)$, $(x_1, x_2, x_3, ...) \in l_p$.

Then $\lambda^{-1} - S_l^{-1}$ is a Saphar relation in l_p whenever $0 < | \lambda | < 1$.

Proof. It is clear that S_r and S_l are bounded operators, the null space of S_l coincides with the subspace generated by $(1, 0, 0, ...)$ and $R(S_l) = l_p$. Hence S_l^{-1} is a closed an everywhere defined linear relation in l_p.

For each $\lambda \in \mathbb{K} \backslash \{0\}$ we have that $N(S_l^{-1} - \lambda^{-1}) = N(S_l - \lambda)$ and $R(S_l^{-1} - \lambda^{-1}) = R(S_l - \lambda)$ ([4, Proposition VI.2.3 and Theorem VI.4.2]) and clearly $S_l - \lambda = S_l - \lambda S_l S_r = \lambda S_l(\lambda^{-1} - S_r)$. These properties combined with the fact that $\sigma(S_l) = \sigma(S_r) = \{\lambda \in \mathbb{K} : | \lambda | \leq 1\}$ (see, for instance [15, Theorem 4.5]) lead to the desired result. □

On the other hand, we note that the notion of regular relation in a Banach space was introduced in [1]. The following theorem is the main result of the paper [1, Theorem 21].

Theorem 1.3. *Let T be a closed linear relation in a complex Banach space X such that $\rho(T) \neq \emptyset$. Let n and $m_i, 1 \leq i \leq n$ be positive integers and let $\lambda_i, 1 \leq i \leq n$ be some distinct constants Assume that for all $i \in \{1, 2, ..., n\}$, $T - \lambda_i$ is regular. Then $\sqcap_{i=1}^{n} (T - \lambda_i)^{m_i}$ is regular.*

In the present paper we continue the investigation initiated in [1]. So our first main objective is to prove the validity of the converse of Theorem 1.1. The second main purpose of this paper is to show that under suitable conditions $\sqcap_{i=1}^{n} (T - \lambda_i)^{m_i}$ is a Saphar relation if and only if for all $1 \leq i \leq n$, $T - \lambda_i$ is a Saphar relation, (see, Theorem 4.4 below).

2. Some Algebraic Properties of a Polynomial in a Linear Relation

In this section we present some purely algebraic properties of a polynomial in a linear relation in a linear space which will be used to prove the main results of this paper.

Lemma 2.1. *Let A be a linear relation in a linear space E. We have:*

(i) *The following properties are equivalent:*
(a) $N(A^n) \subset R(A)$ *for all* $n \in \mathbb{N}$.
(b) $N(A) \subset R(A^m)$ *for all* $m \in \mathbb{N}$.
(c) $N(A^n) \subset R(A^m)$ *for all* $n, m \in \mathbb{N}$.

(ii) *For all* $\alpha, \beta \in \mathbb{K}$ *and for all* $n, m \in \mathbb{N}$ *we have*
(1) $(A-\alpha)^n(A-\beta)^m = (A-\beta)^m(A-\alpha)^n$,
(2) $D((A-\alpha)^n(A-\beta)^m)) = D(A^{n+m})$,
(3) $(A-\alpha)^n(A-\beta)^m(0) = A^{n+m}(0)$,
(4) If A is bijective and everywhere defined then

$$(A-\alpha)^{-n}(A-\beta)^m \subset (A-\beta)^m(A-\alpha)^{-n}.$$

(iii) *Assume that A is everywhere defined and let* $\lambda \in \mathbb{K}$. *Then*

$$(A-\lambda)^n = \textstyle\sum_{i=0}^{n} \binom{n}{i}(-1)^i\lambda^i A^{n-i},\ n \in \mathbb{N}.$$

(iv) *If A has a nonempty resolvent set, then*

$$\{0\} = N(A^n) \cap A^m(0) \text{ for all } n, m \in \mathbb{N}.$$

Proof. (i) See, for instance [8, Lemma 2.7].

(ii) The first three properties are established in [10, 1, (1.2) and (1.3)]. Assume now that $A-\alpha$ is bijective and that $D(A) = E$. It is clear that $(A-\alpha)^n$ is also bijective and its domain is the whole space E which leads to

$$(A-\alpha)^{-n}(A-\alpha)^n = E \subset (A-\alpha)^n(A-\alpha)^{-n}.$$

Hence, we have that
$(A-\alpha)^{-n}(A-\beta)^m \subset (A-\alpha)^{-n}(A-\beta)^m(A-\alpha)^n(A-\alpha)^{-n} =$
$(A-\alpha)^{-n}(A-\alpha)^n(A-\beta)^m(A-\alpha)^{-n} = (A-\beta)^m(A-\alpha)^{-n}$.

(iii) We proceed by induction. For $n = 1$ it is trivial. Assume (iii) holds for some positive integer k. Then one deduces from (ii) combined with [4, Proposition I.4.2] that
$(A-\lambda)^{k+1} = (A-\lambda)^k A - (A-\lambda)^k\lambda = A(A-\lambda)^k - \lambda(A-\lambda)^k =$
$\sum_{i=0}^{k}\binom{k}{i}(-1)^i\lambda^i A^{k+1-i} - \sum_{i=0}^{k}\binom{k}{i}(-1)^i\lambda^{i+1}A^{k-i} =$
$A^{k+1-i} + \sum_{i=1}^{k}\binom{k+1}{i}(-1)^i\lambda^i A^{k+1-i} + (-1)^{k+1}\lambda^{k+1} =$
$\sum_{i=0}^{k+1}\binom{k+1}{i}(-1)^i\lambda^i A^{k+1-i}$.
Therefore (iii) holds.

(iv) A proof of this statement can be found in [11, Lemma 6.1]. □

Definition 2.2. *[10] Let A be a linear relation in a linear space E. Fix* $\lambda \in \mathbb{K}$, *let* $P(\lambda) := \alpha \sqcap_{i=1}^{n} (\lambda - \lambda_i)^{m_i}$ *be a polynomial in* λ *where n and* m_i, $1 \le i \le n$, *are positive integers,* $\alpha \in \mathbb{K}$ *and* λ_i, $1 \le i \le n$ *are some distinct constants. Then, the polynomial P in A given by*

$$P(A) := \alpha \sqcap_{i=1}^{n} (A - \lambda_i)^{m_i}$$

is a linear relation in E by virtue of Lemma 2.1.

Recall that if P_1 and P_2 are relatively prime polynomials in $\lambda \in \mathbb{K}$, then there exist polynomials Q_1 and Q_2 in λ such that $1 = Q_1(\lambda)P_1(\lambda) + Q_2(\lambda)P_2(\lambda)$. The following useful lemma can be seen as an extension of this property to the case of polynomials in a linear relation.

Lemma 2.3. *Let A be an everywhere defined linear relation in a linear space E and let* P_1 *and* P_2 *be relatively prime polynomials in* $\lambda \in \mathbb{K}$. *Assume that* Q_1 *and* Q_2 *are two polynomials in* λ *such that* $1 = Q_1(\lambda)P_1(\lambda) + Q_2(\lambda)P_2(\lambda)$. *Then, for all* $x \in E$

$$Q_1(A)P_1(A)x + Q_2(A)P_2(A)x = x + A^n(0)$$

where n is the degree of the polynomial Q_1P_1.

Proof. The use of Lemma 2.1 makes us to write

$$Q_1(A)P_1(A) = \alpha_o + \sum_{i=1}^{n} \alpha_i A^i \text{ and } Q_2(A)P_2(A) = \delta_o + \sum_{i=1}^{n} \delta_i A^i$$

for some nonzero scalars $\alpha_i, \delta_i, 0 \leq i \leq n$. Hence, for all $x \in E$ we have that

$$Q_1(A)P_1(A)x + Q_2(A)P_2(A)x = (\alpha_o + \delta_o)x + \sum_{i=1}^{n}(\alpha_i + \delta_i)A^i x$$

and since $\alpha_o + \delta_o = 1$ and $\alpha_i + \delta_i = 0$, $1 \leq i \leq n$, we obtain that $Q_1(A)P_1(A)x + Q_2(A)P_2(A)x = x + A(0) + A^2(0) + ... + A^n(0) = x + A^n(0)$. The proof is completed. □

The behaviour of the domain, the range, the null space and the multivalued part of a polynomial in a linear relation is described in the following useful lemma

Lemma 2.4. *Let A be an everywhere defined linear relation in a linear space E having a nonempty resolvent set and let $P(A) := \alpha \sqcap_{i=1}^{n} (A - \lambda_i)^{m_i}$ as in Definition 2.2. Then*

(i) $D(P(A)) = E$ *and* $P(A)(0) = A^{\sum_{i=1}^{n} m_i}(0)$.

(ii) $R(P(A)) = \cap_{i=1}^{n} R(A - \lambda_i)^{m_i}$.

(iii) $N(P(A)) = \oplus_{i=1}^{n} N(A - \lambda_i)^{m_i}$.

Proof. (i) It is covered by the part (ii) in Lemma 2.1.

(ii) It is proved in [10, Theorem 3.3].

(iii) Since $N(P(A)) = \sum_{i=1}^{n} N(A-\lambda_i)^{m_i}$ by virtue of [10, Theorem 3.4], it only remains to verify that such sum is direct which will be a consequence of Lemmas 2.1 and 2.3. Indeed, we shall show that $\cap_{i=1}^{n} N(A-\lambda_i)^{m_i} = \{0\}$ is true for $n = 2$ since the general case then follows by induction. Let Q_1 and Q_2 be polynomials in A such that

$$\text{for every } x \in E,\ Q_1(A)(A - \lambda_1)^{m_1} x + Q_2(A)(A - \lambda_2)^{m_2} x = x + A^r(0)$$

where r denotes the degree of $Q_1(A)(A - \lambda_1)^{m_1}$. So that, for $x \in N(A - \lambda_1)^{m_1} \cap N(A - \lambda_2)^{m_2}$ we obtain that $x + A^r(0) = A^r(0)$ which implies that $N(A - \lambda_1)^{m_1} \cap N(A - \lambda_2)^{m_2} \subset A^r(0)$. This inclusion together with Lemma 2.1 allowed us to conclude that $N(A - \lambda_1)^{m_1} \cap N(A - \lambda_2)^{m_2} = \{0\}$, as desired. □

3. Product of Relatively Regular Linear Relations

At the beginning of this section we present some properties of the topological complementation which are essential to obtain the main results of this paper.

Following [5] we say that a linear relation A in a linear space E is a multivalued projection in E if $A^2 = A$ and $R(A) \subset D(A)$. Multivalued projections in E can be characterized in terms of subspaces pairs, as follows: Let M_1 and M_2 be subspaces of E and let $A \in LR(E)$ defined by $G(A) = \{(m_1 + m_2, m_1) : m_1 \in M_1, m_2 \in M_2\}$. Then A is a multivalued projection in E with $D(A) = M_1 + M_2$, $R(A) = M_1$, $N(A) = M_2$ and $A(0) = M_1 \cap M_2$. Conversely, if A is a multivalued projection in E, then A determines a pair of subspaces M_1 and M_2 of E such that $G(A) = \{(m_1 + m_2, m_1) : m_1 \in M_1, m_2 \in M_2\}$, $D(A) = M_1 + M_2$, $R(A) = M_1$, $N(A) = M_2$ and $A(0) = M_1 \cap M_2$.

The following lemma shows that the notion of topological complementation may be expressed in terms of multivalued projections under suitable restrictions.

Lemma 3.1. *[5, Corollary 3.5 and Proposition 3.13] Let M_1 and M_2 be subspaces of a Banach space X and let S denote the multivalued projection in X with $D(S) = M_1 + M_2$, $R(S) = M_1$, $N(S) = M_2$ and $S(0) = M_1 \cap M_2$. We have:*

(i) *If M_1 and M_2 are closed, then S is continuous if and only if $M_1 + M_2$ is closed.*

(ii) *If S is continuous and $M_1 + M_2$ and $M_1 \cap M_2$ are topologically complemented in X and $M_1 + M_2$ respectively, then M_1 and M_2 are topologically complemented in X.*

Lemma 3.2. *Let M_1 and M_2 be subspaces of a Banach space X such that M_2 is closed and it is contained in M_1. Then*

(i) *M_1 is closed in X if and only if M_1/M_2 is closed in X/M_2.*

(ii) *If M_1 is closed then $(X/M_2)/(M_1/M_2) = X/M_1$ and $Q_{M_1/M_2}Q_{M_2} = Q_{M_1}$ where the equality is a canonical isometry.*

(iii) *For any closed subspace F of X/M_2, the closed subspace G of X given by $G := Q^{-1}_{M_2}F$ satisfies $M_2 \subset G$ and $(X/M_2)/F = X/G$.*

(iv) *If M_1/M_2 and M_2 are topologically complemented in X/M_2 and X, respectively, then M_1 is topologically complemented in X.*

Proof. (i) Follows immediately from the definitions.

(ii) and (iii) These statements are proved in [4, Lemma IV.5.2].

(iv) Since M_1/M_2 is topologically complemented in X/M_2, we infer from the above assertions that M_1 is closed and $X/M_2 = (M_1/M_2) \oplus (M_3/M_2)$ for some closed subspace M_3 of X with $M_2 \subset M_3$. So that $X = M_1 + M_3$ and $M_2 = M_1 \cap M_3$ by virtue of [4, Lemma I.6.8]. Let S be the multivalued projection with $D(S) = M_1 + M_3$, $R(S) = M_1$, $N(S) = M_3$ and $S(0) = M_1 \cap M_3$. According to Lemma 3.1 (i), S is continuous and since M_2 is topologically complemented in X, we deduce from Lemma 3.1 (ii) that M_1 is topologically complemented in X. □

Lemma 3.3. *Let X and Y be Banach spaces and let T be a closed and everywhere defined linear relation from X to Y. We have:*

(i) *If N is a topological complement of $R(T)$, then Q_TN is a topological complement of $R(Q_TT)$.*

(ii) *If $T(0)$ and $R(Q_TT)$ are topologically complemented in Y and $Y/T(0)$ respectively, then $R(T)$ is topologically complemented in Y.*

(iii) *Let N be a closed subspace of Y such that $T(0) \subset N$. Then $T^{-1}N$ is a closed subspace of X.*

(iv) *Assume that $R(T)$ is closed and let M be a subspace of X for which $N(T) \oplus M$ is closed. Then TM is a closed subspace of Y.*

(v) *Let M be a closed subspace of X such that $T|_M$ is injective and TM is topologically complemented in Y. If N is a topological complement of TM, then $T^{-1}N$ is a topological complement of M.*

Proof. Note that by virtue of [4, Proposition II.5.3], $T(0)$ is closed and Q_TT is a bounded and closed operator, so that from Lemma 3.2 (i), we obtain that $R(Q_TT)$ is closed if and only if $R(T)$ is closed.

(i) Applying [4, Lemma I.6.8], we get

$$Q_TY = R(Q_TT) + Q_TN \text{ and } \{0\} = R(Q_TT) \cap Q_TN.$$

Now, by Lemma 3.2 (i) it is enough to show that $N + T(0)$ is a closed subspace of Y. To do this, let $(z_n) \subset N + T(0)$ such that $z_n \to z$ for some $z \in Y$. Then there are $(a_n) \subset N$ and $(b_n) \subset T(0)$ such that $z_n = a_n + b_n = (I - P_{R(T)})z_n + P_{R(T)}z_n \to (I - P_{R(T)})z + P_{R(T)}z$ where $P_{R(T)}$ denotes the bounded operator projection of Y onto $R(T)$ along N. So that, as $T(0)$ is a closed subspace of $R(T)$, we have that $z \in N + T(0)$, as desired.

(ii) It is an immediate consequence of Lemma 3.2 (iv).

(iii) We first claim that

(3.1) $T^{-1}N = \{x \in X : Q_TTx \in Q_TN\}$.

Let $x \in T^{-1}N$. Then there is $y \in N \cap Tx$ which implies by [4, Proposition I.2.8] that $Tx = y + T(0)$, so that $Q_TTx = Q_Ty$ with $y \in N$. Hence $T^{-1}N \subset \{x \in X : Q_TTx \in Q_TN\}$. Conversely, let $x \in X$ such that $Q_TTx = Q_Ty$ for some $y \in N$. Using [4, Propositions I.2.8 and I.3.1] we infer that $y \in N \cap Tx$ and hence $x \in T^{-1}N$. Therefore (3.1) holds.

On the other hand, the set $\{x \in X : Q_TTx \in Q_TN\}$ is closed because Q_TN is closed by (i) and Q_TT is a bounded operator. This fact implies by the use of (3.1) that $T^{-1}N$ is closed.

(iv) If T is an operator then the assertion follows from [7, Lemma IV.2.9]. Turning to the general case, we have that Q_TT is an everywhere defined closed operator with closed range and $N(Q_TT) = T^{-1}Q_T^{-1}(0) = T^{-1}T(0) = T^{-1}(0)$ ([4, Corollary I.2.10]) $= N(T)$. Hence, from what has been showed for the operator case, Q_TM is a closed subspace of $Y/T(0)$ and thus TM is a closed subspace of Y by Lemma 3.2 (i).

(v) Let us consider two cases for T:

Case I: T operator. If N is a topological complement of TM and $T\mid_M$ is injective, then it is clear that $T^{-1}N \cap M = \{0\}$, both $T^{-1}N$ and M are closed subspaces and $X = T^{-1}N + M$.

Case II: T linear relation. Then one has from (i) that $Q_T TM \oplus Q_T N = Q_T Y$ and since $N(Q_T T) = N(T)$, we deduce from the case I applied to $Q_T T$ that $(Q_T T)^{-1} Q_T N$ is a topological complement of M. But $(Q_T T)^{-1} Q_T N = T^{-1} Q_T^{-1} Q_T N = T^{-1}(N + T(0)) = T^{-1}N$ ([4, Corollary I.2.10 and Proposition I.3.1]). Therefore (v) holds. □

The behaviour of the notion of relatively regular in products is given in the following two results.

Proposition 3.4. *Let X, Y and Z be Banach spaces and let $T \in RR(X, Y)$ and $S \in RR(Y, Z)$ such that $N(S) \cap T(0) = \{0\}$, $N(S) \subset R(T)$ and $ST(0)$ is topologically complemented in Z. Then $ST \in RR(X, Z)$.*

Proof. Let us consider two possibilities for S and T:

Case I: S, T operators. By a very known result (see, for instance [3, p.10]), there are bounded operators S_1 from Z to Y and T_1 from Y to X such that $SS_1S = S$, $TT_1T = T$, $I - S_1S$ is the bounded operator projection of Y onto $N(S)$ and TT_1 is the bounded operator projection of Y onto $R(T)$. Furthermore, $TT_1(I - S_1S) = I - S_1S$ because $N(S) \subset R(T)$ and hence

$$ST(T_1S_1)ST = STT_1T - STT_1(I - S_1S)T = ST.$$

This together with [3, p.10] yields to $ST \in RR(X, Z)$.

Case II: S, T linear relations. From [4, Lemma V.2.9] we have that

$$Q_{ST}ST = UV \text{ where } U := Q_{ST}SQ_T^{-1} \text{ and } V := Q_T T.$$

Then

(3.2) *U is a bounded operator from $Y/T(0)$ to $Z/ST(0)$.*

Indeed, as $Q_{ST}SQ_T^{-1}(0) = Q_{ST}ST(0) = \{0\}$ is $U(0) = \{0\}$ equivalently U is an operator. Moreover, from the equality $Q_{ST}S = Q_{ST(0)/S(0)}Q_S S$ (Lemma 3.2 (ii)) it is obvious that $Q_{ST}S$ is a bounded operator and thus applying [4, Corollary II.3.13], we infer that U is a bounded operator and hence it also is closed. Therefore (3.2) holds.

(3.3) *U is relatively regular*.

Since $ST(0)$ is topologically complemented in Z and $R(S)$ is closed, there exists a closed subspace Z_1 of Z such that $Z_1 \subset R(S)$ and $R(S) = ST(0) \oplus Z_1$. This equality combined with [4, Proposition I.3.1 and Lemma I.6.8] implies that

$$Y = (N(S) + T(0)) + S^{-1}Z_1 \text{ and } N(S) = (N(S) + T(0)) \cap S^{-1}Z_1.$$

But, since $S\mid_{T(0)}$ is injective (as $N(S) \cap T(0) = \{0\}$) and $ST(0)$ is topologically complemented in Z by hypothesis, it follows by virtue of Lemma 3.3 (v) that $S^{-1}Z_1$ is closed. On the other hand, one finds by [4, Proposition I.3.1] that $N(U) = (N(S) + T(0))/T(0)$, so that $N(S) + T(0)$ is closed by Lemma 3.2 (i) and the property (3.2).

After that, using Lemma 3.1 (ii), we obtain that $N(S) + T(0)$ is topologically complemented in Y and since $N(S)$ is contained in the closed subspace $R(T)$, we have that

$(N(S) + T(0)) \oplus Y_1 = R(T)$ for some closed subspace Y_1 with $Y_1 \subset R(T)$.

This last property together with the fact that $Q_T\mid_{Y_1}$ is an injective operator shows that $R(Q_T T) = Q_T(N(S) + T(0)) \oplus Q_T Y_1 = N(U) \oplus Q_T Y_1$, that is, $Q_T Y_1$ is a topological complement of $N(U)$ in $R(Q_T T)$. So that $N(U)$ is topologically complemented in $Y/T(0)$ by the part (ii) in Lemma 3.3.

On the other hand, it is clear that $R(U) = R(S)/ST(0)$ and since $R(S)$ is topologically complemented in Z, reasoning as in Lemma 3.3 (i), we deduce that $R(U)$ is topologically complemented in $Z/ST(0)$. Therefore (3.3) holds.

Now, from (3.2) and (3.3) combined with the case I applied to U and V, we infer that $Q_{ST}ST$ is relatively regular. In this situation, the use of Lemma 3.3 (ii) together the fact that $N(Q_{ST}ST) = N(ST)$ makes us to conclude that $ST \in RR(X, Z)$, as desired. □

Proposition 3.5. *Let X be a Banach space and let S, T be closed and everywhere defined linear relations in X such that $ST = TS$, $N(ST) = N(S) \oplus N(T)$, $R(ST) = R(S) \cap R(T)$, $N(T) \subset R(S)$ and $N(S) \subset R(T)$. Then*

(i) *$R(ST)$ is closed if and only if $R(S)$ and $R(T)$ are closed.*
(ii) *If ST is relatively regular then S and T are relatively regular linear relations.*

Proof. (i) Suppose that $R(ST)$ is closed. Since $S(0) \subset ST(0) \subset R(ST)$ it follows immediately from the part (iii) in Lemma 3.2 together with [4, Proposition I.3.1] and the fact that $N(S) \subset R(T)$, that $R(T)$ is closed. As $ST = TS$, analogously we obtain that $R(S)$ is closed. The converse is obvious.

(ii) Since $N(ST) = N(S) \oplus N(T)$ is topologically complemented in X, also $N(S)$ and $N(T)$ are topologically complemented in X and by virtue of the inclusion $N(S) \subset R(T)$, it follows that $N(S)$ is topologically complemented in $R(T)$. Hence $R(T) = N(S) \oplus M$ for some closed subspace M of X with $M \subset R(T)$. This implies that $S|_M$ is injective with $SM = R(ST)$ which is topologically complemented in X. So that, according to Lemma 3.3 (v) we have that $X = S^{-1}L \oplus M$ where L is a topological complement of SM. Therefore, if M_1 is a topological complement of $N(S)$ in $S^{-1}M_1$ we have that $X = M \oplus N(S) \oplus M_1 = R(T) \oplus M_1$ which shows that T is relatively regular in X. Similarly we obtain that $S \in RR(X)$. □

4. Polynomial in a Saphar Relation

Throughout this section we are concerned with the study of the behaviour of a polynomial in a Saphar relation in a Banach space. The analysis is essentially based on the results developed in the previous sections.

In the sequel X will be a complex Banach space and T will always denote an everywhere defined closed linear relation in X having a nonempty resolvent set. We note that by [6, Lemma 3.1] and [4, Corollary III.5.4 and Theorem VI.5.4] we have that

(5.1) *For every $n \in \mathbb{N}$, T^n is closed, bounded and $\rho(T^n) \neq \emptyset$.*

The following result relates the regularity of T to that its powers.

Proposition 4.1. *The following properties are equivalent:*

(i) *T is regular.*
(ii) *T^n is regular for all $n \in \mathbb{N}$.*
(iii) *T^m is regular for some $m \in \mathbb{N}$.*

Proof. (i)⇒(ii) Combine (5.1) and [1, Propositions 11 and 12].

(ii) ⇒ (iii) It is obvious.

(iii) ⇒(i) Assume that T^m is regular for some positive integer m. By Lemma 2.1 (i), $N(T) \subset N(T^m) \subset R(T^n)$ for all $n \in \mathbb{N}$, so that it only remains to verify that $R(T)$ is closed.

Let $\beta \in \rho(T)$. Then $T - \beta$ is closed, bounded and bijective and thus by [4, Proposition VI.5.2], $(T - \beta)^{m-1}$ has the same properties. Define

$$W := (T - \beta)^{-(m-1)} T^{m-1}.$$

Then

(5.2) *W is a bounded operator*.

Note that $W(0) = (T - \beta)^{-(m-1)} T^{m-1}(0) = (T - \beta)^{-(m-1)}(T - \beta)^{m-1}(0)$ (Lemma 2.1 (ii)) $= (T - \beta)^{-(m-1)}(0)$ ([4, Corollary I.2.10])= $\{0\}$, so that W is an operator. Further, it is clear that $D(W) = X$ and the continuity of W follows from [4, Corollary II.13]. Therefore (5.2) holds.

Let $(y_n) \subset R(T)$ such that $y_n \to y$ for some $y \in X$. Then (5.2) leads to $Wy_n \to Wy$ and since $Wy_n \subset R((T - \beta)^{-(m-1)} T^m) \subset R(T^m (T - \beta)^{-(m-1)})$ (Lemma 2.1 (ii))$\subset R(T^m)$ we obtain that $Wy \in R(T^m)$. Let $z \in X$ for which $Wy \in T^m z$. Then, the use of Lemma 2.1 (ii) and [4, Proposition I.4.2 (e)] yields to

$$0 \in T^{m-1}(-(T - \beta)^{-(m-1)} y + Tz)$$

which implies that $(T - \beta)^{-(m-1)} y \in R(T)$ and since $(T - \beta)^{m-1}$ is surjective we deduce that

$y \in (T-\beta)^{m-1}(T-\beta)^{-(m-1)}y \subset (T-\beta)^{m-1}R(T) = R(T(T-\beta)^{m-1}) \subset R(T)$.

The proof is completed. □

We are now in a position to prove the first main result of the present paper.

Theorem 4.2. *Let $P(T) = \alpha \sqcap_{i=1}^{n} (T-\lambda_i)^{m_i}$ as in Definition 2.2. Then $P(T)$ is regular if and only if $T-\lambda_i$ is regular, $1 \leq i \leq n$.*

Proof. Suppose that $P(T)$ is regular. Then, using Lemma 2.4, for all $m \in \mathbb{N}$

$$\oplus_{i=1}^{n} N(T-\lambda_i)^{m_i} \subset \cap_{i=1}^{n} R(T-\lambda_i)^{m_i m}$$

so, for $i \in \{1, 2, ..., n\}$ we have that

$$N(T-\lambda_i) \subset N(T-\lambda_i)^{m_i} \subset R(T-\lambda_i)^{m_i m} \subset R(T-\lambda_i)^{m}.$$

This fact together Proposition 4.1 ensures that in order to prove the regularity of $T-\lambda_i$ is enough to show that $R(T-\lambda_i)^{m_i}$ is closed. For this end, let $\beta \in \rho(T)$ and we write $W_1 := (T-\beta)^{-(r-m_i)} \sqcap_{j=1, j\neq i}^{n} (T-\lambda_j)^{m_j}$ where $r := m_1 + m_2 + ... + m_n$. Then proceeding as in the proof of Proposition 4.1 we obtain that W_1 is a bounded operator wich allowed us to conclude that $R(T-\lambda_i)^{m_i}$ is closed, as required.

The other implication was established in [1, Theorem 21]. □

Our next objective is to obtain an analogous result to Saphar relations.

Proposition 4.3. *Assume that $T^n(0)$ is topologically complemented in X for every $n \in \mathbb{N}$. Then the following properties are equivalent.*

(i) *T is Saphar.*
(ii) *T^n is Saphar for all $n \in \mathbb{N}$.*
(iii) *T^m is Saphar for some $m \in \mathbb{N}$.*

Proof. (i) ⇒(ii) By Proposition 4.1, T^n is regular. We shall prove that T^n is relatively regular proceeding by induction. The case $n = 1$ is evident. Assume that T^k is relatively regular for some positive integer k. Then, as $N(T) \cap T^k(0) = \{0\}$ (Lemma 2.1 (iv)), $N(T) \subset R(T^k)$(as T is regular), it follows from Proposition 3.4 that T^{k+1} is relatively regular, as desired.

(ii) ⇒(iii) It is trivial.

(iii) ⇒ (i) Suppose that there is $m \in \mathbb{N}$ for which T^m is Saphar. So that T is regular by Proposition 4.1; in particular $R(T)$ is closed and since $N(T^m)$ is topologically complemented in X and it is contained in $R(T)$, we have that

$$N(T^m) \oplus M = R(T) \text{ for some closed subspace } M \subset R(T).$$

This fact together with [4, Proposition I.3.1 and Lemma I.6.8] leads to

$$R(T^m) = T^{m-1}N(T^m) + T^{m-1}M = (N(T) + T^m(0)) + T^{m-1}M \text{ and } T^{m-1}(0) = (N(T) + T^m(0)) \cap T^{m-1}M.$$

Furthermore, since T^m is closed with closed range and $N(T^m) \oplus M$ is closed we deduce from Lemma 3.3 (iv) that $T^m M$ is closed. This last property combined with the part (iii) in Lemma 3.3 yields to $T^{m-1}M$ closed. Again applying Lemma 3.3 (iii) we obtain that $N(T) + T^m(0)$ is closed.

After that, using Lemmas 2.1 (v) and 3.1 we have that

$$N(T) \oplus T^m(0) \text{ and } N(T) \text{ are topologically complemented in } X.$$

On the other hand, as $T^m M = R(T^{m+1})$ which is topologically complemented by the implication (i) ⇒ (ii) and $T^m |_M$ is injective, one has from Lemma 3.3 that $T^{-m}L \oplus M = X$ where L is a topological complement of $T^m M$. Hence, if N is a topological complement of $N(T^m)$ in $T^{-m}L$ we have that $X = M \oplus N(T^m) \oplus N = R(T) \oplus N$ which shows that $R(T)$ is topologically complemented in X. The proof is completed. □

Now we are ready to state our second main result of this paper.

Theorem 4.4. *Assume that for each $n \in \mathbb{N}$, $T^n(0)$ is topologically complemented in X and let $P(T) = \alpha \sqcap_{i=1}^{n} (T - \lambda_i)^{m_i}$ as in Definition 2.2. Then $P(T)$ is a Saphar relation if and only if $T - \lambda_i$ is a Saphar relation, $1 \leq i \leq n$.*

Proof. Suppose that $P(T)$ is a Saphar relation in X. Then $T - \lambda_i$ is regular, $1 \leq i \leq n$ by Theorem 4.2. Applying Lemma 2.4 and Proposition 3.5 we infer that each $(T - \lambda_i)^{m_i}$ is relatively regular and thus the use of Proposition 4.3 gives $T - \lambda_i$ relatively regular.

Assume now that $T - \lambda_i$ is Saphar for every $i \in \{1, 2, ..., n\}$. Again applying Theorem 4.2 we obtain that $P(T)$ is regular. Finally, as $(T - \lambda_i)^{m_i}$ is Saphar by virtue of Proposition 4.3, we deduce as an immediate consequence of Lemma 2.4 and Proposition 3.4 that $P(T)$ is a Saphar relation in X. The proof is completed. □

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