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Analytic Representations of Sequences in L^p Spaces, $1 \leq p < \infty$

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Abstract. In this paper we consider a sequence of functions in $L^p(\mathbb{R}), 1 \leq p < \infty$ and, in the second part, we include a sequence of real analytic functions without real roots. We obtain several results regarding their convergence or the convergence of the sequence of their analytic representations. We, also, give results about the analytic representation of the product of the boundary functions and other additional results.

1. Introduction

The boundary values representation has been studied for a long time and from different points of view. One of the first results is that if $f \in L^1(\mathbb{R})$, then the function

$$\widehat{f}(z) = \frac{1}{2\pi i}\langle f(t), \frac{1}{t-z}\rangle,\ z = x+iy,\ x \notin \operatorname{supp} f \subset \mathbb{R}, y \in \mathbb{R}$$

is the Cauchy representation of f i.e.

$$\lim_{y \to 0^+} \int_{-\infty}^{+\infty} [\widehat{f}(x+iy) - \widehat{f}(x-iy)]\varphi(x)dx = \int_{-\infty}^{+\infty} f(x)\varphi(x)dx \tag{1}$$

for every $\varphi \in D = D(\mathbb{R})$, the Schwartz space of test functions.

The results are formulated in four theorems.

In Theorem 2.1, it is proved that the convergence of a sequence of functions in $L^p(\mathbb{R})$ implies the uniform convergence of the sequence of their analytic representations on compact subsets of the complex plane.

The proof of the first part of Theorem 2.1 uses a well known Lemma, given in [2].

Lemma 1.1. *Let f be a function of the class C^m, such that it is bounded on $\mathbb{R}$. Then*

$$f^*(x+iy) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{f(t)}{(t-x)^2+y^2}dt,\ x+iy \in \mathbb{C},\ y \neq 0,$$

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converges uniformly to $f(x)$ on each compact subset of $\mathbb{R}$ as $y \to 0^+$.

In Theorem 2.2, it is proved that the uniform convergence of a sequence of analytic representations, $\{\widehat{f_n}(z)\}$ over the sets of the form $[a,b]\times[\delta,T]$ and $[a,b]\times[-T,-\delta]$, $0<\delta<T$, $a<b$ of functions, f_n in $L^p(\mathbb{R})$, $1 \le p < \infty$ that form bounded sequence in $L^p(\mathbb{R})$ implies the weak convergence of the sequence $\{f_n\}$ in $L^p(\mathbb{R})$ with analytic representation of the boundary value of $\{f_n\}$ being the boundary value of $\{\widehat{f_n}(z)\}$.

Theorem 2.3 deals with a convergent sequence of functions in $L^p(\mathbb{R})$ and a uniformly convergent sequence of real analytic functions on compact subsets of $\mathbb{R}$ and gives results about the analytic representation of the product of their boundary functions.

Theorem 2.4 considers a sequence of functions $\{f_n\}$ in $L^p(\mathbb{R})$ and a uniformly convergent sequence of polynomials $\{P_n\}$ on compact subsets of $\mathbb{R}$, bounded by some polynomial P_0. In the theorem, it is proved the existence of a function $f \in L^p(\mathbb{R})$ such that $\left\{\dfrac{f_n \cdot P_n}{P_0}\right\}$ converges weakly to $\left\{\dfrac{f \cdot P}{P_0}\right\}$, as $n \to \infty$.

2. Main Results

Theorem 2.1. *Let $\{f_n\}$ be a sequence of functions in $L^p(\mathbb{R})$, $1 \le p < \infty$ that converges to f in $L^p(\mathbb{R})$ and let $\widehat{f_n}(z) = \frac{1}{2\pi i}\langle f_n(t), \frac{1}{t-z}\rangle$, $z \in \mathbb{C}\setminus\mathbb{R}$ for $n \in \mathbb{N}$. Then:*

i) *For every $n \in \mathbb{N}$, $\widehat{f_n}(z)$, $z \in \mathbb{C}\setminus\mathbb{R}$, is an analytic representation of f_n.*

ii) *The sequence $\{\widehat{f_n}(z)\}$ uniformly converges on compact subsets of $\mathbb{C}\setminus\mathbb{R}$ to the function $\widehat{f}(z)$, where $\widehat{f}(z) = \frac{1}{2\pi i}\langle f(t), \frac{1}{t-z}\rangle$, $Imz \neq 0$.*

iii) *The function $\widehat{f}(z)$, $z \in \mathbb{C}\setminus\mathbb{R}$, is an analytic representation of the function f.*

Proof. We only give the proof of *i)* and *ii)*.
i) Let $z = x+iy \in \mathbb{C}$ be such that $Imz \neq 0$. For arbitrary chosen $\varphi \in D$, $n \in \mathbb{N}$,

$$I_n = \int_{\mathbb{R}} [\widehat{f_n}(x+iy) - \widehat{f_n}(x-iy)]\varphi(x)dx$$
$$= \int_{\mathbb{R}} \frac{1}{2\pi i}\left[\int_{\mathbb{R}} f_n(t)\frac{2iy}{(t-x)^2+y^2}dt\right]\varphi(x)dx.$$

Applying Fubini's theorem, we get that

$$I_n = \int_{\mathbb{R}}\left[\frac{y}{\pi}\int_{\mathbb{R}}\frac{\varphi(x)}{(t-x)^2+y^2}dx\right]f_n(t)dt.$$

Using Lemma 1 and the Lebesgue dominated convergence theorem, we obtain that $\lim\limits_{y\to 0^+} I_n = \int\limits_{\mathbb{R}} f_n(t)\varphi(t)dt$.

So, we proved that $\widehat{f_n}(z)$ is analytic representation of f_n for every $n \in \mathbb{N}$.
ii) Let $z = x+iy \in \mathbb{C}$ be such that $Imz \neq 0$. Then

$$\widehat{f_n}(z) - \widehat{f}(z) = \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{f_n(t)-f(t)}{t-z}dt, z \in \mathbb{C}\setminus\mathbb{R}.$$

We apply the Hölder's inequality and obtain the assertion since $\left[\int\limits_{\mathbb{R}} \frac{1}{|t-z|^q}dt\right]^{\frac{1}{q}}$ is bounded. $\square$

Theorem 2.2. *Let $\{f_n\}$ be a bounded sequence of functions in $L^p(\mathbb{R})$, $1 \le p < \infty$, and for every $n \in \mathbb{N}$, let $\hat{f}_n(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$ be the analytic representation of f_n. If the sequence $\{\hat{f}_n(z)\}$ uniformly converges on sets of the form $[a,b] \times [\delta, T]$ and $[a,b] \times [-T, -\delta]$, $0 < \delta < T, a < b$ in the upper and lower half plane, respectively, to the function $\hat{f}(z)$, then there exists a function $f \in L^p(\mathbb{R})$ such that the sequence $\{f_n\}$ converges weakly to f in $L^p(\mathbb{R})$ and $\hat{f}(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$ is the analytic representation of f.*

Proof. We will show that there exists a function $f \in L^p(\mathbb{R})$, such that, for every $\varphi \in D$, $\lim_{n\to\infty} \langle f_n, \varphi \rangle = \langle f, \varphi \rangle$. Let $\varphi \in D$ and supp$\varphi = [-a, a]$, $a > 0$. Then, for $n, m \in \mathbb{N}$, we have

$$\begin{aligned}\langle f_n, \varphi\rangle - \langle f_m, \varphi\rangle =& \{\langle f_n, \varphi\rangle - \langle \hat{f}_n(z) - \hat{f}_n(\overline{z}), \varphi\rangle\} + \{\langle \hat{f}_n(z) - \hat{f}_n(\overline{z}), \varphi\rangle - \langle \hat{f}_m(z) - \hat{f}_m(\overline{z}), \varphi\rangle\} \\ &+ \{\langle \hat{f}_m(z) - \hat{f}_m(\overline{z}), \varphi\rangle - \langle f_m, \varphi\rangle\} = I_1 + I_2 + I_3.\end{aligned}$$

Let $\varepsilon > 0$. By the boundedness of f_n, we get that there exists y_0 such that

$$|I_1| < \frac{\varepsilon}{3} \text{ and } |I_3| < \frac{\varepsilon}{3}, \text{ for } 0 < y \le y_0, n \in \mathbb{N}. \tag{2}$$

The integral I_2 can be written in the form

$$I_2 = \langle \hat{f}_n(z) - \hat{f}_m(z), \varphi\rangle - \langle \hat{f}_n(\overline{z}) - \hat{f}_m(\overline{z}), \varphi\rangle.$$

By the uniform convergence of $\{\hat{f}_n(z)\}$ on $[a,b] \times [\delta, T]$ and $\{\hat{f}_n(\overline{z})\}$ on $[a,b] \times [-T, -\delta]$, we have that $\{\hat{f}_n(z)\}$ and $\{\hat{f}_n(\overline{z})\}$ are Cauchy sequences. This means that there exists $n_0 \in \mathbb{N}$, such that

$$|I_2| < \frac{\varepsilon}{3}, \text{ for } n, m > n_0. \tag{3}$$

(2) and (3) imply that $\{f_n\}$ is a Cauchy sequence in D', so there exists $\Lambda \in D'$, such that

$$\lim_{n\to\infty} \langle f_n, \varphi\rangle = \langle \Lambda, \varphi\rangle, \ \varphi \in D. \tag{4}$$

We will prove that $\Lambda \in L^p$. Clearly Λ is linear on D. For $\varphi \in D \subset L^q(\mathbb{R})$, applying the Hölder's inequality, we get

$$\left| \int_{-\infty}^{+\infty} f_n(x)\varphi(x)dx \right| \le \|f_n\|_p \cdot \|\varphi\|_q. \tag{5}$$

The definition (4) of Λ, the estimates (5) and the assumption of the theorem that $\{f_n\}$ is bounded sequence in $L^p(\mathbb{R})$, imply that Λ is bounded on D. By the Hahn-Banach theorem, Λ can be extended to a linear and continuous functional f on L^q, $q = \frac{p}{p-1}$. Thus, $\Lambda = f \in L^p$ and $\{f_n\}$ converges weakly to f in $L^p(\mathbb{R})$.

Last, we will prove that $\hat{f}(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$, is the analytic representation of this function f.

For arbitrary $\varphi \in D$, we have

$$\begin{aligned}\langle f, \varphi\rangle - \langle \hat{f}(z) - \hat{f}(\overline{z}), \varphi\rangle =& \{\langle f, \varphi\rangle - \langle f_n, \varphi\rangle\} + \{\langle f_n, \varphi\rangle - \langle \hat{f}_n(z) - \hat{f}_n(\overline{z}), \varphi\rangle\} \\ &+ \{\langle \hat{f}_n(z) - \hat{f}_n(\overline{z}), \varphi\rangle - \langle \hat{f}(z) - \hat{f}(\overline{z}), \varphi\rangle\} = J_1 + J_2 + J_3.\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary chosen.

Because $\{f_n\}$ converges weakly to f, we get that there exist $n_0 \in \mathbb{N}$, such that $|J_1| < \frac{\varepsilon}{3}$ for $n > n_0$.

Since $\hat{f}_n(z)$ is the analytic representation of f_n, for $n \in \mathbb{N}$, we get that there exists y_0 such that $|J_2| < \frac{\varepsilon}{3}$, for $y \le y_0$.

Finally, the assumption for the uniform convergence of the sequence $\{\hat{f}_n(z)\}$ to the function $\hat{f}(z)$, and the fact that $J_3 = \langle \hat{f}_n(z) - \hat{f}(z), \varphi\rangle - \langle \hat{f}_n(\overline{z}) - \hat{f}(\overline{z}), \varphi\rangle$, imply that $|J_3| < \frac{\varepsilon}{3}$, for $n > n_0$.

All together implies that $\hat{f}_n(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$, is the analytic representation of f. $\square$

Theorem 2.3. *Let $\{f_n(t)\}$ be a sequence of functions in $L^p(\mathbb{R})$, $1 \le p < \infty$ that converges to $f(t)$ in $L^p(\mathbb{R})$. Let $\{P_n(t)\}$ be a sequence of real analytic functions without real roots, and such that it converges to $P(t)$ uniformly on compact subsets of $\mathbb{R}$. Suppose that $P_0(t)$ is real analytic function without real roots such that $|P_n(t)| \le P_0(t)$, $t \in \mathbb{R}$, for every $n \in \mathbb{N}$.*

Then, for every $n \in \mathbb{N}$,

$$\frac{P_0(z)}{2\pi i}\left\langle \frac{f_n(t)P_n(t)}{P_0(t)}, \frac{1}{t-z}\right\rangle - \frac{P_0(\overline{z})}{2\pi i}\left\langle \frac{f_n(t)P_n(t)}{P_0(t)}, \frac{1}{t-\overline{z}}\right\rangle, \ z = x+iy \in \mathbb{C}\setminus\mathbb{R},$$

converges to $f_n(t)P_n(t)$ in D' as y tends to 0^+.

Moreover,

$$\frac{P_0(z)}{2\pi i}\left\langle \frac{f_n(t)P_n(t)}{P_0(t)}, \frac{1}{t-z}\right\rangle - \frac{P_0(\overline{z})}{2\pi i}\left\langle \frac{f_n(t)P_n(t)}{P_0(t)}, \frac{1}{t-\overline{z}}\right\rangle$$

converges uniformly on $[a,b]\times[\delta,T]$, $a<b$, $0<\delta<T$ to the function

$$\frac{P_0(z)}{2\pi i}\left\langle \frac{f(t)P(t)}{P_0(t)}, \frac{1}{t-z}\right\rangle - \frac{P_0(\overline{z})}{2\pi i}\left\langle \frac{f(t)P(t)}{P_0(t)}, \frac{1}{t-\overline{z}}\right\rangle \qquad (*)$$

as n tends to ∞.

By $(*)$ is given the analytic representation of fP i.e.

$$\frac{P_0(z)}{2\pi i}\left\langle \frac{f(t)P(t)}{P_0(t)}, \frac{1}{t-z}\right\rangle - \frac{P_0(\overline{z})}{2\pi i}\left\langle \frac{f(t)P(t)}{P_0(t)}, \frac{1}{t-\overline{z}}\right\rangle$$

converges to $f(t)P(t)$ in D' as y tends to 0^+.

Proof. Since $P_n(t) \to P(t)$ as $n \to \infty$ uniformly on compact sets, it follows that for $n > n_0$, all are of the same degree as P. So, we assume that it holds for every $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be fixed. We will prove that, for $\varphi \in D$

$$\lim_{y\to 0^+}\frac{1}{2\pi i}\int_{\mathbb{R}}\left[P_0(z)\left\langle \frac{f_n(t)P_n(t)}{P_0(t)}, \frac{1}{t-z}\right\rangle - P_0(\overline{z})\left\langle \frac{f_n(t)P_n(t)}{P_0(t)}, \frac{1}{t-\overline{z}}\right\rangle\right]\varphi(x)dx$$
$$=\langle f_nP_n, \varphi\rangle.$$

The integral on the left hand side equals

$$\frac{1}{2\pi i}\int_{\mathbb{R}} P_0(z)\left\langle \frac{f_n(t)P_n(t)}{P_0(t)}, \frac{1}{t-z} - \frac{1}{t-\overline{z}}\right\rangle\varphi(x)dx + \frac{1}{2\pi i}\int_{\mathbb{R}}[P_0(z)-P_0(\overline{z})]\left\langle \frac{f_n(t)P_n(t)}{P_0(t)}, \frac{1}{t-\overline{z}}\right\rangle\varphi(x)dx = I_1 + I_2.$$

The use of Fubini's theorem gives

$$I_1 = \int_{\mathbb{R}} \frac{f_n(t)P_n(t)}{P_0(t)}\left[\frac{y}{\pi}\int_{\mathbb{R}}\frac{P_0(z)\varphi(x)}{|t-z|^2}dx\right]dt.$$

Since $\varphi \in D$, there exists $a > 0$, such that $supp\varphi \subset [-a,a]$. So, we have

$$\int_{\mathbb{R}}\frac{P_0(z)\varphi(x)}{|t-z|^2}dx = \int_{-a}^{a}\frac{P_0(z)\varphi(x)}{|t-z|^2}dx = \int_{-a}^{a}\frac{P_0(x+iy)-P_0(x)}{|t-z|^2}\varphi(x)dx + \int_{-a}^{a}\frac{P_0(x)}{|t-z|^2}\varphi(x)dx = J_1 + J_2.$$

Thus,

$$I_1 = \int_{\mathbb{R}} \frac{f_n(t)P_n(t)}{P_0(t)} \frac{y}{\pi}(J_1 + J_2)dt. \tag{6}$$

$$|J_1| = \left| \int_{-a}^{a} \frac{P_0(x+iy) - P_0(x)}{|t-z|^2} \varphi(x)dx \right| \leq \int_{-a}^{a} \frac{|P_0(x+iy) - P_0(x)|}{|t-z|^2} |\varphi(x)|dx.$$

Let $\delta > 0$ be such that $|y| < \delta$. Then, for every $\varepsilon > 0$, we have that $|P_0(x+iy) - P_0(x)| < \varepsilon$, which implies that $\lim\limits_{y\to 0+} J_1 = 0$.

Now,

$$\lim_{y\to 0^+} \int_{\mathbb{R}} \frac{f_n(t)P_n(t)}{P_0(t)} \frac{y}{\pi} J_1 dt = 0 \tag{7}$$

$$\int_{\mathbb{R}} \frac{f_n(t)P_n(t)}{P_0(t)} \frac{y}{\pi} J_2 dt = \int_{\mathbb{R}} \frac{f_n(t)P_n(t)}{P_0(t)} \left[\frac{y}{\pi} \int_{-a}^{a} \frac{P_0(x)}{|t-z|^2} \varphi(x)dx \right] dt$$

converges to $\int_{\mathbb{R}} \frac{f_n(t)P_n(t)}{P_0(t)} P_0(t)\varphi(t)dt = \int_{\mathbb{R}} f_n(t)P_n(t)\varphi(t)dt$ as $y \to 0^+$ since $\left\langle \frac{f_n(t)P_n(t)}{P_0(t)}, \frac{1}{t-z} \right\rangle$ is analytic representation for $\frac{f_n(t)P_n(t)}{P_0(t)}$.

So,

$$\lim_{y\to 0^+} \int_{\mathbb{R}} \frac{f_n(t)P_n(t)}{P_0(t)} \frac{y}{\pi} J_2 dt = \int_{\mathbb{R}} f_n(t)P_n(t)\varphi(t)dt. \tag{8}$$

By Fubini's theorem, we get

$$I_2 = \int_{\mathbb{R}} [P_0(z) - P_0(\overline{z})] \left[\int_{\mathbb{R}} \frac{f_n(t)P_n(t)}{P_0(t)(t-\overline{z})} dt \right] \varphi(x)dx = \int_{\mathbb{R}} \frac{f_n(t)P_n(t)}{P_0(t)} \left[\int_{\mathbb{R}} \frac{P_0(z) - P_0(\overline{z})}{t-\overline{z}} \varphi(x)dx \right] dt.$$

Consider

$$\int_{\mathbb{R}} \frac{P_0(z) - P_0(\overline{z})}{t-\overline{z}} \varphi(x)dx = \int_{\mathbb{R}} \frac{P_0(z) - P_0(\overline{z})}{t-\overline{z}} [\varphi(x) - \varphi(t)]dx + \int_{\mathbb{R}} \frac{P_0(z) - P_0(\overline{z})}{t-\overline{z}} \varphi(t)dx = K_1 + K_2.$$

So,

$$I_2 = \int_{\mathbb{R}} \frac{f_n(t)P_n(t)}{P_0(t)} (K_1 + K_2)dt. \tag{9}$$

For $\varepsilon > 0$ there exists $\delta > 0$ such that $|P_0(z) - P_0(\overline{z})| < \varepsilon$, for $|y| < \delta$. Then $t \mapsto \frac{\varphi(x) - \varphi(t)}{x - t}$ is integrable, and $\left| \frac{\varphi(x) - \varphi(t)}{t - \overline{z}} \right| = \left| \frac{\varphi(x) - \varphi(t)}{t - x + iy} \right| \leq \left| \frac{\varphi(x) - \varphi(t)}{t - x} \right|$, for all $y \in \mathbb{R}$. These arguments and the use of the Lebesque dominated convergence theorem, give that

$$\lim_{y\to 0^+} \int_{\mathbb{R}} \frac{P_0(z) - P_0(\overline{z})}{t-\overline{z}} [\varphi(x) - \varphi(t)]dx = 0,$$

i.e.

$$\lim_{y\to 0^+} K_1 = 0. \tag{10}$$

For K_2, we have that

$$\lim_{y\to 0^+} K_2 = 0. \tag{11}$$

Now by (6), (7), (8), (9),(10) and (11) we have

$$\lim_{y\to 0^+} \int\limits_{\mathbb{R}} \left[\frac{P_0(z)}{2\pi i} \left\langle \frac{f_n(t)P_n(t)}{P_0(t)}, \frac{1}{t-z} \right\rangle - \frac{P_0(\overline{z})}{2\pi i} \left\langle \frac{f_n(t)P_n(t)}{P_0(t)}, \frac{1}{t-\overline{z}} \right\rangle \right] \varphi(x)dx = \langle f_nP_n, \varphi\rangle,\ \varphi \in D.$$

Now we fix the set $[a,b]\times[\delta,T]$. For $z \in [a,b]\times[\delta,T]$ we have

$$\begin{aligned}
&\frac{P_0(z)}{2\pi i} \left\langle \frac{f_n(t)P_n(t)}{P_0(t)}, \frac{1}{t-z} \right\rangle - \frac{P_0(\overline{z})}{2\pi i} \left\langle \frac{f_n(t)P_n(t)}{P_0(t)}, \frac{1}{t-\overline{z}} \right\rangle \\
&-\left[\frac{P_0(z)}{2\pi i} \left\langle \frac{f(t)P(t)}{P_0(t)}, \frac{1}{t-z} \right\rangle - \frac{P_0(\overline{z})}{2\pi i} \left\langle \frac{f(t)P(t)}{P_0(t)}, \frac{1}{t-\overline{z}} \right\rangle \right] \\
&=\left[\frac{P_0(z)}{2\pi i} \left\langle \frac{f_n(t)P_n(t)}{P_0(t)}, \frac{1}{t-z} \right\rangle - \frac{P_0(z)}{2\pi i} \left\langle \frac{f(t)P(t)}{P_0(t)}, \frac{1}{t-z} \right\rangle \right] \\
&-\left[\frac{P_0(\overline{z})}{2\pi i} \left\langle \frac{f_n(t)P_n(t)}{P_0(t)}, \frac{1}{t-\overline{z}} \right\rangle - \frac{P_0(\overline{z})}{2\pi i} \left\langle \frac{f(t)P(t)}{P_0(t)}, \frac{1}{t-\overline{z}} \right\rangle \right] \\
&=A+B.
\end{aligned}$$

$$\begin{aligned}
A &= \frac{P_0(z)}{2\pi i} \int\limits_{\mathbb{R}} \frac{f_n(t)P_n(t)}{P_0(t)(t-z)}dt - \frac{P_0(z)}{2\pi i} \int\limits_{\mathbb{R}} \frac{f(t)P(t)}{P_0(t)(t-z)}dt = \frac{P_0(z)}{2\pi i} \int\limits_{\mathbb{R}} \frac{f_n(t)P_n(t) - f(t)P(t)}{P_0(t)(t-z)}dt \\
&= \frac{P_0(z)}{2\pi i} \int\limits_{\mathbb{R}} \frac{f_n(t)-f(t)}{P_0(t)(t-z)}P_n(t)dt + \frac{P_0(z)}{2\pi i} \int\limits_{\mathbb{R}} \frac{P_n(t)-P(t)}{P_0(t)(t-z)}f_n(t)dt = A_1 + A_2.
\end{aligned}$$

Since $\{f_n\}$ converges to f in $L^p(\mathbb{R})$, $\dfrac{1}{t-z} \in L^q$ and $\dfrac{P_n(t)}{P_0(t)}$ is bounded function, applying Holder's inequality, we get that $\lim\limits_{n\to\infty} A_1 = 0$. Similarly, $\lim\limits_{n\to\infty} A_2 = 0$ i.e. $\lim\limits_{n\to\infty} A = 0$.
Analogously,

$$\begin{aligned}
B &= \frac{P_0(\overline{z})}{2\pi i} \int\limits_{\mathbb{R}} \frac{f_n(t)P_n(t)}{P_0(t)} \frac{1}{(t-\overline{z})}dt - \frac{P_0(\overline{z})}{2\pi i} \int\limits_{\mathbb{R}} \frac{f(t)P(t)}{P_0(t)} \frac{1}{(t-\overline{z})}dt \\
&= \frac{P_0(\overline{z})}{2\pi i} \int\limits_{\mathbb{R}} \frac{f_n(t)-f(t)}{P_0(t)(t-\overline{z})}P_n(t)dt + \frac{P_0(\overline{z})}{2\pi i} \int\limits_{\mathbb{R}} \frac{P_n(t)-P(t)}{P_0(t)(t-\overline{z})}f(t)dt
\end{aligned}$$

and $\lim\limits_{n\to\infty} B = 0$.

The proof of the last part of the theorem is similar to the proof of the first part and we omit it. □

Theorem 2.4. *Let $\{f_n\}$ be a sequence of functions $L^p(\mathbb{R})$, $1 \le p < \infty$. Let $\{P_n\}$ be a sequence of polynomials (of the same degree) that converges to P uniformly on compact subsets of $\mathbb{R}$ as $n \to \infty$. Let P_0 be a polynomial without real roots on $\mathbb{R}$ such that $|P_n| \le P_0$, for every $n \in \mathbb{N}$ i.e. $\{\frac{P_n}{P_0}\}$ is bounded sequence in $L^\infty(\mathbb{R})$.*

For $H_n(z) = \dfrac{P_0(z)}{2\pi i}\left\langle \dfrac{f_n(t)P_n(t)}{P_0(t)}, \dfrac{1}{t-z}\right\rangle$ and

$H_n(\overline{z}) = \dfrac{P_0(\overline{z})}{2\pi i}\left\langle \dfrac{f_n(t)P_n(t)}{P_0(t)}, \dfrac{1}{t-\overline{z}}\right\rangle$, assume that $\{H_n(z) - H_n(\overline{z})\}$ converges uniformly on sets of the form $[a,b] \times [\delta, T]$ and $[a,b]\times[-T,-\delta]$, $a < b$, $0 < \delta < T$, to $H(z)-H(\overline{z})$ as $n \to \infty$. Then there exists a function $f \in L^p(\mathbb{R})$ such that $\left\{\dfrac{f_n \cdot P_n}{P_0}\right\}$ converges weakly to $\dfrac{f\cdot P}{P_0}$ as $n \to \infty$ and $H(z)-H(\overline{z}) = \dfrac{P_0(z)}{2\pi i}\left\langle \dfrac{f(t)P(t)}{P_0(t)}, \dfrac{1}{t-z}\right\rangle - \dfrac{P_0(\overline{z})}{2\pi i}\left\langle \dfrac{f(t)P(t)}{P_0(t)}, \dfrac{1}{t-\overline{z}}\right\rangle$.

Proof. Let $H_n^*(z) = H_n(z) - H_n(\overline{z})$, $n \in \mathbb{N}$, $z = x + iy$, and $H^*(z) = H(z) - H(\overline{z})$. Let$\varepsilon > 0$ be arbitrary chosen. The uniform convergence of $\{H_n^*(z)\}$ on sets of the form $[a,b] \times [\delta, T]$ and $[a,b] \times [-T, -\delta]$, $a < b$, $0 < \delta < T$ implies that there exists $n_0 \in \mathbb{N}$, such that $|H_n^*(z) - H^*(z)| < \varepsilon$, for all $n > n_0$, and that, for $y \ne 0$, the boundary function $H^*(z)$ is continuous. So, for $y \ne 0$, $H^*(z) = H^*(x+iy)$ is continuous on x and bounded on closed intervals.

Let $\varphi \in D$ be arbitrary chosen and let supp$\varphi \subset [-\alpha, \alpha]$, $\alpha > 0$. The integral

$$\int_{-a}^{a} H_n^*(x+iy)\varphi(x)dx = \int_{-a}^{a} [H_n(x+iy) - H_n(x-iy)]\varphi(x)dx$$

converges uniformly to the integral

$$\int_{-a}^{a} H^*(x+iy)\varphi(x)dx = \int_{-a}^{a} [H(x+iy) - H(x-iy)]\varphi(x)dx$$

as $n \to \infty$, so, there exists $n_1 \in \mathbb{N}$, such that

$$\left|\int_{-a}^{a} H_n^*(x+iy)\varphi(x)dx - \int_{-a}^{a} H^*(x+iy)\varphi(x)dx\right| < \varepsilon,$$

for all $n > n_1$. Let us now fix $n \in \mathbb{N}$, and let $y \to 0^+$. The uniform convergence of $\{H_n^*(z)\}$ implies that

$$\lim_{y\to 0^+} \int_{\mathbb{R}} [H_n(x+iy) - H_n(x-iy)]\varphi(x)dx = \langle \Lambda, \varphi \rangle.$$

We will prove that Λ is continuous linear functional. Since $D \subset L^q$ is dense and D is of second category, the Banach-Steinhaus theorem implies that $\lim_{y\to 0^+} \int_{\mathbb{R}} H_n^*(x+iy)g(x)dx$, for $g \in L^q$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, exists. One more application of the same theorem gives that Λ is continuous linear functional on L^q. So, there exists $f \in (L^q)' = L^p$ such that $\langle \Lambda, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x)dx$, for $\varphi \in D \subset L^q$.

Since, $f \in L^p$, it has a Cauchy representation $\widehat{f}(z) = \dfrac{1}{2\pi i}\left\langle f(t), \dfrac{1}{t-z}\right\rangle$, which implies that $\dfrac{P_0(z)}{2\pi i}\left\langle \dfrac{f(t)\cdot P(t)}{P_0(t)}, \dfrac{1}{t-z}\right\rangle$ is analytic representation for $f \cdot P$. In the same way, by the assumption, for $n \in \mathbb{N}$, $f_n \in L^p$, so $\dfrac{P_0(z)}{2\pi i}\left\langle \dfrac{f_n(t)\cdot P_n(t)}{P_0(t)}, \dfrac{1}{t-z}\right\rangle$ is analytic representation for $f_n \cdot P_n$.

Now, we will prove that $\left\{\dfrac{f_n \cdot P_n}{P_0}\right\}$ converges weakly to $\dfrac{f \cdot P}{P_0}$. For $\varphi \in D$, we have

$$\begin{aligned}\left\langle \frac{f_n \cdot P_n}{P_0}, \varphi \right\rangle - \left\langle \frac{f \cdot P}{P_0}, \varphi \right\rangle = &\left\{ \left\langle \frac{f_n \cdot P_n}{P_0}, \varphi \right\rangle - \left[\frac{P_0(z)}{2\pi i} \left\langle \frac{f_n \cdot P_n}{P_0}, \frac{1}{t-z} \right\rangle - \frac{P_0(\overline{z})}{2\pi i} \left\langle \frac{f_n \cdot P_n}{P_0}, \frac{1}{t-\overline{z}} \right\rangle \right] \right\} \\ &+ \left\{ \left[\frac{P_0(z)}{2\pi i} \left\langle \frac{f_n \cdot P_n}{P_0}, \frac{1}{t-z} \right\rangle - \frac{P_0(\overline{z})}{2\pi i} \left\langle \frac{f_n \cdot P_n}{P_0}, \frac{1}{t-\overline{z}} \right\rangle \right] \right. \\ &\left. - \left[\frac{P_0(z)}{2\pi i} \left\langle \frac{f \cdot P}{P_0}, \frac{1}{t-z} \right\rangle - \frac{P_0(\overline{z})}{2\pi i} \left\langle \frac{f \cdot P}{P_0}, \frac{1}{t-\overline{z}} \right\rangle \right] \right\} \\ &+ \left\{ \left[\frac{P_0(z)}{2\pi i} \left\langle \frac{f \cdot P}{P_0}, \frac{1}{t-z} \right\rangle - \frac{P_0(\overline{z})}{2\pi i} \left\langle \frac{f \cdot P}{P_0}, \frac{1}{t-\overline{z}} \right\rangle \right] - \left\langle \frac{f \cdot P}{P_0}, \varphi \right\rangle \right\} \\ =& I_1 + I_2 + I_3. \end{aligned}$$

The first integral I_1 converges to 0 because $\frac{P_0(z)}{2\pi i} \left\langle \frac{f_n P_n}{P_0}, \frac{1}{t-z} \right\rangle$ is analytic representation for $f_n \cdot P_n$. In the same way, the third integral I_3 converges to 0 because $\frac{P_0(z)}{2\pi i} \left\langle \frac{fP}{P_0}, \frac{1}{t-z} \right\rangle$ is analytic representation for $f \cdot P$. As for the second integral I_2, it converges to 0 because of the assumption of the theorem for the uniform convergence of the sequence $\{H_n(z) - H_n(\overline{z})\}$. $\square$

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