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Pseudo-Differential Operators Involving Fractional Fourier Cosine (Sine) Transform

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Abstract. A brief introduction to the fractional Fourier cosine transform as well as fractional Fourier sine transform and their basic properties are given. Fractional Fourier cosine (fractional Fourier sine) transform of tempered distributions is studied. Pseudo-differential operators involving these transformations are investigated and discussed the continuity on certain spaces $\mathcal{S}_e$ and $\mathcal{S}_o$.

1. Introduction

The fractional Fourier cosine (sine) transform is a generalization of ordinary Fourier cosine (sine) transform with an angle θ, has many applications in the several areas including Physics, Signal Processing, Mathematical Analysis and other fields ([3, 8]). Motivated by the above works, we have defined one dimensional fractional Fourier cosine (fractional Fourier sine) transform for $\theta\,(0 < \theta < \pi)$ as follows:

Definition 1.1. *The fractional Fourier cosine transform of a function $f \in L_1(\mathbb{R}_+)$; $\mathbb{R}_+ = (0, \infty)$ is defined as*

$$\hat{f}_c^{\theta}(y) = \mathcal{F}_c^{\theta}\left(f(x)\right)(y) = \int_0^{\infty} K_{\theta}^c(x, y) f(x)dx, \tag{1}$$

where

$$K_{\theta}^c(x, y) = \begin{cases} C_{\theta}\, e^{i(x^2+y^2)\cot\theta/2} \cos(xy\csc\theta) & \theta \neq n\pi, \\ \sqrt{\frac{2}{\pi}}\, \cos xy, & \theta = \frac{\pi}{2}, \\ \delta(x-y) & \theta = n\pi, n \in \mathbb{Z}, \end{cases} \tag{2}$$

where

$$C_{\theta} = \sqrt{\frac{2(1 - i\cot\theta)}{\pi}}. \tag{3}$$

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The corresponding inverse fractional Fourier cosine transform is given by

$$\left(\left(\mathcal{F}_c^{\theta}\right)^{-1} \hat{f_c^{\theta}}(y)\right)(x) = f(x) = \int_0^{\infty} \overline{K_{\theta}^c(x,y)}\, \hat{f_c^{\theta}}(y)dy, \tag{4}$$

where

$$\overline{K_{\theta}^c(x,y)} = \overline{C_{\theta}}\, e^{-i(x^2+y^2)\cot\theta/2} \cos(xy\csc\theta) = K_{-\theta}^c(x,y),$$

and

$$\overline{C_{\theta}} = \sqrt{\frac{2(1+i\cot\theta)}{\pi}} = C_{-\theta}. \tag{5}$$

Similarly, we define fractional Fourier sine transform as:

Definition 1.2. *If $f \in L_1(\mathbb{R}_+)$, then its fractional Fourier sine transform is defined as*

$$\hat{f_s^{\theta}}(y) = \mathcal{F}_s^{\theta}\left(f(x)\right)(y) = \int_0^{\infty} K_{\theta}^s(x,y)f(x)dx, \tag{6}$$

where

$$K_{\theta}^s(x,y) = \begin{cases} e^{i(\theta-\pi/2)}C_{\theta}\, e^{i(x^2+y^2)\cot\theta/2} \sin(xy\csc\theta) & \theta \neq n\pi, \\ \sqrt{\frac{2}{\pi}}\, \sin xy, & \theta = \frac{\pi}{2}, \\ \delta(x-y) & n \in \mathbb{Z}. \end{cases} \tag{7}$$

The corresponding inverse fractional Fourier sine transform is given by

$$\left(\left(\mathcal{F}_s^{\theta}\right)^{-1} \hat{f_s^{\theta}}(y)\right)(x) = f(x) = \int_0^{\infty} \overline{K_{\theta}^s(x,y)}\, \hat{f_s^{\theta}}(y)dy, \tag{8}$$

where

$$\overline{K_{\theta}^s(x,y)} = K_{-\theta}^s(x,y),$$

and C_{θ} is given in (3).

Definition 1.3. *([1, 4]) The space $\mathcal{S}_e$ ($\mathcal{S}_o$) is the subset of all even (odd) functions in $\mathcal{S}$ (Schwartz space). Thus $\phi \in \mathcal{S}_e$ ($\mathcal{S}_o$) function and it satisfies*

$$\gamma_{\beta,\nu}(\phi) = \sup_{x\in\mathbb{R}_+} \left|x^{\beta}\, D_x^{\nu}\phi(x)\right| < \infty, \quad \forall\, \beta,\nu \in \mathbb{N}_0. \tag{9}$$

If f is of polynomial growth and is locally integrable function on $\mathbb{R}_+$, then it generates a distribution in $\mathcal{S}'$ as follows:

$$\left\langle f,\phi\right\rangle = \int_0^{\infty} f(x)\phi(x)dx\ ;\ \phi \in \mathcal{S}(\mathbb{R}_+). \tag{10}$$

Lemma 1.4. ([5]) *A function $\phi \in C^{\infty}(\mathbb{R}_+)$ satisfies* (9) *if and only if*

$$\tau_{m,\beta}(\phi) = \sup_{x\in\mathbb{R}_+} \left|(1+|x|^2)^{m/2}\, D_x^{\beta}\phi(x)\right| < \infty, \quad \forall\, m,\beta \in \mathbb{N}_0. \tag{11}$$

The fractional Fourier cosine and fractional Fourier sine transforms are powerful tools in Mathematical Analysis, Physics, Signal Processing etc. Many fundamental results of these transforms are already known, but applications in Pure Mathematics are still missing. In this correspondence, the continuity of above said transforms and the transforms of tempered distributions have been investigated. Moreover, we obtained differential operators Δ_x and $\Delta_x^{\star}$ and also defined the pseudo-differential operators. Then, we discussed the continuity of pseudo-differential operators on spaces $\mathcal{S}_e$ and $\mathcal{S}_o$.

2. Properties of Fractional Fourier Cosine (Sine) Transform

Theorem 2.1. *Let $K^c_\theta(x,y)$ be the kernel of fractional Fourier cosine transform then*

$$\Delta_x = \frac{d^2}{dx^2} - 2ix\cot\theta\frac{d}{dx} - x^2\cot^2\theta - i\cot\theta, \tag{12}$$

and

$$\Delta_x K^c_\theta(x,y) = -(y\csc\theta)^2 K^c_\theta(x,y). \tag{13}$$

Proof: Since $K^c_\theta(x,y) = C_\theta e^{i(x^2+y^2)\cot\theta/2}\cos(xy\csc\theta)$, then

$$\frac{d}{dx}\left(K^c_\theta(x,y)\right) = C_\theta e^{i(x^2+y^2)\cot\theta/2}\left[(ix\cot\theta)\cos(xy\csc\theta) - \sin(xy\csc\theta)y\csc\theta\right]. \tag{14}$$

Rearranging (14), we obtain

$$\sin(xy\csc\theta) = \frac{1}{C_\theta}\, e^{-i(x^2+y^2)\cot\theta/2}(y\csc\theta)^{-1}\left[ix\cot\theta\, K^c_\theta(x,y) - \frac{d}{dx}K^c_\theta(x,y)\right]. \tag{15}$$

Differentiating (14) with respect to x and using (15), we have

$$\begin{aligned}\frac{d^2}{dx^2}K^c_\theta(x,y) &= -x^2\cot\theta\, K^c_\theta(x,y) + i\cot\theta\, K^c_\theta(x,y) - y^2\csc^2\theta\, K^c_\theta(x,y)\\ &\quad +2ix\cot\theta\frac{d}{dx}K^c_\theta(x,y) + 2x^2\cot^2\theta\, K^c_\theta(x,y).\end{aligned}$$

Hence

$$\left[\frac{d^2}{dx^2} - 2ix\cot\theta\frac{d}{dx} - x^2\cot^2\theta - i\cot\theta\right]K^c_\theta(x,y) = -(y\csc\theta)^2 K^c_\theta(x,y).$$

Therefore,

$$\Delta_x K^c_\theta(x,y) = -(y\csc\theta)^2 K^c_\theta(x,y).$$

The above theorem can be generalized as follows:

Remark 2.2. *Let $r \in \mathbb{N}_0$ and $K^c_\theta(x,y)$ be the kernel of fractional Fourier cosine transform, then*

$$\Delta^r_x K^c_\theta(x,y) = \left(-(y\csc\theta)^2\right)^r K^c_\theta(x,y). \tag{16}$$

Theorem 2.3. *For all $\phi \in \mathcal{S}_e(\mathbb{R}_+)$, we have*

$$\left\langle \Delta_x K^c_\theta(x,y),\ \phi(x)\right\rangle = \left\langle K^c_\theta(x,y),\ \Delta^\star_x\phi(x)\right\rangle, \tag{17}$$

where $\Delta^\star_x$ is conjugate complex of Δ_x.

Proof: Using integration by parts, we have

$$\begin{aligned}\left\langle \Delta_x K^c_\theta(x,y), \phi(x)\right\rangle &= \int_0^\infty \Delta_x K^c_\theta(x,y)\,\phi(x)dx\\ &= \int_0^\infty \left(\frac{d^2}{dx^2}K^c_\theta(x,y)\right)\phi(x)dx - \int_0^\infty\left(\frac{d}{dx}K^c_\theta(x,y)\right)2ix\cot\theta\,\phi(x)dx\\ &\quad -\cot^2\theta\int_0^\infty K^c_\theta(x,y)\left(x^2\phi(x)\right)dx - i\cot\theta\int_0^\infty K^c_\theta(x,y)\phi(x)dx\\ &= \int_0^\infty K^c_\theta(x,y)\left(\frac{d^2}{dx^2} + 2ix\cot\theta\,\frac{d}{dx} - x^2\cot^2\theta + i\cot\theta\right)\phi(x)dx\\ &= \int_0^\infty K^c_\theta(x,y)\,\Delta^\star_x\phi(x)dx\\ &= \left\langle K^c_\theta(x,y),\ \Delta^\star_x\phi(x)\right\rangle.\end{aligned}$$

The above theorem can be generalized as follows:

Remark 2.4. *If $\phi \in \mathcal{S}_e$, then*

$$\left\langle \Delta_x^r K_\theta^c(x,y), \phi(x) \right\rangle = \left\langle K_\theta^c(x,y), (\Delta_x^\star)^r \phi(x) \right\rangle, \tag{18}$$

where $r \in \mathbb{N}_0$ and Δ_x is given by (12) *and $\Delta_x^\star$ is conjugate complex of Δ_x .*

Remark 2.5. *Similar results of above Theorems and Remarks of this Section can be found for kernel $K_\theta^s(x,y)$.*

Theorem 2.6. *Let $\phi \in \mathcal{S}_e(\mathbb{R}_+)$, then*

$$\mathcal{F}_c^\theta \left((\Delta_x^\star)^r \phi(x) \right)(y) = \left(-(y \csc\theta)^2 \right)^r \left(\mathcal{F}_c^\theta \phi \right)(y). \tag{19}$$

Proof: By (1), we have

$$\begin{aligned}
\mathcal{F}_c^\theta \left((\Delta_x^\star)^r \phi(x) \right)(y) &= \int_0^\infty K_\theta^c(x,y) \, (\Delta_x^\star)^r \phi(x) dx \\
&= \left\langle K_\theta^c(x,y), (\Delta_x^\star)^r \phi(x) \right\rangle \\
&= \left\langle (\Delta_x)^r K_\theta^c(x,y), \phi(x) \right\rangle \\
&= \left(-(y \csc\theta)^2 \right)^r \left\langle K_\theta^c(x,y), \phi(x) \right\rangle \\
&= \left(-(y \csc\theta)^2 \right)^r \left(\mathcal{F}_c^\theta \phi \right)(y).
\end{aligned}$$

Theorem 2.7. *If $\phi \in \mathcal{S}_e(\mathbb{R}_+)$, then*

$$\Delta_y^r \left(\mathcal{F}_c^\theta \phi \right)(y) = \mathcal{F}_c^\theta \left[\left(-(x \csc\theta)^2 \right)^r \phi(x) \right](y). \tag{20}$$

Proof: Using Remark 2.2, we have

$$\begin{aligned}
\Delta_y^r \left(\mathcal{F}_c^\theta \phi \right)(y) &= \Delta_y^r \int_0^\infty K_\theta^c(x,y) \, \phi(x) dx \\
&= \int_0^\infty \Delta_y^r \, K_\theta^c(x,y) \, \phi(x) dx \\
&= \int_0^\infty \left(-(x \csc\theta)^2 \right)^r K_\theta^c(x,y) \phi(x) dx \\
&= \mathcal{F}_c^\theta \left[\left(-(x \csc\theta)^2 \right)^r \phi(x) \right](y).
\end{aligned}$$

Definition 2.8. *The test function space $\mathcal{S}_\theta$ is defined as follows: an infinitely differential complex valued function ϕ on $\mathbb{R}_+$ belongs to $\mathcal{S}_\theta$ if and only if for every choice of β and ν of non-negative integers, it satisfies*

$$\Gamma_{\beta,\nu}^\theta(\phi) = \sup_{x \in \mathbb{R}_+} \left| x^\beta \Delta_x^\nu \phi(x) \right| < \infty, \tag{21}$$

where Δ_x is given in (12).

Theorem 2.9. *The mapping $\mathcal{F}_c^\theta : \mathcal{S}_e(\mathbb{R}_+) \to \mathcal{S}_\theta(\mathbb{R}_+)$ is linear and continuous.*

Proof: Linearity of $\mathcal{F}_c^\theta$ is obvious. Let β, ν be any two non-negative integers and $\{\phi_n\}_{n \in \mathbb{N}} \in \mathcal{S}_e(\mathbb{R}_+)$. Using (20), we have

$$\sup_{y \in \mathbb{R}_+} \left| y^\beta \, \Delta_y^\nu \left(\mathcal{F}_c^\theta(\phi_n)(y) \right) \right| = \sup_{y \in \mathbb{R}_+} \left| y^\beta \mathcal{F}_c^\theta \left(\left(-(x \csc\theta)^2 \right)^\nu \phi_n \right)(y) \right|.$$

Since $\phi_n \in \mathcal{S}_e(\mathbb{R}_+), \left((-x \csc\theta)^2 \right)^\nu \phi_n \in \mathcal{S}_e(\mathbb{R}_+) \Rightarrow \mathcal{F}_c^\theta \left(\left(-(x \csc\theta)^2 \right)^\nu \phi_n \right) \in \mathcal{S}_e(\mathbb{R}_+)$.

Hence

$$\sup_{y\in\mathbb{R}_+}\left|y^{\beta}\left(\Delta_y^{\nu}\mathcal{F}_c^{\theta}(\phi_n)(y)\right)\right| \to 0 \text{ if } \phi_n \to 0 \text{ in } \mathcal{S}_\theta \text{ as } n\to\infty,$$

which implies the continuity of $\mathcal{F}_c^{\theta}$.

Similar results can be found of above Theorems 2.6-2.9 for fractional Fourier sine transform as follows:

Theorem 2.10. *Let $\phi \in \mathcal{S}_o(\mathbb{R}_+)$, then*

(i) $\mathcal{F}_s^{\theta}\left((\Delta_x^{\star})^r\,\phi(x)\right)(y) = \left(-(y\csc\theta)^2\right)^r\left(\mathcal{F}_s^{\theta}\phi(x)\right)(y),\ \forall\, r\in\mathbb{N}_0,$

(ii) $\Delta_y^r\left(\mathcal{F}_s^{\theta}\phi\right)(y) = \mathcal{F}_s^{\theta}\left[\left(-(x\csc\theta)^2\right)^r\phi(x)\right](y),\ \forall\, r\in\mathbb{N}_0,$

(iii) The mapping $\mathcal{F}_s^{\theta}:\mathcal{S}_o(\mathbb{R}_+)\to\mathcal{S}_\theta(\mathbb{R}_+)$ is linear and continuous.

Theorem 2.11. *Let ϕ be a measurable function defined on $\mathbb{R}_+$. For any fixed $a>0$, we define the function*

$$\left(T_{a,\theta}\phi\right)(x) = \phi(x+a)\,e^{iax\cot\theta}, \tag{22}$$

then

(i) $\mathcal{F}_c^{\theta}\left(T_{a,\theta}\phi\right)(y) + \mathcal{F}_c^{\theta}\left(T_{-a,\theta}\phi\right)(y) = 2\,e^{-\frac{i}{2}a^2\cot\theta}\cos(ay\csc\theta)\left(\mathcal{F}_c^{\theta}\phi\right)(y),$

(ii) $\mathcal{F}_s^{\theta}\left(T_{a,\theta}\phi\right)(y) + \mathcal{F}_s^{\theta}\left(T_{-a,\theta}\phi\right)(y) = 2\,e^{-\frac{i}{2}a^2\cot\theta}\cos(ay\csc\theta)\left(\mathcal{F}_s^{\theta}\phi\right)(y).$

Proof: Proof of this theorem is straight forward and thus avoided.

3. Fractional Fourier Cosine (Sine) Transform of Tempered Distribution

Theorem 3.1. *The fractional Fourier cosine transform is a continuous linear map of $\mathcal{S}_e(\mathbb{R}_+)$ onto itself.*

Proof: Let $\phi\in\mathcal{S}_e(\mathbb{R}_+)\subseteq L_1(\mathbb{R}_+)$, then

$$\begin{aligned}
\left(\mathcal{F}_c^{\theta}\phi\right)(y) &= \int_0^\infty K_\theta^c(x,y)\,\phi(x)dx\\
&= C_\theta\int_0^\infty e^{i(x^2+y^2)\cot\theta/2}\cos(xy\csc\theta)\,\phi(x)dx\\
&= C_\theta e^{iy^2\cot\theta/2}\int_0^\infty e^{ix^2\cot\theta/2}\cos(xy\csc\theta)\,\phi(x)dx\\
&= C_\theta e^{iy^2\cot\theta/2}\mathcal{F}_c\left[e^{ix^2\cot\theta/2}\phi(x)\right](y\csc\theta),\\
&= C_\theta e^{iy^2\cot\theta/2}\,\Phi_\theta(y),
\end{aligned}$$

where $\Phi_\theta(y) = \mathcal{F}_c\left[e^{ix^2\cot\theta/2}\phi(x)\right](y\csc\theta)$.
Since $\phi\in\mathcal{S}_e(\mathbb{R}_+)$, $\Phi_\theta(y) = \mathcal{F}_c\left[e^{ix^2\cot\theta/2}\phi(x)\right](y\csc\theta)\in\mathcal{S}_e(\mathbb{R}_+)$.
Therefore,

$$\begin{aligned}
D_y^{\nu}\left(\mathcal{F}_c^{\theta}\phi\right)(y) &= C_\theta D_y^{\nu}\left[e^{iy^2\cot\theta/2}\,\Phi_\theta(y)\right]\\
&= C_\theta\sum_{\eta=0}^{\nu}\binom{\nu}{\eta}D_y^{\eta}\left(e^{iy^2\cot\theta/2}\right)\,D_y^{\nu-\eta}\Phi_\theta(y)\\
&= C_\theta\sum_{\eta=0}^{\nu}\binom{\nu}{\eta}e^{iy^2\cot\theta/2}P_\eta\left(y, i\cot\theta/2\right)\,D_y^{\nu-\eta}\Phi_\theta(y)
\end{aligned}$$

where $P_\eta\left(y, i\cot\theta/2\right)$ is a polynomial of maximum degree η, and using the technique [7, 9]. Thus

$$D_y^\nu\left(\mathcal{F}_c^\theta\phi\right)(y) = C_\theta\sum_{\eta=0}^{\nu}\binom{\nu}{\eta}e^{iy^2\cot\theta/2}\left(\sum_{s=0}^{\eta}a_s(\cot\theta)y^s\right)D_y^{\nu-\eta}\Phi_\theta(y),$$

therefore

$$\left|y^\beta D_y^\nu\left(\mathcal{F}_c^\theta\phi\right)(y)\right| \le |C_\theta|\sum_{\eta=0}^{\nu}\binom{\nu}{\eta}\sum_{s=0}^{\eta}|a_s(\cot\theta)|\left|y^{\beta+s}D_y^{\nu-\eta}\Phi_\theta(y)\right|,$$

hence,

$$\begin{aligned}\gamma_{\nu,\beta}\left[\left(\mathcal{F}_c^\theta\phi\right)(y)\right] &\le |C_\theta|\sum_{\eta=0}^{\nu}\binom{\nu}{\eta}\sum_{s=0}^{\eta}|a_s(\cot\theta)|\sup_{y\in\mathbb{R}_+}\left|y^{\beta+s}D_y^{\nu-\eta}\Phi_\theta(y)\right| \\ &< \infty,\end{aligned} \tag{23}$$

because $\Phi_\theta(y)\in\mathcal{S}_e(\mathbb{R}_+)$. Thus $\left(\mathcal{F}_c^\theta\phi\right)(y)\in\mathcal{S}_e(\mathbb{R}_+)$.
Also from (1) and (4), we observe that for all $\phi\in\mathcal{S}_e(\mathbb{R}_+)$,

$$\left(\left(\mathcal{F}_c^\theta\right)^{-1}\mathcal{F}_c^\theta\right)\phi = \phi = \left(\mathcal{F}_c^\theta\left(\mathcal{F}_c^\theta\right)^{-1}\right)\phi.$$

It follows that $\mathcal{F}_c^\theta$ is an 1-1 function of $\mathcal{S}_e(\mathbb{R}_+)$ onto itself. Clearly, $\mathcal{F}_c^\theta$ is also a linear map of $\mathcal{S}_e(\mathbb{R}_+)$ onto itself. Also for every sequence $\{\phi_n\}_{n\in\mathbb{N}}$ which converges to zero as $n\to\infty$ in $\mathcal{S}_e(\mathbb{R}_+)$ then by (23), $\{\mathcal{F}_c^\theta\phi_n\}\to 0$ in $\mathcal{S}_e(\mathbb{R}_+)$ as $n\to\infty$, which implies the continuity of fractional Fourier cosine transform.

Theorem 3.2. *The fractional Fourier sine transform is a continuous linear map of $\mathcal{S}_o(\mathbb{R}_+)$ onto itself.*

Theorem 3.3. *(Parseval identity of fractional Fourier cosine transform) Let $\phi,\psi\in\mathcal{S}_e(\mathbb{R}_+)$, then the following equalities hold*

$$\int_0^\infty\left(\mathcal{F}_c^\theta\phi\right)(y)\,\overline{\left(\mathcal{F}_c^\theta\psi\right)(y)}dy = \int_0^\infty\phi(x)\,\overline{\psi(x)}dx, \tag{24}$$

and

$$\int_0^\infty\left|\mathcal{F}_c^\theta\phi(y)\right|^2dy = \int_0^\infty\left|\phi(x)\right|^2dx. \tag{25}$$

Proof: Since $\phi,\psi\in\mathcal{S}_e(\mathbb{R}_+)$, hence using (2), we have

$$\begin{aligned}\int_0^\infty\left(\mathcal{F}_c^\theta\phi\right)(y)\,\overline{\left(\mathcal{F}_c^\theta\psi\right)(y)}dy &= C^\theta\int_0^\infty\left[\int_0^\infty e^{i(x^2+y^2)\cot\theta/2}\cos(xy\csc\theta)\phi(x)dx\right]\overline{\left(\mathcal{F}_c^\theta\psi\right)(y)}dy \\ &= \int_0^\infty\phi(x)\overline{\left[\overline{C^\theta}\int_0^\infty e^{-i(x^2+y^2)\cot\theta/2}\cos(xy\csc\theta)\left(\mathcal{F}_c^\theta\psi\right)(y)dy\right]}dx,\end{aligned}$$

as an application of inverse fractional Fourier-cosine transform, we obtain

$$\int_0^\infty\left(\mathcal{F}_c^\theta\phi\right)(y)\,\overline{\left(\mathcal{F}_c^\theta\psi\right)(y)}dy = \int_0^\infty\phi(x)\,\overline{\psi(x)}dx.$$

If $\phi=\psi$ in the last expression, we obtain

$$\int_0^\infty\left|\mathcal{F}_c^\theta\phi(y)\right|^2dy = \int_0^\infty\left|\phi(x)\right|^2dx.$$

Theorem 3.4. *(Parseval identity of fractional Fourier sine transform) Let $\phi, \psi \in \mathcal{S}_o(\mathbb{R}_+)$, then the following equalities hold*

$$\int_0^\infty \left(\mathcal{F}_s^\theta \phi\right)(y)\, \overline{\left(\mathcal{F}_s^\theta \psi\right)(y)} dy = \int_0^\infty \phi(x)\, \overline{\psi(x)} dx, \tag{26}$$

and

$$\int_0^\infty \left|\mathcal{F}_s^\theta \phi(y)\right|^2 dy = \int_0^\infty \left|\phi(x)\right|^2 dx. \tag{27}$$

Definition 3.5. *The generalized fractional Fourier cosine transform $\left(\mathcal{F}_c^\theta f\right)$ of $f \in \mathcal{S}_e'(\mathbb{R}_+)$ is defined by*

$$\left\langle \mathcal{F}_c^\theta f, \phi \right\rangle = \left\langle f, \mathcal{F}_c^\theta \phi \right\rangle, \tag{28}$$

where $\phi \in \mathcal{S}_e(\mathbb{R}_+)$.

By Theorem 3.1, $\left(\mathcal{F}_c^\theta \phi\right) \in \mathcal{S}_e(\mathbb{R}_+)\ \ \forall\ \phi \in \mathcal{S}_e(\mathbb{R}_+)$, so R.H.S. of (28) is well defined.

Definition 3.6. *The inverse fractional Fourier cosine transform $\left(\left(\mathcal{F}_c^\theta\right)^{-1} f\right)$ of $f \in \mathcal{S}_e'(\mathbb{R}_+)$ is defined as*

$$\left\langle \left(\mathcal{F}_c^\theta\right)^{-1} f, \phi \right\rangle = \left\langle f, \left(\mathcal{F}_c^\theta\right)^{-1} \phi \right\rangle; \ \ \phi \in \mathcal{S}_e(\mathbb{R}_+). \tag{29}$$

Similarly, we define the generalized fractional Fourier sine transform and and its inverse for $\mathcal{S}_o'(\mathbb{R}_+)$ as follows:

Definition 3.7. *The generalized fractional Fourier sine transform $\left(\mathcal{F}_s^\theta f\right)$ of $f \in \mathcal{S}_o'(\mathbb{R}_+)$ is defined by*

$$\left\langle \mathcal{F}_s^\theta f, \phi \right\rangle = \left\langle f, \mathcal{F}_s^\theta \phi \right\rangle, \tag{30}$$

where $\phi \in \mathcal{S}_o(\mathbb{R}_+)$.

Definition 3.8. *The inverse fractional Fourier cosine transform $\left(\left(\mathcal{F}_s^\theta\right)^{-1} f\right)$ of $f \in \mathcal{S}_o'(\mathbb{R}_+)$ is defined as*

$$\left\langle \left(\mathcal{F}_s^\theta\right)^{-1} f, \phi \right\rangle = \left\langle f, \left(\mathcal{F}_s^\theta\right)^{-1} \phi \right\rangle; \ \ \phi \in \mathcal{S}_o(\mathbb{R}_+). \tag{31}$$

Theorem 3.9. *The generalized fractional Fourier cosine transform $\mathcal{F}_c^\theta$ is a continuous linear map of $\mathcal{S}_e'(\mathbb{R}_+)$ onto itself.*

Proof: Let $f \in \mathcal{S}_e'(\mathbb{R}_+)$ and if the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ converges to zero in $\mathcal{S}_e(\mathbb{R}_+)$, then by continuity of fractional Fourier cosine transform $\{\mathcal{F}_c^\theta \phi_n\} \to 0$ as $n \to \infty$. Hence

$$\left\langle \mathcal{F}_c^\theta f, \phi_n \right\rangle = \left\langle f, \mathcal{F}_c^\theta \phi_n \right\rangle \to 0 \text{ as } n \to \infty.$$

Therefore, $\left(\mathcal{F}_c^\theta\right)$ is continuous on $\mathcal{S}_e'(\mathbb{R}_+)$.
Also for $f, g \in \mathcal{S}_e'(\mathbb{R}_+)$, we have

$$\begin{aligned} \left\langle \mathcal{F}_c^\theta (f+g), \phi \right\rangle &= \left\langle f+g, \mathcal{F}_c^\theta \phi \right\rangle \\ &= \left\langle f, \mathcal{F}_c^\theta \phi \right\rangle + \left\langle g, \mathcal{F}_c^\theta \phi \right\rangle \\ &= \left\langle \mathcal{F}_c^\theta f, \phi \right\rangle + \left\langle \mathcal{F}_c^\theta g, \phi \right\rangle. \end{aligned}$$

Hence, $\mathcal{F}_c^\theta$ is linear on $\mathcal{S}_e'(\mathbb{R}_+)$.

Also,

$$\begin{aligned}\left\langle f,\phi\right\rangle &= \left\langle \mathcal{F}_c^\theta f, \left(\mathcal{F}_c^\theta\right)^{-1}\phi\right\rangle\\ &= \left\langle f, \left(\mathcal{F}_c^\theta\left(\mathcal{F}_c^\theta\right)^{-1}\right)\phi\right\rangle,\end{aligned}$$

so that

$$\left(\left(\mathcal{F}_c^\theta\right)^{-1}\mathcal{F}_c^\theta\right)f = f.$$

Similarly,

$$\left(\mathcal{F}_c^\theta\left(\mathcal{F}_c^\theta\right)^{-1}\right)f = f.$$

Therefore, $\mathcal{F}_c^\theta$ and $\left(\mathcal{F}_c^\theta\right)^{-1}$ are 1-1 map of $\mathcal{S}_e'(\mathbb{R}_+)$ onto itself.

Example 3.10. *If $x \in \mathbb{R}_+,\ a > 0$, then*

(i) $\mathcal{F}_c^\theta\,[\delta(x-a)] = K_\theta^c(a,y)$,
(ii) $\mathcal{F}_s^\theta\,[\delta(x-a)] = K_\theta^s(a,y)$.

Proof: (*i*) Let $\phi \in \mathcal{S}_e(\mathbb{R}_+)$, then

$$\begin{aligned}\left\langle \mathcal{F}_c^\theta\,[\delta(x-a)]\,,\phi\right\rangle &= \left\langle \delta(x-a), \left(\mathcal{F}_c^\theta\phi\right)(x)\right\rangle\\ &= \left(\mathcal{F}_c^\theta\phi\right)(a)\\ &= C_\theta\int_0^\infty e^{i(a^2+y^2)\cot\theta/2}\ \cos(ay\csc\theta)\ \phi(y)dy\\ &= \left\langle C_\theta e^{i(a^2+y^2)\cot\theta/2}\ \cos(ay\csc\theta),\ \phi(y)\right\rangle\\ &= \left\langle K_\theta^c(a,y),\phi(y)\right\rangle,\end{aligned}$$

hence, $\mathcal{F}_c^\theta\,[\delta(x-a)] = K_\theta^c(a,y)$.
(*ii*) Similarly if $\phi \in \mathcal{S}_o(\mathbb{R}_+)$ and proceeding as in (i), we obtain (ii).

Example 3.11. *If $x \in \mathbb{R}_+$, then*

(i) $\mathcal{F}_c^\theta\,[\delta(x)] = C_\theta\ e^{iy^2\cot\theta/2}$.
(ii) $\mathcal{F}_s^\theta\,[\delta(x)] = 0$.

Proof: Put $a = 0$ in Example 3.10, we get the desired results.

4. Pseudo-Differential Operators (P.D.O.'s)

A linear partial differential operator $A(x,\Delta_x^\star)$ on $\mathbb{R}_+$ is given by

$$A(x,\Delta_x^\star) = \sum_{r=0}^m a_r(x)\,(\Delta_x^\star)^r\,, \tag{32}$$

where the coefficient $a_r(x)$ are functions defined on $\mathbb{R}_+$ and $\Delta_x^\star$ is conjugate complex of Δ_x, given in (12). If we replace $\Delta_x^\star$ by monomial $\left(-(y\csc\theta)^2\right)$ in $\mathbb{R}_+$, then we obtain the so called symbol

$$A(x,y) = \sum_{r=0}^m a_r(x)\left(-(y\csc\theta)^2\right)^r. \tag{33}$$

In order to get another representation of the operator $A(x, \Delta_x^\star)$, let us take any function $\phi \in \mathcal{S}_e$, then we have

$$(A(x, \Delta_x^\star))\,\phi(x) = \sum_{r=0}^{m} a_r(x)\left(\left(\mathcal{F}_c^\theta\right)^{-1}\mathcal{F}_c^\theta\right)(\Delta_x^\star)^r\,\phi(x) \tag{34}$$

$$= \sum_{r=0}^{m} a_r(x)\left(\mathcal{F}_c^\theta\right)^{-1}\left(\left(-(y\csc\theta)^2\right)^r\left(\mathcal{F}_c^\theta\phi\right)\right)(x) \tag{35}$$

$$= \int_0^\infty \overline{K_\theta^c(x,y)}A(x,y)\left(\mathcal{F}_c^\theta\phi\right)(y)dy, \tag{36}$$

where $\overline{K_\theta^c(x,y)}$ is as in (2). If we replace the symbol $A(x,y)$ by more general symbol $a(x,y)$, which is no longer a polynomial in y necessarily , we get the pseudo-differential operator $A_{a,\theta}^c$ defined below. For p.d.o., involving Fourier transform, Hankel transform, fractional Fourier transform, Fourier-Jacobi transform and a singular differential operator, we may refer respectively [11, 13], [6], [7, 9], [10, 12] and [2].

Definition 4.1. *Let $m \in \mathbb{R}$, then we define the symbol class S^m to be the set of all functions $a(x,y) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ such that for $\mu, \nu \in \mathbb{N}_0$, there exists a positive constant $C_{\mu,\nu}$ depending upon μ and ν only, such that*

$$\left|D_x^\mu\, D_y^\nu a(x,y)\right| \le C_{\mu,\nu}\,(1+y)^{m-\nu}. \tag{37}$$

Definition 4.2. *Let a be a symbol satisfying* (37), *then the pseudo-differential operator (p.d.o.) involving fractional Fourier cosine transform, $A_{a,\theta}^c$ is defined by*

$$\left(A_{a,\theta}^c\phi\right)(x) = \int_0^\infty \overline{K_\theta^c(x,y)}\,a(x,y)\left(\mathcal{F}_c^\theta\phi\right)(y)dy;\ \forall\phi \in \mathcal{S}_e(\mathbb{R}_+). \tag{38}$$

Similarly, we can define p.d.o. involving fractional Fourier sine transform as follows:

Definition 4.3. *Let a be a symbol satisfying* (37), *then the p.d.o. involving fractional Fourier sine transform, $A_{a,\theta}^s$ is defined by*

$$\left(A_{a,\theta}^s\phi\right)(x) = \int_0^\infty \overline{K_\theta^s(x,y)}\,a(x,y)\left(\mathcal{F}_s^\theta\phi\right)(y)dy;\ \forall\phi \in \mathcal{S}_o(\mathbb{R}_+). \tag{39}$$

Theorem 4.4. *Let a be a symbol belonging to the symbol class S^m, $m < -(\beta+\nu+1)$, then $A_{a,\theta}^c$ maps $\mathcal{S}_e(\mathbb{R}_+)$ into itself.*

Proof: Let $\phi \in \mathcal{S}_e(\mathbb{R}_+)$ and $\beta, \nu \in \mathbb{N}_0$. In order to prove that $A_{a,\theta}^c$ maps $\mathcal{S}_e(\mathbb{R}_+)$ into itself we need to prove that

$$\sup_{x\in\mathbb{R}_+}\left|x^\beta\, D_x^\nu\left(A_{a,\theta}^c\phi(x)\right)\right| < \infty.$$

Using (38), we have

$$\begin{aligned}
x^\beta D_x^\nu\left(A_{a,\theta}^c\phi\right)(x) &= x^\beta\int_0^\infty D_x^\nu\left(\overline{K_\theta^c(x,y)}\,a(x,y)\right)\left(\mathcal{F}_c^\theta\phi\right)(y)dy\\
&= \overline{C_\theta}\,x^\beta\int_0^\infty\left\{\sum_{\eta=0}^{\nu}\binom{\nu}{\eta}D_x^\eta\left(e^{-i(x^2+y^2)\cot\theta/2}\,\cos(xy\csc\theta)\right)D_x^{\nu-\eta}a(x,y)\right\}\\
&\times\left(\mathcal{F}_c^\theta\phi\right)(y)\,dy\\
&= \overline{C_\theta}\,x^\beta\sum_{\eta=0}^{\nu}\binom{\nu}{\eta}\sum_{\xi=0}^{\eta}\binom{\eta}{\xi}(-1)^{\left[\frac{\eta-\xi+1}{2}\right]}\int_0^\infty P_\xi\left(x, i\cot\theta/2\right)\,e^{-i(x^2+y^2)\cot\theta/2}\\
&\times\ A(xy\csc\theta)\ (y\csc\theta)^{\eta-\xi}\,D_x^{\nu-\eta}\left(a(x,y)\right)\ \left(\mathcal{F}_c^\theta\phi\right)(y)\,dy,
\end{aligned}$$

where $P_\xi(x, i\cot\theta/2)$ is a polynomial and $A(xy\csc\theta) = \cos(xy\csc\theta)$ or $\sin(xy\csc\theta)$ depending upon $(\eta-\xi)$ is even or odd. Thus

$$\begin{aligned}
x^\beta D_x^\nu\left(A_{a,\theta}^c\phi\right)(x) &= \overline{C_\theta}\, x^\beta \sum_{\eta=0}^{\nu}\binom{\nu}{\eta}\sum_{\xi=0}^{\eta}\binom{\eta}{\xi}(-1)^{\left[\frac{\eta-\xi+1}{2}\right]}\int_0^\infty\left(\sum_{r=0}^{\xi}a_r(\cot\theta)x^r\right)\\
&\times\ e^{-i(x^2+y^2)\cot\theta/2}A(xy\csc\theta)\ (y\csc\theta)^{\eta-\xi}\, D_x^{\nu-\eta}\left(a(x,y)\right)\ \left(\mathcal{F}_c^\theta\phi\right)(y)\,dy\\
&= (-1)^{\frac{\eta-\xi+\beta+r+2}{2}}\,\overline{C_\theta}\sum_{\eta=0}^{\nu}\binom{\nu}{\eta}\sum_{\xi=0}^{\eta}\binom{\eta}{\xi}\sum_{r=0}^{\xi}a_r(\cot\theta)\int_0^\infty e^{-i(x^2+y^2)\cot\theta/2}\\
&\times\ D_y^{\beta+r}\left(A(xy\csc\theta)\right)\ y^{\eta-\xi}\csc^{\eta-\xi-r-\beta} D_x^{\nu-\eta}\left(a(x,y)\right)\left(\mathcal{F}_c^\theta\phi\right)(y)\,dy.
\end{aligned}$$

Now, using integration by parts, we have

$$\begin{aligned}
x^\beta D_x^\nu\left(A_{a,\theta}^c\phi\right)(x) &= \overline{C_\theta}\sum_{\eta=0}^{\nu}\binom{\nu}{\eta}\sum_{\xi=0}^{\eta}\binom{\eta}{\xi}\sum_{r=0}^{\xi}a_r(\cot\theta)\ \csc^{\eta-\xi-r-\beta}(-1)^{\frac{\eta-\xi+\beta+r+2}{2}}\\
&\times\ \int_0^\infty A(xy\csc\theta)\sum_{t=0}^{\beta+r}\binom{\beta+r}{t}\{D_y^t\left(e^{-i(x^2+y^2)\cot\theta/2}\,D_x^{\nu-\eta}a(x,y)\right)\\
&\times\ D_y^{\beta+r-t}\left(y^{\eta-\xi}\left(\mathcal{F}_c^\theta\phi\right)(y)\right)\}dy\\
&= \overline{C_\theta}\sum_{\eta=0}^{\nu}\binom{\nu}{\eta}\sum_{\xi=0}^{\eta}\binom{\eta}{\xi}\sum_{r=0}^{\xi}a_r(\cot\theta)\ \csc^{\eta-\xi-r-\beta}(-1)^{\frac{\eta-\xi+\beta+r+2}{2}}\\
&\times\ \sum_{t=0}^{\beta+r}\binom{\beta+r}{t}\sum_{j=0}^{t}\binom{t}{j}\sum_{\gamma=0}^{j}\binom{j}{\gamma}a_\gamma(\cot\theta)\int_0^\infty A(xy\csc\theta)\ y^\gamma\\
&\times\ e^{-i(x^2+y^2)\cot\theta/2}\left(D_y^{t-j}\,D_x^{\nu-\eta}a(x,y)\right)D_y^{\beta+r-t}\left(y^{\eta-\xi}\left(\mathcal{F}_c^\theta\phi\right)(y)\right)dy.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|x^\beta\, D_x^\nu\left(A_{a,\theta}\phi\right)(x)\right| &\le \left|\overline{C_\theta}\right|\ \sum_{\eta=0}^{\nu}\binom{\nu}{\eta}\sum_{\xi=0}^{\eta}\binom{\eta}{\xi}\sum_{r=0}^{\xi}|a_r(\cot\theta)|\,|\csc^{\eta-\xi-r-\beta}|\ \sum_{t=0}^{\beta+r}\binom{\beta+r}{t}\\
&\times\ \sum_{j=0}^{t}\binom{t}{j}\sum_{\gamma=0}^{j}\binom{j}{\gamma}|a_\gamma(\cot\theta)|\ \int_0^\infty\left|D_y^{t-j}\,D_x^{\nu-\eta}a(x,y)\right|\\
&\times\ \left|y^\gamma\, D_y^{\beta+r-t}\left(y^{\eta-\xi}\left(\mathcal{F}_c^\theta\phi\right)(y)\right)\right|\,dy\\
&\le \left|\overline{C_\theta}\right|\ \sum_{\eta=0}^{\nu}\binom{\nu}{\eta}\sum_{\xi=0}^{\eta}\binom{\eta}{\xi}\sum_{r=0}^{\xi}|a_r(\cot\theta)|\,|\csc^{\eta-\xi-r-\beta}|\ \sum_{t=0}^{\beta+r}\binom{\beta+r}{t}\\
&\times\ \sum_{j=0}^{t}\binom{t}{j}\sum_{\gamma=0}^{j}\binom{j}{\gamma}|a_\gamma(\cot\theta)|\,C_{\nu-\eta,t-j}\int_0^\infty(1+y)^{m-(t-j)+\gamma}\\
&\times\ \left|D_y^{\beta+r-t}\left(y^{\eta-\xi}\left(\mathcal{F}_c^\theta\phi\right)(y)\right)\right|dy,
\end{aligned}$$

since $y^{\eta-\xi}\left(\mathcal{F}_c^\theta\phi\right)(y)\in\mathcal{S}_e(\mathbb{R}_+)$, so the last integral is convergent. Hence

$$\sup_{x\in\mathbb{R}_+}\left|x^\beta\, D_x^\nu\left(A_{a,\theta}^c\phi(x)\right)\right| < \infty.$$

Theorem 4.5. *Let a be a symbol belonging to the symbol class S^m, $m < -(\beta + \nu + 1)$, then $A^s_{a,\theta}$ maps $\mathcal{S}_o(\mathbb{R}_+)$ into itself.*

Proof: Proof of this theorem is similar to that of Theorem 4.4 and thus avoided.

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