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Odd-Dimensional Weyl and Pseudo-Weyl spaces with Additional Tensor Structures

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Abstract. Odd-dimensional Weyl and pseudo-Weyl spaces admitting almost contact, almost paracontact and nilpotent structures are considered in this work. The results are obtained by means of the apparatus of the prolonged covariant differentiation. A linear connection with torsion is constructed. With respect to this connection the prolonged covariant derivatives of the fundamental tensors of the Weyl and pseudo-Weyl spaces are found to be zero. The curvature tensor with respect to this connection is considered.

1. Introduction

Riemannian spaces with almost contact and almost paracontact structures have been studied in [1, 3, 5, 6, 13–15]. In [11, 12, 16, 17] Weyl spaces are studied, and in [19] nilpotent structures have been considered.

In this paper, we study odd-dimensional Weyl and pseudo-Weyl spaces endowed with various structures: almost contact, almost paracontact and nilpotent. In our investigations we use the apparatus of the prolonged covariant differentiation which is defined in [4] and developed in [18, 21, 22]. The affinors of the considered structures are defined by means of $2n+1$ independent directional fields $\underset{\sigma}{v}^{\alpha}$ $(\sigma, \alpha = 1, 2, ..., 2n+1)$ and their reciprocal covectors $\overset{\sigma}{v}_{\alpha}$ [2, 23, 24]. We pay special attention to the spaces with parallel structures with respect to the Levi-Civita connection of the metric, i.e. the so called Kähler-like classes. For such spaces we obtain a decomposition of three mutually orthogonal subspaces and also their line elements (fundamental forms).

In the last section, we introduce a linear non-symmetric connection. With respect to this connection the prolonged covariant derivatives of the fundamental tensors of the Weyl and pseudo-Weyl spaces are proved to be zero. We study the curvature tensor corresponding to the introduced connection.

2. Preliminaries

A set of quantities that differ from each other by a non-zero factor is called a *pseudo-quantity*. A particular quantity of this set is called a *representative* of the pseudo-quantity. The choice of a representative from a pseudo-quantity is called *normalization*.

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Let A_n be an n-dimensional space with an affine connection and the pseudo-quantity $A \in A_n$.
The following definitions are given in [7, 10, 20]:

Definition 2.1. *By a pseudo-quantity with weight k it is meant a set of objects A admitting a transformation (renormalization) of the form*

$$\check{A} = \lambda^k A, \tag{1}$$

where $\lambda = \lambda(\overset{1}{u}, \overset{2}{u}, ..., \overset{n}{u})$ is a non-zero function of the point, and $k \in \mathbb{R}$. We denote a pseudo-quantity with weight k by $A\{k\}$.

Definition 2.2. *A normalizer is defined as a covector (1-form) T_σ which is transformed by the rule*

$$\check{T}_\sigma = T_\sigma + \partial_\sigma \ln \lambda, \qquad \partial_\sigma \ln \lambda = \frac{\partial \ln \lambda}{\partial u^\sigma}. \tag{2}$$

In [4], V. Hlavatý introduced the notion of prolonged derivative of a pseudo-quantity $A\{k\}$ by

$$\partial^\bullet_\sigma A = \partial_\sigma A - k T_\sigma A. \tag{3}$$

Because of (1) and (3) we have $\partial^\bullet_\sigma \check{A} = \lambda^k \partial^\bullet_\sigma A$ from which it follows that the prolonged differentiation preservers the weight of a pseudo-quantity. It is known that if $A\{p\}$ and $B\{q\}$, then $AB\{p+q\}$ and $\partial^\bullet_\sigma(AB) = (\partial^\bullet_\sigma A)B + A(\partial^\bullet_\sigma B)$.

Definition 2.3. *The prolonged covariant derivative of a pseudo-quantity $A\{k\}$ is called the object [7, 10, 20]*

$$\overset{\bullet}{\nabla}_\sigma A = \nabla_\sigma A - k T_\sigma A, \tag{4}$$

where $\nabla_\sigma A$ is the usual covariant derivative of A.

Let $M_{2n+1}(g_{\alpha\beta}, T_\sigma)$ be a $(2n+1)$-dimensional smooth manifold with a Weyl connection ∇, a symmetric pseudo-tensor $g_{\alpha\beta}$ and an additional covector T_σ. The space $M_{2n+1}(g_{\alpha\beta}, T_\sigma)$ will be denoted by W_{2n+1} and will be called a *Weyl space*. The coefficients $\Gamma^\sigma_{\alpha\beta}$ of the Weyl connection are given by [7](p. 154)

$$\Gamma^\sigma_{\alpha\beta} = \{^{\sigma}_{\alpha\beta}\} - \left(T_\alpha \delta^\sigma_\beta + T_\beta \delta^\sigma_\alpha - T_\nu g^{\nu\sigma} g_{\alpha\beta}\right), \tag{5}$$

where $\{^{\sigma}_{\alpha\beta}\}$ are the Christoffel symbols of the tensor $g_{\alpha\beta}$.
According to [7](p. 152), the fundamental tensor $g_{\alpha\beta}$ admits a transformation of the form

$$\check{g}_{\alpha\beta} = \lambda^2 g_{\alpha\beta}, \tag{6}$$

where $\lambda = \lambda(\overset{1}{u}, \overset{2}{u}, ..., \overset{2n+1}{u})$, $\lambda \neq 0$, is an arbitrary smooth function of the point.
By the renormalization (6) of $g_{\alpha\beta}$, the additional covector T_σ is transformed by formula (2) ([7], p. 152). According to [7](p. 152), the following hold

$$\nabla_\sigma g_{\alpha\beta} = 2\, T_\sigma g_{\alpha\beta}, \qquad \nabla_\sigma g^{\alpha\beta} = -2\, T_\sigma g^{\alpha\beta}, \tag{7}$$

where $g_{\alpha\beta} g^{\alpha\nu} = \delta^\nu_\beta$.
From (6) it follows that $g_{\alpha\beta}\{2\}$ and $g^{\alpha\beta}\{-2\}$. Then, according to (4) and (7), we obtain

$$\overset{\bullet}{\nabla}_\sigma g_{\alpha\beta} = 0, \qquad \overset{\bullet}{\nabla}_\sigma g^{\alpha\beta} = 0. \tag{8}$$

Since the identity affinor has zero weight, i.e. $\delta^\beta_\alpha\{0\}$, then $\overset{\bullet}{\nabla}_\sigma \delta^\beta_\alpha = 0$.

Let us introduce the notations

$$\alpha,\beta,\gamma,\sigma,\tau,\nu,\delta = 1,2,...,2n+1; \quad p,q,r,t = 1,2,...,2n;$$
$$j,s,k,l,m = 1,2,...,n; \quad \bar{j},\bar{s},\bar{k},\bar{l},\bar{m} = n+1,n+2,...,2n. \tag{9}$$

Let $\underset{\alpha}{v}^{\beta}(\alpha = 1,2,...,2n+1)$ be $2n+1$ independent directional fields over M_{2n+1}. The pseudo-vectors $\underset{\alpha}{v}^{\beta}$ are renormalized by the condition

$$g_{\alpha\beta}\underset{\sigma}{v}^{\alpha}\underset{\sigma}{v}^{\beta} = 1. \tag{10}$$

From (10) it follows that $\underset{\alpha}{v}^{\beta}\{-1\}$. According to [7](p. 153) and (10), we have

$$g_{\alpha\beta}\underset{\sigma}{v}^{\alpha}\underset{\nu}{v}^{\beta} = \cos\underset{\sigma\nu}{\omega}, \tag{11}$$

where $\underset{\sigma\nu}{\omega}\{0\}$ is the angle between the pseudo-vectors $\underset{\sigma}{v}^{\alpha}$ and $\underset{\nu}{v}^{\beta}$.

Let the following conditions hold

$$g_{\alpha\beta}\underset{k}{v}^{\alpha}\underset{\bar{s}}{v}^{\beta} = 0, \qquad g_{\alpha\beta}\underset{p}{v}^{\alpha}\underset{2n+1}{v}^{\beta} = 0. \tag{12}$$

The net defined by the pseudo-vectors $\underset{\alpha}{v}^{\beta}$ is denoted by $\{\underset{\alpha}{v}\}$. The pseudo-covectors $\overset{\alpha}{v}_{\beta}$ are given by

$$\underset{\sigma}{v}^{\beta}\overset{\sigma}{v}_{\alpha} = \delta^{\beta}_{\alpha} \Longleftrightarrow \underset{\alpha}{v}^{\sigma}\overset{\beta}{v}_{\sigma} = \delta^{\beta}_{\alpha}, \tag{13}$$

from which it follows that $\overset{\alpha}{v}_{\beta}\{1\}$.

We choose $\{\underset{\alpha}{v}\}$ to be the coordinate net. Then, from (10), (11), (12) and (13) we have

$$\underset{1}{v}^{\beta}\left(\frac{1}{\sqrt{g_{11}}},0,0,...,0\right), \underset{2}{v}^{\beta}\left(0,\frac{1}{\sqrt{g_{22}}},0,...,0\right),..., \underset{2n+1}{v}^{\beta}\left(0,0,...,0,\frac{1}{\sqrt{g_{2n+1\ 2n+1}}}\right);$$
$$\overset{1}{v}_{\beta}\left(\sqrt{g_{11}},0,0,...,0\right), \overset{2}{v}_{\beta}\left(0,\sqrt{g_{22}},0,...,0\right),..., \overset{2n+1}{v}_{\beta}\left(0,0,...,0,\sqrt{g_{2n+1\ 2n+1}}\right). \tag{14}$$

In the parameters of the coordinate net $\{\underset{\alpha}{v}\}$ the matrix of the fundamental tensor $g_{\alpha\beta}$ has the following block diagonal form

$$\left\|g_{\alpha\beta}\right\| = \left\| \begin{matrix} g_{ks} & 0 & 0 \\ 0 & g_{\bar{k}\bar{s}} & 0 \\ 0 & 0 & g_{2n+1\ 2n+1} \end{matrix} \right\|, \qquad \det g_{\alpha\beta} \neq 0, \quad g_{\alpha\alpha} > 0. \tag{15}$$

Let us consider the pseudo-tensor $\widetilde{g}_{\alpha\beta}$ whose matrix has the following form in the parameters of the coordinate net $\{\underset{\alpha}{v}\}$:

$$\left\|\widetilde{g}_{\alpha\beta}\right\| = \left\| \begin{matrix} g_{ks} & 0 & 0 \\ 0 & -g_{\bar{k}\bar{s}} & 0 \\ 0 & 0 & g_{2n+1\ 2n+1} \end{matrix} \right\|. \tag{16}$$

By (8), (15) and (16) we get

$$\nabla_{\sigma}\widetilde{g}_{\alpha\beta} = 2T_{\sigma}\widetilde{g}_{\alpha\beta}, \qquad \nabla_{\sigma}\widetilde{g}^{\alpha\beta} = -2T_{\sigma}\widetilde{g}^{\alpha\beta}, \tag{17}$$

where $\widetilde{g}_{\alpha\beta}\widetilde{g}^{\alpha\sigma} = \delta^{\sigma}_{\beta}$. According to (16) and $\widetilde{g}_{\alpha\beta}\widetilde{g}^{\alpha\sigma} = \delta^{\sigma}_{\beta}$, it follows that $\widetilde{g}_{\alpha\beta}\{2\}$ and $\widetilde{g}^{\alpha\beta}\{-2\}$. Then, by (4) and (17) we obtain

$$\overset{\bullet}{\nabla}_{\sigma}\widetilde{g}_{\alpha\beta} = 0, \qquad \overset{\bullet}{\nabla}_{\sigma}\widetilde{g}^{\alpha\beta} = 0. \tag{18}$$

The space $W_{2n+1}(\widetilde{g}_{\alpha\beta}, T_\sigma)$ will be denoted by $\widetilde{W}_{2n+1}$ and will be called a *pseudo-Weyl space* with fundamental tensor $\widetilde{g}_{\alpha\beta}$ and additional covector T_σ. The coefficients of the connection of $\widetilde{W}_{2n+1}$ coincide with the coefficients of the connection of the space W_{2n+1}.

From (10) and (16) it follows

$$\widetilde{g}_{\alpha\beta}\underset{k}{v}^{\alpha}\underset{k}{v}^{\beta} = 1, \qquad \widetilde{g}_{\alpha\beta}\underset{\bar{k}}{v}^{\alpha}\underset{\bar{k}}{v}^{\beta} = -1, \qquad \widetilde{g}_{\alpha\beta}\underset{2n+1}{v}^{\alpha}\underset{2n+1}{v}^{\beta} = 1. \tag{19}$$

In the parameters of the coordinate net $\{\underset{\alpha}{v}\}$ it it easy to prove that $g_{\alpha\beta}\underset{2n+1}{v}^{\alpha} = \overset{2n+1}{v}_{\beta}$ and $\widetilde{g}_{\alpha\beta}\underset{2n+1}{v}^{\alpha} = \overset{2n+1}{v}_{\beta}$.

The direction fields $\underset{\alpha}{v}^{\beta}$ satisfy the following derivative equations [22]:

$$\overset{\bullet}{\nabla}_\sigma \underset{\alpha}{v}^{\beta} = \underset{\alpha}{\overset{\nu}{T}}_\sigma \underset{\nu}{v}^{\beta}, \qquad \overset{\bullet}{\nabla}_\sigma \overset{\alpha}{v}_{\beta} = -\underset{\nu}{\overset{\alpha}{T}}_\sigma \overset{\nu}{v}_{\beta}, \tag{20}$$

where $\underset{\alpha}{\overset{\nu}{T}}_\sigma\{0\}$.

Lemma 2.4. *In the parameters of the coordinate net $\{\underset{\alpha}{v}\}$ the coefficients of the derivative equations (20) have the form:*

$$\begin{aligned} &\underset{\alpha}{\overset{\gamma}{T}}_\sigma = \frac{\sqrt{g_{\gamma\gamma}}}{\sqrt{g_{\alpha\alpha}}}\Gamma^{\gamma}_{\sigma\alpha}, \quad \gamma \neq \alpha, \\ &\underset{\alpha}{\overset{\alpha}{T}}_\sigma = \Gamma^{\alpha}_{\sigma\alpha} - \frac{1}{2}\frac{\partial_\sigma g_{\alpha\alpha}}{g_{\alpha\alpha}} + T_\sigma \quad \textit{(no summation over } \alpha). \end{aligned} \tag{21}$$

Proof. *According to (4) and the first of the equalities (20), we have* $\overset{\bullet}{\nabla}_\sigma \underset{\alpha}{v}^{\beta} = \nabla_\sigma \underset{\alpha}{v}^{\beta} + T_\sigma \underset{\alpha}{v}^{\beta} = \underset{\alpha}{\overset{\nu}{T}}_\sigma \underset{\nu}{v}^{\beta}$, *from which we get*

$$\underset{\alpha}{\overset{\nu}{T}}_\sigma \underset{\nu}{v}^{\beta} = \partial_\sigma \underset{\alpha}{v}^{\beta} + \Gamma^{\beta}_{\sigma\nu}\underset{\alpha}{v}^{\nu} + T_\sigma \underset{\alpha}{v}^{\beta}. \tag{22}$$

Having in mind (13), after contracting (22) with $\overset{\gamma}{v}_{\beta}$, *we obtain*

$$\underset{\alpha}{\overset{\gamma}{T}}_\sigma = \partial_\sigma \underset{\alpha}{v}^{\beta}\,\overset{\gamma}{v}_{\beta} + \Gamma^{\beta}_{\sigma\nu}\,\underset{\alpha}{v}^{\nu}\,\overset{\gamma}{v}^{\beta} + T_\sigma \delta^{\gamma}_{\alpha}. \tag{23}$$

By (23) we get

$$\underset{\alpha}{\overset{\gamma}{T}}_\sigma = \partial_\sigma \underset{\alpha}{v}^{\beta}\,\overset{\gamma}{v}_{\beta} + \Gamma^{\beta}_{\sigma\nu}\,\underset{\alpha}{v}^{\nu}\,\overset{\gamma}{v}_{\beta}, \quad \gamma \neq \alpha, \qquad \underset{\alpha}{\overset{\alpha}{T}}_\sigma = \partial_\sigma \underset{\alpha}{v}^{\beta}\,\overset{\alpha}{v}_{\beta} + \Gamma^{\beta}_{\sigma\nu}\,\underset{\alpha}{v}^{\nu}\,\overset{\alpha}{v}_{\beta} + T_\sigma \quad \textit{(no summation over } \alpha). \tag{24}$$

We choose $\{\underset{\alpha}{v}\}$ for the coordinate net. Then, according to (14), equalities (24) take the form (21). □

Let us consider the affinor field a^{β}_{α} defined by [2, 23, 24]

$$a^{\beta}_{\alpha} = \underset{p}{v}^{\beta}\,\overset{p}{v}_{\alpha} - \underset{2n+1}{v}^{\beta}\,\overset{2n+1}{v}_{\alpha}. \tag{25}$$

By (13) and (25) it follows that $a^{\beta}_{\alpha}a^{\sigma}_{\beta} = \delta^{\sigma}_{\alpha}$. Hence, the affinor a^{β}_{α} defines a composition $X_{2n} \times X_1$ of the basic manifolds X_{2n} and X_1 [8]. The positions (tangent planes) of the basic manifolds X_{2n} and X_1 are denoted by $P(X_{2n})$ and $P(X_1)$, respectively [8]. According to [8, 9], the affinors

$$\overset{1}{a}{}^{\beta}_{\alpha} = \frac{1}{2}(\delta^{\beta}_{\alpha} + a^{\beta}_{\alpha}) = \underset{p}{v}^{\beta}\,\overset{p}{v}_{\alpha}, \qquad \overset{2}{a}{}^{\beta}_{\alpha} = \frac{1}{2}(\delta^{\beta}_{\alpha} - a^{\beta}_{\alpha}) = \underset{2n+1}{v}^{\beta}\,\overset{2n+1}{v}_{\alpha}$$

are the projecting affinors of the composition $X_{2n}\times X_1$. If v^β is an arbitrary vector, we have $v^\beta = \overset{1}{a}{}^{\beta}_{\alpha}\, v^\alpha + \overset{2}{a}{}^{\beta}_{\alpha}\, v^\alpha = \overset{1}{V}{}^\beta + \overset{2}{V}{}^\beta$, where $\overset{1}{V}{}^\beta = \overset{1}{a}{}^{\beta}_{\alpha}\, v^\alpha \in P(X_{2n})$ and $\overset{2}{V}{}^\beta = \overset{2}{a}{}^{\beta}_{\alpha}\, v^\alpha \in P(X_1)$. Obviously, $\underset{p}{v}{}^\alpha \in P(X_{2n})$, and $\underset{2n+1}{v}{}^\alpha \in P(X_1)$. The affinors a^β_α, $\overset{1}{a}{}^{\beta}_{\alpha}$, $\overset{2}{a}{}^{\beta}_{\alpha}$ have zero weights.

Let $X_a \times X_b$ $(a + b = 2n + 1)$ be an arbitrary composition in the space W_{2n+1}, and $P(X_a)$ and $P(X_b)$ be the positions of the differentiable manifolds X_a and X_b, respectively. According to [9], the composition $X_a \times X_b$ is of the type (c, c), i.e. (*Cartesian*, *Cartesian*), if the positions $P(X_a)$ and $P(X_b)$ are translated parallelly along any line in the space M_{2n+1}.

3. Almost Contact and Almost Paracontact Structures in W_{2n+1} and $\widetilde{W}_{2n+1}$

Let us consider the following affinor fields

$$\underset{\varkappa}{b}{}^\beta_\alpha = \varkappa\left(\underset{k}{v}{}^\beta \overset{k}{v}_\alpha - \underset{\bar{k}}{v}{}^\beta \overset{\bar{k}}{v}_\alpha\right), \tag{26}$$

where $\varkappa = 1, i$ $(i^2 = -1)$. According to (13) and (26), we have $\underset{\varkappa}{b}{}^\beta_\alpha \underset{2n+1}{v}{}^\alpha = 0$ and $\underset{\varkappa}{b}{}^\beta_\alpha \overset{2n+1}{v}_\beta = 0$.

Let $\varkappa = 1$. From (13) and (26) we obtain $\underset{1}{b}{}^\beta_\alpha \underset{1}{b}{}^\sigma_\beta = \delta^\sigma_\alpha - \underset{2n+1}{v}{}^\sigma \overset{2n+1}{v}_\alpha$, i.e. the affinor $\underset{1}{b}{}^\beta_\alpha$ defines an almost paracontact structure on W_{2n+1}.

Let $\varkappa = i$. By (13) and (26) it follows that $\underset{i}{b}{}^\beta_\alpha \underset{i}{b}{}^\sigma_\beta = -\delta^\sigma_\alpha + \underset{2n+1}{v}{}^\sigma \overset{2n+1}{v}_\alpha$, i.e. $\underset{i}{b}{}^\beta_\alpha$ defines an almost contact structure on W_{2n+1}.

Theorem 3.1. *The affinor $\underset{\varkappa}{b}{}^\beta_\alpha$ is parallel with respect to the Weyl connection ∇, i.e.*

$$\nabla_\sigma \underset{\varkappa}{b}{}^\beta_\alpha = 0 \tag{27}$$

if and only if the coefficients of the derivative equations (20) satisfy the conditions:

$$\underset{k}{\overset{\bar{s}}{T}}{}_\sigma = \underset{\bar{k}}{\overset{s}{T}}{}_\sigma = \underset{2n+1}{\overset{p}{T}}{}_\sigma = \underset{p}{\overset{2n+1}{T}}{}_\sigma = 0. \tag{28}$$

Proof. *Because of (4) and $\underset{\varkappa}{b}{}^\beta_\alpha\{0\}$, the condition (27) is equivalent to*

$$\overset{\bullet}{\nabla}_\sigma \underset{\varkappa}{b}{}^\beta_\alpha = 0. \tag{29}$$

According to (20) and (26), equality (29) has the form

$$\underset{k}{\overset{\nu}{T}}{}_\sigma \underset{\nu}{v}{}^\beta \overset{k}{v}_\alpha - \underset{\nu}{\overset{k}{T}}{}_\sigma \underset{k}{v}{}^\beta \overset{\nu}{v}_\alpha - \underset{\bar{k}}{\overset{\nu}{T}}{}_\sigma \underset{\nu}{v}{}^\beta \overset{\bar{k}}{v}_\alpha + \underset{\nu}{\overset{\bar{k}}{T}}{}_\sigma \underset{\bar{k}}{v}{}^\beta \overset{\nu}{v}_\alpha = 0.$$

If we write the sums over the index ν in a more detailed form and regroup the addends, the last equality has the form

$$\left(2\,\underset{k}{\overset{\bar{s}}{T}}{}_\sigma \underset{\bar{s}}{v}{}^\beta + \underset{k}{\overset{2n+1}{T}}{}_\sigma \underset{2n+1}{v}{}^\beta\right)\overset{k}{v}_\alpha - \left(2\,\underset{\bar{k}}{\overset{s}{T}}{}_\sigma \underset{s}{v}{}^\beta + \underset{\bar{k}}{\overset{2n+1}{T}}{}_\sigma \underset{2n+1}{v}{}^\beta\right)\overset{\bar{k}}{v}_\alpha + \left(\underset{2n+1}{\overset{\bar{s}}{T}}{}_\sigma \underset{\bar{s}}{v}{}^\beta - \underset{2n+1}{\overset{s}{T}}{}_\sigma \underset{s}{v}{}^\beta\right)\overset{2n+1}{v}_\alpha = 0. \tag{30}$$

The independence of the pseudo-covectors $\overset{\nu}{v}_\alpha$ yields that (30) holds true if and only if the following equalities are valid:

$$2\,\underset{k}{\overset{\bar{s}}{T}}{}_\sigma \underset{\bar{s}}{v}{}^\beta + \underset{k}{\overset{2n+1}{T}}{}_\sigma \underset{2n+1}{v}{}^\beta = 0, \qquad 2\,\underset{\bar{k}}{\overset{s}{T}}{}_\sigma \underset{s}{v}{}^\beta + \underset{\bar{k}}{\overset{2n+1}{T}}{}_\sigma \underset{2n+1}{v}{}^\beta = 0, \qquad \underset{2n+1}{\overset{\bar{s}}{T}}{}_\sigma \underset{\bar{s}}{v}{}^\beta - \underset{2n+1}{\overset{s}{T}}{}_\sigma \underset{s}{v}{}^\beta = 0. \tag{31}$$

Since the pseudo-vectors $\underset{\nu}{v}^{\beta}$ *are mutually independent, equalities (31) are valid if and only if* $\overset{\bar{s}}{\underset{k}{T}}{}_{\sigma} = \overset{s}{\underset{\bar{k}}{T}}{}_{\sigma} = 0$ *and* $\overset{s}{\underset{2n+1}{T}}{}_{\sigma} = \overset{\bar{s}}{\underset{2n+1}{T}}{}_{\sigma} = \overset{2n+1}{\underset{k}{T}}{}_{\sigma} = \overset{2n+1}{\underset{\bar{k}}{T}}{}_{\sigma} = 0$. *Because of (9) the latter conditions are equivalent to* $\overset{p}{\underset{2n+1}{T}}{}_{\sigma} = \overset{2n+1}{\underset{p}{T}}{}_{\sigma} = 0$. *Thus we proved that equalities (31), and hence (27), are valid if and only if conditions (28) hold.* □

Corollary 3.2. *If* $\nabla_{\sigma} \underset{\varkappa}{b}{}_{\alpha}^{\beta} = 0$, *in the parameters of the coordinate net* $\{\underset{\alpha}{v}\}$, *the coefficients* $\Gamma^{\nu}_{\alpha\beta}$ *of the Weyl connection satisfy:*

$$\Gamma^{\bar{k}}_{\sigma s} = \Gamma^{k}_{\sigma \bar{s}} = \Gamma^{p}_{\sigma 2n+1} = \Gamma^{2n+1}_{\sigma p} = 0. \tag{32}$$

Proof. *According to (21), equalities (28) imply (32).* □

Let the space M_{2n+1} be a topological product of three smooth manifolds X_a, X_b and X_c ($a+b+c = 2n+1$), i.e. let M_{2n+1} be the space of the composition $X_a \times X_b \times X_c$. We denote by $P(X_a)$, $P(X_b)$ and $P(X_c)$, respectively, the positions of the manifolds X_a, X_b and X_c.

Definition 3.3. *The composition* $X_a \times X_b \times X_c$ *is said to be of type* (c, c, c) *if the positions* $P(X_a)$, $P(X_b)$ *and* $P(X_c)$ *are translated parallelly along any line in* W_{2n+1}.

Definition 3.4. *The composition* $X_a \times X_b \times X_c$ *is said to be orthogonal if the positions* $P(X_a)$, $P(X_b)$ *and* $P(X_c)$ *are mutually orthogonal.*

Theorem 3.5. *If* $\overset{\bullet}{\nabla}_{\sigma} \underset{\varkappa}{b}{}_{\alpha}^{\beta} = 0$, *the space* W_{2n+1} *is a space of orthogonal compositions* $X_n \times \overline{X}_n \times X_1$ *of the type* (c, c, c).

Proof. *Let condition (29) hold true. Then, in the parameters of the net* $\{\underset{\alpha}{v}\}$ *the conditions (32) are valid. Let us consider the composition* $X_{2n} \times X_1$ *defined by the affinor (25). According to [9], by (32) it follows that the composition* $X_{2n} \times X_1$ *is of the type* (c, c). *Hence, the position* $P(X_1)$ *of the manifold* X_1 *(which is a curve) is translated parallelly along any line in* M_{2n+1}.

Let us consider the following affinors

$$c_{\alpha}^{\beta} = \underset{k}{v}^{\beta} \overset{k}{v}_{\alpha} - \underset{\bar{k}}{v}^{\beta} \overset{\bar{k}}{v}_{\alpha} - \underset{2n+1}{v}{}^{\beta} \overset{2n+1}{v}{}_{\alpha}, \qquad d_{\alpha}^{\beta} = \underset{k}{v}^{\beta} \overset{k}{v}_{\alpha} - \underset{\bar{k}}{v}^{\beta} \overset{\bar{k}}{v}_{\alpha} + \underset{2n+1}{v}{}^{\beta} \overset{2n+1}{v}{}_{\alpha}. \tag{33}$$

By (13) and (33) it follows that $c_{\alpha}^{\beta} c_{\beta}^{\sigma} = \delta_{\alpha}^{\sigma}$ *and* $d_{\alpha}^{\beta} d_{\beta}^{\sigma} = \delta_{\alpha}^{\sigma}$. *Hence, affinors* c_{α}^{β} *and* d_{α}^{β} *define the compositions* $X_n \times Y_{n+1}$ *and* $\overline{X}_n \times Z_{n+1}$, *respectively, where* Y_{n+1} *and* Z_{n+1} *are smooth* $(n+1)$*-dimensional manifolds. In the parameters of the coordinate net the affinors* a_{α}^{β}, c_{α}^{β} *and* d_{α}^{β} *have the form, respectively:*

$$(a_{\alpha}^{\beta}) = \begin{pmatrix} \delta_{p}^{q} & 0 \\ 0 & -1 \end{pmatrix}, \qquad (c_{\alpha}^{\beta}) = \begin{pmatrix} \delta_{k}^{s} & 0 & 0 \\ 0 & -\delta_{\bar{k}}^{\bar{s}} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad (d_{\alpha}^{\beta}) = \begin{pmatrix} \delta_{k}^{s} & 0 & 0 \\ 0 & -\delta_{\bar{k}}^{\bar{s}} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{34}$$

By (34) it follows that $Y_{n+1} = \overline{X}_n \times X_1$ *and* $Z_{n+1} = X_n \times X_1$, *therefore* W_{2n+1} *is a space of the composition* $X_n \times \overline{X}_n \times X_1$. *We denote by* $P(X_n)$ *and* $P(\overline{X}_n)$, *respectively, the positions of the manifolds* X_n *and* $\overline{X}_n$. *According to [9], equality (32) yields that the compositions* $X_n \times Y_{n+1}$ *and* $\overline{X}_n \times Z_{n+1}$ *are of the type* (c, c). *Hance, the positions* $P(X_n)$ *and* $P(\overline{X}_n)$ *are translated parallelly along any line in* W_{2n+1}, *i.e. we proved that the composition* $X_n \times \overline{X}_{\times} X_1$ *is of the type* (c, c, c). *The projecting affinors of the composition* $X_n \times \overline{X}_n \times X_1$ *are:*

$$\overset{1}{c}{}_{\alpha}^{\beta} = \underset{k}{v}^{\beta} \overset{k}{v}_{\alpha}, \qquad \overset{1}{d}{}_{\alpha}^{\beta} = \underset{\bar{k}}{v}^{\beta} \overset{\bar{k}}{v}_{\alpha}, \qquad \overset{2}{a}{}_{\alpha}^{\beta} = \underset{2n+1}{v}{}^{\beta} \overset{2n+1}{v}{}_{\alpha}.$$

If w^β is an arbitrary vector, we have

$$w^\beta = \overset{1}{c}{}^{\beta}_{\alpha}\, w^\alpha + \overset{1}{d}{}^{\beta}_{\alpha}\, w^\alpha + \overset{2}{a}{}^{\beta}_{\alpha}\, w^\alpha = \overset{1}{W}{}^\beta + \overset{2}{W}{}^\beta + \overset{3}{W}{}^\beta,$$

where $\overset{1}{W}{}^\beta = \overset{1}{c}{}^{\beta}_{\alpha}\, w^\alpha \in P(X_n)$, $\overset{2}{W}{}^\beta = \overset{1}{d}{}^{\beta}_{\alpha}\, w^\alpha \in P(\overline{X}_n)$ *and* $\overset{3}{W}{}^\beta = \overset{2}{a}{}^{\beta}_{\alpha}\, w^\alpha \in P(X_1)$. *Because* $\underset{k}{v}{}^\alpha \in P(X_n)$, $\underset{\bar{k}}{v}{}^\alpha \in P(\overline{X}_n)$ *and* $\underset{2n+1}{v}{}^\alpha \in P(X_1)$, *from (15) or (16) it follows that the positions* $P(X_n)$, $P(\overline{X}_n)$ *and* $P(X_1)$ *are mutually orthogonal.* □

We denote by $\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}\overset{\alpha}{u}\mathrm{d}\overset{\beta}{u}$ and $\mathrm{d}\widetilde{s}^2 = \widetilde{g}_{\alpha\beta}\mathrm{d}\overset{\alpha}{u}\mathrm{d}\overset{\beta}{u}$ the fundamental forms of the spaces W_{2n+1} and $\widetilde{W}_{2n+1}$, respectively.

Theorem 3.6. *If* $\overset{\bullet}{\nabla}_\sigma \underset{\varkappa}{b}{}^{\beta}_{\alpha} = 0$, *in the parameters of the coordinate net* $\{\underset{\alpha}{v}\}$, *the fundamental forms of the space* W_{2n+1} *and* $\widetilde{W}_{2n+1}$ *are given by*

$$\begin{aligned} \mathrm{d}s^2 &= \underset{1}{f}\overset{\circ}{g}_{ks}\mathrm{d}\overset{k}{u}\mathrm{d}\overset{s}{u} + \underset{2}{f}\overset{\circ}{g}_{\bar{k}\bar{s}}\mathrm{d}\overset{\bar{k}}{u}\mathrm{d}\overset{\bar{s}}{u} + \underset{3}{f}\overset{\circ}{g}_{2n+1\ 2n+1}\mathrm{d}(\overset{2n+1}{u})^2 \\ \mathrm{d}\widetilde{s}^2 &= \underset{1}{f}\overset{\circ}{g}_{ks}\mathrm{d}\overset{k}{u}\mathrm{d}\overset{s}{u} - \underset{2}{f}\overset{\circ}{g}_{\bar{k}\bar{s}}\mathrm{d}\overset{\bar{k}}{u}\mathrm{d}\overset{\bar{s}}{u} + \underset{3}{f}\overset{\circ}{g}_{2n+1\ 2n+1}\mathrm{d}(\overset{2n+1}{u})^2, \end{aligned} \tag{35}$$

where

$$T_s = \tfrac{1}{2}\partial_s \ln \underset{2}{f} = \tfrac{1}{2}\partial_s \ln \underset{3}{f}, \qquad T_{\bar{s}} = \tfrac{1}{2}\partial_{\bar{s}} \ln \underset{1}{f} = \tfrac{1}{2}\partial_{\bar{s}} \ln \underset{3}{f}, \qquad T_{2n+1} = \tfrac{1}{2}\partial_{2n+1} \ln \underset{1}{f} = \tfrac{1}{2}\partial_{2n+1} \ln \underset{2}{f},$$

and $\underset{1}{f} = \underset{1}{f}(\overset{\alpha}{u})$, $\underset{2}{f} = \underset{2}{f}(\overset{\alpha}{u})$, $\underset{3}{f} = \underset{3}{f}(\overset{\alpha}{u})$, $\overset{\circ}{g}_{ks} = \overset{\circ}{g}_{ks}(\overset{j}{u})$, $\overset{\circ}{g}_{\bar{k}\bar{s}} = \overset{\circ}{g}_{\bar{k}\bar{s}}(\overset{\bar{j}}{u})$, $\overset{\circ}{g}_{2n+1\ 2n+1} = \overset{\circ}{g}_{2n+1\ 2n+1}(\overset{2n+1}{u})$, $\frac{\partial f}{\partial \overset{\alpha}{u}} = \partial_\alpha f$.

Proof. *Let condition (29) hold. Then, in the parameters of the coordinate net* $\{\underset{\alpha}{v}\}$, *conditions (32) will be valid. From (7), (17) and (32) we obtain*

$$\begin{aligned} &\partial_j g_{\bar{k}\bar{s}} = 2T_j g_{\bar{k}\bar{s}}, \quad \partial_j g_{2n+1\ 2n+1} = 2T_j g_{2n+1\ 2n+1}, \quad \partial_{\bar{j}} g_{ks} = 2T_{\bar{j}} g_{ks}, \quad \partial_{\bar{j}} g_{2n+1\ 2n+1} = 2T_{\bar{j}} g_{2n+1\ 2n+1}, \\ &\partial_{2n+1} g_{ks} = 2T_{2n+1} g_{ks}, \quad \partial_{2n+1} g_{\bar{k}\bar{s}} = 2T_{2n+1} g_{\bar{k}\bar{s}}. \end{aligned} \tag{36}$$

The truthfulness of the theorem follows after integration of equations (36). □

By (35) it follows that the positions $P(X_n)$ (or $P(\overline{X}_n)$, or $P(X_1)$) are in conformal correspondence under parallel translation.

According to (14), a direction field w^α defines an isotropic direction in $\widetilde{W}_{2n+1}$ if $\widetilde{g}_{\alpha\beta}w^\alpha w^\beta = 0$. Then, by (12) and (19) it is easy to prove that the direction fields $\underset{k}{v}{}^\alpha \pm \underset{\bar{s}}{v}{}^\alpha$ and $\underset{2n+1}{v}{}^\alpha \pm \underset{\bar{s}}{v}{}^\alpha$ define isotropic directions in the space $\widetilde{W}_{2n+1}$.

Let the functions $\underset{1}{f}$, $\underset{2}{f}$ and $\underset{3}{f}$ involved in (35) satisfy the condition $\underset{1}{f} = \underset{2}{f} = \underset{3}{f}$. Then, $T_\alpha = \mathrm{grad}$, and according to [7](p. 157), the spaces W_{2n+1} and $\widetilde{W}_{2n+1}$ are Riemannian and pseudo-Riemannian, respectively, which we denote by V_{2n+1} and $\widetilde{V}_{2n+1}$. After renormalization of the fundamental tensor $g_{\alpha\beta}$ we get [7](p. 157) $\nabla_\sigma g_{\alpha\beta} = \nabla_\sigma \widetilde{g}_{\alpha\beta} = 0$. By (35) it follows that the line elements $\mathrm{d}S^2$ and $\mathrm{d}\widetilde{S}^2$ of the spaces V_{2n+1} and $\widetilde{V}_{2n+1}$ have the form, respectively:

$$\begin{aligned} \mathrm{d}S^2 &= g_{ks}(\overset{j}{u})\mathrm{d}\overset{k}{u}\mathrm{d}\overset{s}{u} + g_{\bar{k}\bar{s}}(\overset{\bar{j}}{u})\mathrm{d}\overset{\bar{k}}{u}\mathrm{d}\overset{\bar{s}}{u} + g_{2n+1\ 2n+1}(\overset{2n+1}{u})\mathrm{d}(\overset{2n+1}{u})^2, \\ \mathrm{d}\widetilde{S}^2 &= g_{ks}(\overset{j}{u})\mathrm{d}\overset{k}{u}\mathrm{d}\overset{s}{u} - g_{\bar{k}\bar{s}}(\overset{\bar{j}}{u})\mathrm{d}\overset{\bar{k}}{u}\mathrm{d}\overset{\bar{s}}{u} + g_{2n+1\ 2n+1}(\overset{2n+1}{u})\mathrm{d}(\overset{2n+1}{u})^2. \end{aligned} \tag{37}$$

Equalities (37) imply that the positions $P(X_n)$ (or $P(\overline{X}_n)$, or $P(X_1)$) are in conformal correspondence under parallel translation.

Theorem 3.7. *Condition (29) is equivalent to the following conditions:*

$$\overset{1}{c}{}^{\sigma}_{\nu}\overset{\bullet}{\nabla}_{\alpha}\overset{1}{c}{}^{\beta}_{\sigma}=0,\qquad \overset{1}{d}{}^{\sigma}_{\nu}\overset{\bullet}{\nabla}_{\alpha}\overset{1}{d}{}^{\beta}_{\sigma}=0,\qquad \overset{2}{a}{}^{\sigma}_{\nu}\overset{\bullet}{\nabla}_{\alpha}\overset{2}{a}{}^{\beta}_{\sigma}=0,\tag{38}$$

where $\overset{1}{c}{}^{\sigma}_{\nu}$, $\overset{1}{d}{}^{\sigma}_{\nu}$ *and* $\overset{2}{a}{}^{\sigma}_{\nu}$ *are the projecting affinors of the composition* $X_n\times\overline{X}_n\times X_1$.

Proof. *Since* $\overset{1}{c}{}^{\sigma}_{\nu}=\underset{k}{v}{}^{\sigma}\overset{k}{v}{}_{\nu}$, $\overset{1}{d}{}^{\sigma}_{\nu}=\underset{\bar k}{v}{}^{\sigma}\overset{\bar k}{v}{}_{\nu}$ *and* $\overset{2}{a}{}^{\sigma}_{\nu}=\underset{2n+1}{v}{}^{\sigma}\overset{2n+1}{v}{}_{\nu}$, *we obtain*

$$\overset{1}{c}{}^{\sigma}_{\nu}\overset{\bullet}{\nabla}_{\alpha}\overset{1}{c}{}^{\beta}_{\sigma}=\underset{k}{v}{}^{\sigma}\overset{k}{v}{}_{\nu}\overset{\bullet}{\nabla}_{\alpha}\left(\underset{s}{v}{}^{\beta}\overset{s}{v}{}_{\sigma}\right),\qquad \overset{1}{d}{}^{\sigma}_{\nu}\overset{\bullet}{\nabla}_{\alpha}\overset{1}{d}{}^{\beta}_{\sigma}=\underset{\bar k}{v}{}^{\sigma}\overset{\bar k}{v}{}_{\nu}\overset{\bullet}{\nabla}_{\alpha}\left(\underset{\bar s}{v}{}^{\beta}\overset{\bar s}{v}{}_{\sigma}\right),\qquad \overset{2}{a}{}^{\sigma}_{\nu}\overset{\bullet}{\nabla}_{\alpha}\overset{2}{a}{}^{\beta}_{\sigma}=\underset{2n+1}{v}{}^{\sigma}\overset{2n+1}{v}{}_{\nu}\overset{\bullet}{\nabla}_{\alpha}\left(\underset{2n+1}{v}{}^{\beta}\overset{2n+1}{v}{}_{\sigma}\right).\tag{39}$$

By (20) and (39) we get

$$\overset{1}{c}{}^{\sigma}_{\nu}\overset{\bullet}{\nabla}_{\alpha}\overset{1}{c}{}^{\beta}_{\sigma}=\left(\overset{\bar s}{\underset{k}{T}}{}_{\alpha}\underset{\bar s}{v}{}^{\beta}+\overset{2n+1}{\underset{k}{T}}{}_{\alpha}\underset{2n+1}{v}{}^{\beta}\right)\overset{k}{v}{}_{\nu},\qquad \overset{1}{d}{}^{\sigma}_{\nu}\overset{\bullet}{\nabla}_{\alpha}\overset{1}{d}{}^{\beta}_{\sigma}=\left(\overset{s}{\underset{\bar k}{T}}{}_{\alpha}\underset{s}{v}{}^{\beta}+\overset{2n+1}{\underset{\bar k}{T}}{}_{\alpha}\underset{2n+1}{v}{}^{\beta}\right)\overset{\bar k}{v}{}_{\nu},\qquad \overset{2}{a}{}^{\sigma}_{\nu}\overset{\bullet}{\nabla}_{\alpha}\overset{2}{a}{}^{\beta}_{\sigma}=\overset{p}{\underset{2n+1}{T}}{}_{\alpha}\underset{p}{v}{}^{\beta}\overset{2n+1}{v}{}_{\nu}.\tag{40}$$

By (40) it follows that conditions (38) are valid if and only if conditions (28) are valid, too. Then, in accordance to Theorem 3.1, conditions (37) are equivalent to (29). □

4. A Nilpotent Structure on W_{2n+1} and $\widetilde{W}_{2n+1}$

Let us consider the affinor

$$f^{\beta}_{\alpha}=\underset{i}{v}{}^{\beta}\overset{n+i}{v}{}_{\alpha}.\tag{41}$$

Obviously, $f^{\beta}_{\alpha}\{0\}$ and hence $\overset{\bullet}{\nabla}_{\sigma}f^{\beta}_{\alpha}=\nabla_{\sigma}f^{\beta}_{\alpha}$. By (13) and (15) we get $f^{\beta}_{\alpha}f^{\sigma}_{\beta}=0$, i.e. the affinor f^{β}_{α} is nilpotent. In the parameters of the coordinate net $\{\underset{\alpha}{v}\}$ the matrix of f^{β}_{α} is given by

$$\left\|f^{\beta}_{\alpha}\right\|=\begin{Vmatrix}0&\dots&0&\frac{\sqrt{g_{n+1\ n+1}}}{\sqrt{g_{11}}}&0&\dots&0&0\\0&\dots&0&0&\frac{\sqrt{g_{n+2\ n+2}}}{\sqrt{g_{22}}}&\dots&0&0\\\dots&\dots&\dots&\dots&\dots&\dots&\dots&\dots\\0&\dots&0&0&0&\dots&\frac{\sqrt{g_{2n\ 2n}}}{\sqrt{g_{nn}}}&0\\0&\dots&0&0&\dots&0&\dots&0\\\dots&\dots&\dots&\dots&\dots&\dots&\dots&\dots\\0&\dots&0&0&\dots&0&\dots&0\end{Vmatrix}.$$

Theorem 4.1. *The affinor* f^{β}_{α} *is parallel with respect to* ∇, *i.e.*

$$\nabla_{\sigma}f^{\beta}_{\alpha}=0\tag{42}$$

if and only if the coefficients of the derivative equations (20) satisfy the following conditions:

$$\overset{n+k}{\underset{s}{T}}{}_{\sigma}=\overset{n+k}{\underset{2n+1}{T}}{}_{\sigma}=\overset{2n+1}{\underset{s}{T}}{}_{\sigma}=\overset{k}{\underset{s}{T}}{}_{\sigma}-\overset{n+k}{\underset{n+s}{T}}{}_{\sigma}=0.\tag{43}$$

Proof. *Since* $\overset{\bullet}{\nabla}_{\sigma}f^{\beta}_{\alpha}=\nabla_{\sigma}f^{\beta}_{\alpha}$, *and because of (20) and (41), equality (42) has the form*

$$\overset{\nu}{\underset{s}{T}}{}_{\sigma}\underset{\nu}{v}{}^{\beta}\overset{n+s}{v}{}_{\alpha}-\overset{n+s}{\underset{\nu}{T}}{}_{\sigma}\underset{s}{v}{}^{\beta}\overset{\nu}{v}{}_{\alpha}=0.$$

If we write the sums over the index ν in a more detailed form and regroup the addends, the last equality is equivalent to

$$\left(\underset{s}{\overset{n+k}{T}}{}_{\sigma}\underset{k}{v}^{\beta}\right)\overset{s}{v}_{\alpha}-\left[(\underset{s}{\overset{k}{T}}{}_{\sigma}-\underset{n+s}{\overset{n+k}{T}}{}_{\sigma})\underset{k}{v}^{\beta}+\underset{s}{\overset{n+k}{T}}{}_{\sigma}\underset{n+k}{v}{}^{\beta}+\underset{s}{\overset{2n+1}{T}}{}_{\sigma}\underset{2n+1}{v}{}^{\beta}\right]\overset{n+s}{v}{}_{\alpha}+\left(\underset{2n+1}{\overset{n+k}{T}}{}_{\sigma}\underset{k}{v}^{\beta}\right)\overset{2n+1}{v}{}_{\alpha}=0. \tag{44}$$

Because of the independence of the pseudo-covectors $\overset{\nu}{v}_{\alpha}$, equality (44) is valid if and only if the following equalities hold:

$$\underset{s}{\overset{n+k}{T}}{}_{\sigma}\underset{k}{v}^{\beta}=0,\quad \underset{2n+1}{\overset{n+k}{T}}{}_{\sigma}\underset{k}{v}^{\beta}=0,\quad (\underset{s}{\overset{k}{T}}{}_{\sigma}-\underset{n+s}{\overset{n+k}{T}}{}_{\sigma})\underset{k}{v}^{\beta}+\underset{s}{\overset{n+k}{T}}{}_{\sigma}\underset{n+k}{v}{}^{\beta}+\underset{s}{\overset{2n+1}{T}}{}_{\sigma}\underset{2n+1}{v}{}^{\beta}=0.$$

The independence of the pseudo-vectors $\underset{\nu}{v}^{\beta}$ yields that the last equalities, and hence (42), are valid if and only if conditions (43) hold true. □

Corollary 4.2. *If $\nabla_\sigma\ f_\alpha^\beta=0$, the coefficients of the Weyl connection $\Gamma^\nu_{\alpha\beta}$ satisfy the following conditions in the parameters of the coordinate net $\{\underset{\alpha}{v}\}$:*

$$\Gamma^{\bar{k}}_{\sigma s}=\Gamma^{\bar{k}}_{\sigma 2n+1}=\Gamma^{2n+1}_{\sigma s}=0, \tag{45}$$

$$\frac{\sqrt{g_{kk}}}{\sqrt{g_{ss}}}\Gamma^{k}_{\sigma s}-\frac{\sqrt{g_{n+k\ n+k}}}{\sqrt{g_{n+s\ n+s}}}\Gamma^{n+k}_{\sigma n+s}=0,\quad k\neq s,\qquad \Gamma^{k}_{\sigma k}-\frac{1}{2}\frac{\partial_\sigma g_{kk}}{g_{kk}}=\Gamma^{n+k}_{\sigma n+k}-\frac{1}{2}\frac{\partial_\sigma g_{n+k\ n+k}}{g_{n+k\ n+k}}. \tag{46}$$

Proof. *According to (21), equalities (43) take the form (45) and (46).* □

Proposition 4.3. *If $\nabla_\sigma\ f_\alpha^\beta=0$, in the parameters of the coordinate net $\{\underset{\alpha}{v}\}$, the fundamental tensor $g_{\alpha\beta}$, the additional covector T_σ and the coefficients of the connection $\Gamma^\sigma_{\alpha\beta}$ satisfy the following conditions:*

$$g_{\bar{k}\bar{s}}=h\overset{\circ}{g}_{\bar{k}\bar{s}},\qquad g_{2n+1\ 2n+1}=h\overset{\circ}{g}_{2n+1\ 2n+1},\qquad T_j=\tfrac{1}{2}\partial_j\ln h,\qquad T_{2n+1}=\tfrac{1}{2}\partial_{2n+1}\ln h, \tag{47}$$

$$\Gamma^{k}_{sk}+T_s=\frac{1}{2}\frac{\partial_s g_{kk}}{g_{kk}},\qquad \Gamma^{k}_{2n+1\ k}+T_{2n+1}=\frac{1}{2}\frac{\partial_{2n+1}g_{kk}}{g_{kk}},\qquad \Gamma^{k}_{\bar{j}k}-\Gamma^{n+k}_{\bar{j}n+k}=\frac{1}{2}\frac{\partial_{\bar{j}}g_{kk}}{g_{kk}}-\frac{1}{2}\frac{\partial_{\bar{j}}g_{n+k\ n+k}}{g_{n+k\ n+k}}, \tag{48}$$

where $h=h(\overset{\alpha}{u})$, $\overset{\circ}{g}_{\bar{k}\bar{s}}=\overset{\circ}{g}_{\bar{k}\bar{s}}(\overset{\bar{j}}{u})$, $\overset{\circ}{g}_{2n+1\ 2n+1}=\overset{\circ}{g}_{2n+1\ 2n+1}(\overset{\bar{i}}{u},\overset{2n+1}{u})$.

Proof. *By (7) and (45) we obtain*

$$\partial_j g_{\bar{k}\bar{s}}=2T_j g_{\bar{k}\bar{s}},\quad \partial_{2n+1}g_{\bar{k}\bar{s}}=2T_{2n+1}g_{\bar{k}\bar{s}},\quad \partial_j g_{2n+1\ 2n+1}=2T_j g_{2n+1\ 2n+1}. \tag{49}$$

After integration of equations (49), we get (47). According to (45) and (49), equalities (46) imply (48). □

5. Transformations of Connections

Let us consider the connections

$$^{1}\Gamma^\nu_{\alpha\beta}=\Gamma^\nu_{\alpha\beta}+S^\nu_{\alpha\beta},\qquad ^{1}\widetilde{\Gamma}^\nu_{\alpha\beta}=\Gamma^\nu_{\alpha\beta}+\widetilde{S}^\nu_{\alpha\beta}, \tag{50}$$

where

$$\begin{aligned} S^\nu_{\alpha\beta}&=\frac{1}{\sqrt{g_{2n+1\ 2n+1}}}\textstyle\sum_{\tau=1}^{2n+1}\overset{\tau}{v}_\alpha\ g_{\beta\delta}\sum_{k=1}^{n}\sum_{\bar{s}=n+1}^{2n}\left(\underset{k}{v}^{\delta}\ \underset{\bar{s}}{v}^{\nu}-\ \underset{k}{v}^{\nu}\ \underset{\bar{s}}{v}^{\delta}\right),\\ \widetilde{S}^\nu_{\alpha\beta}&=\frac{1}{\sqrt{\widetilde{g}_{2n+1\ 2n+1}}}\textstyle\sum_{\tau=1}^{2n+1}\overset{\tau}{v}_\alpha\ \widetilde{g}_{\beta\delta}\sum_{k=1}^{n}\sum_{\bar{s}=n+1}^{2n}\left(\underset{k}{v}^{\delta}\ \underset{\bar{s}}{v}^{\nu}-\ \underset{k}{v}^{\nu}\ \underset{\bar{s}}{v}^{\delta}\right)\end{aligned} \tag{51}$$

Obviously, $S^\nu_{\alpha\beta}\{0\}$, $\widetilde{S}^\nu_{\alpha\beta}\{0\}$.

We denote by $^{1}\nabla$ ($^{1}\widetilde{\nabla}$) and ^{1}R ($^{1}\widetilde{R}$) the covariant derivative and the curvature tensor corresponding to the connection $\Gamma^\nu_{\alpha\beta}$ ($\widetilde{\Gamma}^\nu_{\alpha\beta}$), respectively.

Theorem 5.1. *The fundamental tensors $g_{\alpha\beta}$ and $\widetilde{g}_{\alpha\beta}$ of the spaces W_{2n+1} and $\widetilde{W}_{2n+1}$, respectively, satisfy*

$$ {}^1\overset{\bullet}{\nabla}_\sigma g_{\alpha\beta} = 0, \qquad {}^1\overset{\bullet}{\widetilde{\nabla}}_\sigma \widetilde{g}_{\alpha\beta} = 0. \tag{52} $$

Proof. *By (4) and (50) we obtain*

$$ {}^1\overset{\bullet}{\nabla}_\sigma g_{\alpha\beta} = \overset{\bullet}{\nabla}_\sigma g_{\alpha\beta} - S^\nu_{\sigma\alpha} g_{\nu\beta} - S^\nu_{\sigma\beta} g_{\alpha\nu}, \qquad {}^1\overset{\bullet}{\widetilde{\nabla}}_\sigma \widetilde{g}_{\alpha\beta} = \overset{\bullet}{\nabla}_\sigma \widetilde{g}_{\alpha\beta} - \widetilde{S}^\nu_{\sigma\alpha} \widetilde{g}_{\nu\beta} - \widetilde{S}^\nu_{\sigma\beta} \widetilde{g}_{\alpha\nu}. \tag{53} $$

Let us consider the tensors defined by

$$ T_{\sigma\alpha\beta} = S^\nu_{\sigma\alpha} g_{\nu\beta}, \qquad \widetilde{T}_{\sigma\alpha\beta} = \widetilde{S}^\nu_{\sigma\alpha} \widetilde{g}_{\nu\beta} \tag{54} $$

Obviously, $T_{\sigma\alpha\beta}\{2\}$, $\widetilde{T}_{\sigma\alpha\beta}\{2\}$.

According to (50) and (54), we have

$$ \begin{aligned} T_{\sigma\alpha\beta} &= \frac{1}{\sqrt{g_{2n+1\ 2n+1}}} \sum_{\tau=1}^{2n+1} \overset{\tau}{v}_\sigma\, g_{\alpha\delta} \sum_{k=1}^{n} \sum_{\bar{s}=n+1}^{2n} \Big(\underset{k}{v}^\delta \underset{\bar{s}}{v}^\nu - \underset{k}{v}^\nu \underset{\bar{s}}{v}^\delta \Big) g_{\nu\beta}, \\ \widetilde{T}_{\sigma\alpha\beta} &= \frac{1}{\sqrt{\widetilde{g}_{2n+1\ 2n+1}}} \sum_{\tau=1}^{2n+1} \overset{\tau}{v}_\sigma\, \widetilde{g}_{\alpha\delta} \sum_{k=1}^{n} \sum_{\bar{s}=n+1}^{2n} \Big(\underset{k}{v}^\delta \underset{\bar{s}}{v}^\nu - \underset{k}{v}^\nu \underset{\bar{s}}{v}^\delta \Big) \widetilde{g}_{\nu\beta}. \end{aligned} \tag{55} $$

In the parameters of the coordinate net $\{\underset{\alpha}{v}\}$ *we obtain*

$$ \begin{aligned} T_{\sigma\alpha\beta} &= \frac{1}{\sqrt{g_{2n+1\ 2n+1}}} \sum_{\tau=1}^{2n+1} \overset{\tau}{v}_\sigma \sum_{k=1}^{n} \sum_{\bar{s}=n+1}^{2n} \frac{g_{\alpha k} g_{\beta\bar{s}} - g_{\alpha\bar{s}} g_{\beta k}}{\sqrt{g_{kk}}\sqrt{g_{\bar{s}\bar{s}}}}, \\ \widetilde{T}_{\sigma\alpha\beta} &= \frac{1}{\sqrt{\widetilde{g}_{2n+1\ 2n+1}}} \sum_{\tau=1}^{2n+1} \overset{\tau}{v}_\sigma \sum_{k=1}^{n} \sum_{\bar{s}=n+1}^{2n} \frac{\widetilde{g}_{\alpha k} \widetilde{g}_{\beta\bar{s}} - \widetilde{g}_{\alpha\bar{s}} \widetilde{g}_{\beta k}}{\sqrt{\widetilde{g}_{kk}}\sqrt{\widetilde{g}_{\bar{s}\bar{s}}}}, \end{aligned} \tag{56} $$

from which it follows that

$$ T_{\sigma(\alpha\beta)} = 0, \qquad \widetilde{T}_{\sigma(\alpha\beta)} = 0. \tag{57} $$

Then, (8), (18), (53) and (57) imply (52). □

In the parameters of the coordinate net $\{\underset{\alpha}{v}\}$, by (14) and (51) we obtain the following non-zero components of the tensors $S^\nu_{\alpha\beta}$ and $\widetilde{S}^\nu_{\alpha\beta}$:

$$ \begin{aligned} & S^j_{l\bar{m}} = -\widetilde{S}^j_{l\bar{m}} = -\frac{\sqrt{g_{ll}}}{\sqrt{g_{jj}}\sqrt{g_{2n+1\ 2n+1}}} \sum_{\bar{s}=n+1}^{2n} \frac{g_{\bar{m}\bar{s}}}{\sqrt{g_{\bar{s}\bar{s}}}}, && S^j_{\bar{l}\bar{m}} = -\widetilde{S}^j_{\bar{l}\bar{m}} = -\frac{\sqrt{g_{\bar{l}\bar{l}}}}{\sqrt{g_{jj}}\sqrt{g_{2n+1\ 2n+1}}} \sum_{\bar{s}=n+1}^{2n} \frac{g_{\bar{m}\bar{s}}}{\sqrt{g_{\bar{s}\bar{s}}}}, \\ & S^j_{2n+1\bar{m}} = -\widetilde{S}^j_{2n+1\bar{m}} = -\frac{1}{\sqrt{g_{jj}}} \sum_{\bar{s}=n+1}^{2n} \frac{g_{\bar{m}\bar{s}}}{\sqrt{g_{\bar{s}\bar{s}}}}, && S^{\bar{j}}_{lm} = \widetilde{S}^{\bar{j}}_{lm} = \frac{\sqrt{g_{ll}}}{\sqrt{g_{\bar{j}\bar{j}}}\sqrt{g_{2n+1\ 2n+1}}} \sum_{k=1}^{n} \frac{g_{mk}}{\sqrt{g_{kk}}}, \\ & S^{\bar{j}}_{\bar{l}m} = \widetilde{S}^{\bar{j}}_{\bar{l}m} = \frac{\sqrt{g_{\bar{l}\bar{l}}}}{\sqrt{g_{\bar{j}\bar{j}}}\sqrt{g_{2n+1\ 2n+1}}} \sum_{k=1}^{n} \frac{g_{mk}}{\sqrt{g_{kk}}}, && S^{\bar{j}}_{2n+1\ m} = \widetilde{S}^{\bar{j}}_{2n+1\ m} = \frac{1}{\sqrt{g_{\bar{j}\bar{j}}}} \sum_{k=1}^{n} \frac{g_{mk}}{\sqrt{g_{kk}}}. \end{aligned} \tag{58} $$

In the parameters of the net $\{\underset{\alpha}{v}\}$, by (32), (50) and (58) we obtain the following non-zero coefficients of the connections ${}^1\Gamma^\nu_{\alpha\beta}$ and ${}^1\widetilde{\Gamma}^\nu_{\alpha\beta}$:

$$ \begin{aligned} & {}^1\Gamma^j_{lm} = {}^1\widetilde{\Gamma}^j_{lm} = \Gamma^j_{lm}, \qquad {}^1\Gamma^j_{l\bar{m}} = -{}^1\widetilde{\Gamma}^j_{l\bar{m}} = S^j_{l\bar{m}}, \qquad {}^1\Gamma^j_{\bar{l}\bar{m}} = -{}^1\widetilde{\Gamma}^j_{\bar{l}\bar{m}} = S^j_{\bar{l}\bar{m}}, \\ & {}^1\Gamma^{\bar{j}}_{lm} = {}^1\widetilde{\Gamma}^{\bar{j}}_{lm} = S^{\bar{j}}_{lm}, \qquad {}^1\Gamma^{\bar{j}}_{\bar{l}\bar{m}} = {}^1\widetilde{\Gamma}^{\bar{j}}_{\bar{l}\bar{m}} = \Gamma^{\bar{j}}_{\bar{l}\bar{m}}, \qquad {}^1\Gamma^{\bar{j}}_{\bar{l}m} = {}^1\widetilde{\Gamma}^{\bar{j}}_{\bar{l}m} = S^{\bar{j}}_{\bar{l}m}, \\ & {}^1\Gamma^j_{2n+1\ \bar{m}} = -{}^1\widetilde{\Gamma}^j_{2n+1\ \bar{m}} = S^j_{2n+1\ \bar{m}}, \quad {}^1\Gamma^{\bar{j}}_{2n+1\ m} = {}^1\widetilde{\Gamma}^{\bar{j}}_{2n+1\ m} = S^{\bar{j}}_{2n+1\ m}. \end{aligned} \tag{59} $$

By (59) and straightforward computations we get the following components of the curvature tensors ${}^1R_{\alpha\beta\sigma}^{\ \ \ \ \nu}$ and ${}^1\widetilde{R}_{\alpha\beta\sigma}^{\ \ \ \ \nu}$:

$$
\begin{aligned}
&{}^1R_{skm}^{\ \ \ \ j} = R_{skm}^{\ \ \ \ j} + 2S^{j}_{[s|\bar{l}}S^{\bar{l}}_{\ k]m}, \quad {}^1R_{\bar{s}\bar{k}\bar{m}}^{\ \ \ \ \bar{j}} = R_{\bar{s}\bar{k}\bar{m}}^{\ \ \ \ \bar{j}} + 2S^{\bar{j}}_{[\bar{s}|l}S^{l}_{\ \bar{k}]\bar{m}},\\
&{}^1\widetilde{R}_{skm}^{\ \ \ \ j} = R_{skm}^{\ \ \ \ j} - 2S^{j}_{[s|\bar{l}}S^{\bar{l}}_{\ k]m}, \quad {}^1\widetilde{R}_{\bar{s}\bar{k}\bar{m}}^{\ \ \ \ \bar{j}} = R_{\bar{s}\bar{k}\bar{m}}^{\ \ \ \ \bar{j}} - 2S^{\bar{j}}_{[\bar{s}|l}S^{l}_{\ \bar{k}]\bar{m}},\\
&{}^1R_{2n+1\ \bar{s}\bar{k}}^{\ \ \ \ \ \ \ \ j} = 2\partial_{[2n+1}S^{j}_{\bar{s}]\bar{k}} + S^{j}_{2n+1\bar{l}}\Gamma^{\bar{l}}_{\bar{s}\bar{k}}, \quad {}^1R_{2n+1\ sk}^{\ \ \ \ \ \ \ \ \bar{j}} = 2\partial_{[2n+1}S^{\bar{j}}_{s]k} + S^{\bar{j}}_{2n+1\ l}\Gamma^{l}_{sk},\\
&{}^1\widetilde{R}_{2n+1\ \bar{s}\bar{k}}^{\ \ \ \ \ \ \ \ j} = -2\partial_{[2n+1}S^{j}_{\bar{s}]\bar{k}} - S^{j}_{2n+1\bar{l}}\Gamma^{\bar{l}}_{\bar{s}\bar{k}}, \quad {}^1\widetilde{R}_{2n+1\ sk}^{\ \ \ \ \ \ \ \ \bar{j}} = 2\partial_{[2n+1}S^{\bar{j}}_{s]k} + S^{\bar{j}}_{2n+1\ l}\Gamma^{l}_{sk},
\end{aligned}
$$

where $R_{\alpha\beta\sigma}^{\ \ \ \ \nu}$ is the curvature tensor of the space W_{2n+1}.

References

[1] T. Adati, T. Miyazawa,On paracontact Riemannian manifolds, TRU Math. 13(2) (1977), 27–39.
[2] M. Ajeti, M. Teofilova, G. Zlatanov, Triads of compositions in an even–dimensional space with a symmetric affine connection, Tensor, N.S. 73(3) (2011), 171–187.
[3] D. Blair, Riemannian geometry of contact and symplectic manifolds, Prog. in Math. 203, Birkhäuser Boston, 2002.
[4] V. Hlavatý, Les courbes de la variété W_n, Memor. Sci. Math., Paris, 1934.
[5] S. Kaneyuki, F. L. Williams, Almost paracontact and parahodge structures on manifolds, Nagoya Math. J. 99 (1985), 173–187.
[6] S. Kaneyuki, M. Kozai, Paracomplex structures and affine symmetric spaces, Tokyo J. Math. 8(1) (1985), 81–98.
[7] A. P. Norden, Affinely connected spaces, Izd. GRFML Moscow (1976) (in Russian).
[8] A. P. Norden, Spaces of Cartesian composition, Izv. Vyssh. Uchebn. Zaved., Math. 4 (1963), 117–128 (in Russian).
[9] A. P. Norden, G. N. Timofeev, Invariant criteria for special compositions of multidimensional spaces, Izv. Vyssh. Uchebn. Zaved., Math. 8 (1972), 81–89 (in Russian).
[10] A. P. Norden, A. Sh. Yafarov, Theory of the nongeodesic vector field in two-dimensional affinely connected spaces, Izv. Vyssh. Uchebn. Zaved., Math. 12 (1974), 29–34 (in Russian)
[11] A. Özdeğer, Conformal and generalized concircular mappings on Einstein-Weyl manifolds, Acta Math. Sci. 30B(5) (2010), 1739–1745.
[12] A. Özdeğer, Generalized Einstein tensor for a Weyl manifold and its applications, Acta Math. Sin. (Engl. Ser.) 29(2) (2013), 373–382.
[13] S. Sasaki, On paracontact Riemannian manifolds, TRU Math. 16(2) (1980), 75-86.
[14] S. Sasaki, On differentiable manifolds with certain structures which are closely related to almost contact structure I, Tohoku Math. J.(2) 12(3) (1960), 459-476.
[15] I. Sato, On a structure similar to the almost contact structure, Tensor, N.S. 30(3) (1976), 219–224.
[16] G. N. Timofeev, Invariant characteristics of special compositions in Weyl spaces, Izv. Vyssh. Uchebn. Zaved., Math. 1 (1976), 87–99 (in Russian).
[17] G. N. Timofeev, Orthogonal compositions in Weyl spaces, Izv. Vyssh. Uchebn. Zaved., Math. 3 (1976), 73–85 (in Russian).
[18] B. Tsareva, G. Zlatanov, Proportional nets in Weyl spaces, J. Geom. 71 (2001), 192–196.
[19] V. V. Vishnevskii, Manifolds over plural numbers and semi tangent structures, Itogi Sci. Tech., Geom. 20 (1988), 35–47 (in Russian).
[20] G. Zlatanov, A. P. Norden, Orthogonal trajectories of a geodesic field, Izv. Vyssh. Uchebn. Zaved., Math. 7 (1975), 42–46 (in Russian).
[21] G. Zlatanov, Nets in an n-dimensional space of Weyl, C. R. Acad. Bulg. Sci. 41 (1988), 29–32 (in Russian).
[22] G. Zlatanov, Special networks in the n-dimensional space of Weyl W_n, Tensor, N.S. 48 (1989), 234–240.
[23] G. Zlatanov, Compositions generated by special nets in affinely connected spaces, Serdica Math. J. 28 (2002), 1001–1012.
[24] G. Zlatanov, Special compositions in affinely connected spaces without a torsion, Serdica, Math. J. 37 (2011), 211–220.