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A Fixed Point Theorem for G-Monotone Multivalued Mapping with Application to Nonlinear Integral Equations

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Abstract. We extend notion and theorem of [21] to the case of a multivalued mapping defined on a metric space endowed with a finite number of graphs. We also construct an example to show the generality of our result over existing results. Finally, we give an application to nonlinear integral equations.

1. Introduction

Graph theory is a powerful tool in solving various optimization problems of interest in pure and applied sciences. Thus, we refer to the usual notions and notations of graph theory, see for instance [9]. On the other hand, fixed point theory is also useful in dealing with practical situations, where the problem under investigation can be equivalently formulated as a fixed point problem. Both topics are investigated not only by a theoretical point of view, but also are used to obtain suitable algorithms for approximation of the problems under investigation. Jachymski in [9], merged above theories to have a generalization of the Banach contraction principle for mappings of a metric space endowed with a graph. Then, Beg *et al*. [3] extended some results of Jachymski to multivalued mappings; other generalizations of [9] are available in [1, 4, 5, 10, 15, 17, 18]. For completeness, we recall that Nadler [14] first extended the Banach contraction principle to multivalued mappings; then, Nadler's fixed point theorem has been generalized and extended in several directions, see for example [2, 6, 8, 11–13, 16, 19]. On these bases, we extend the recent results in [21] to the case of multivalued mappings defined on a metric space endowed with a finite number of graphs. Also, we give an example and an application to nonlinear integral equations.
We recollect some concepts for our further use. Let (V, d) be a metric space and $G = \{G_i : 1 \leq i \leq q\}$ be a family of graphs such that $G_i = (V, E_i)$, $E_i \subseteq V \times V$ for each $i \in \{1, 2, \cdots, q\}$. Let $CB(V)$ stand for the set of all non-empty closed and bounded subsets of V.

Definition 1.1. *[20] Let (V, d) be a metric space endowed with a graph $G = (V, E)$. A mapping $f : V \to CB(V)$ is said to be graph-preserving multivalued mapping if*

$$(x, y) \in E \Rightarrow\ (v, w) \in E \ \text{ for each } v \in fx \ \text{ and } w \in fy.$$

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Example 1.2. *Let $G = (V, E)$, where $V = \{1, 2, 3, 4, 6, 8\}$ and*
$E = \{(1, 1), (1, 3), (2, 2), (2, 4), (2, 6), (2, 8), (4, 2), (4, 4), (4, 6), (4, 8), (6, 8)\}$.
Define $f : V \to CB(V)$ by:

$$fx = \begin{cases} \{2,4\} & \text{if } x \in \{1,4\}, \\ \{6,8\} & \text{if } x = 3, \\ \{2\} & \text{if } x \in \{2,6,8\}. \end{cases}$$

Clearly, f is graph-preserving multivalued mapping.

Definition 1.3. *[21] The pair (G, d) is said to be regular if the following condition holds:*
if $\{x_n\}$ is a sequence in V and x is a point in V such that

(i) for all $i \in \{1, 2, \cdots, q\}$, there exists a subsequence $\{x_{m_{i,k}}\}$ of $\{x_n\}$ with $(x_{m_{i,k}}, x_{m_{i,k+1}}) \in E_i$, for all k,

(ii) $d(x_n, x) \to 0$ as $n \to \infty$,

then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a rank $j \in \{1, 2, \cdots, q\}$ such that $(x_{n_k}, x) \in E_j$ for all k.

Example 1.4. *Let $G = (V, E)$ be a graph with $V = C([0, 1], \mathbb{R})$, the set of continuous real functions defined on $[0, 1]$, and $E \subseteq V \times V$ given by*

$$(x, y) \in E \Leftrightarrow x(t) \le y(t) \text{ for all } t \in [0, 1].$$

Choose the metric d on V defined by

$$d(x, y) = \sup_{t \in [0,1]} |x(t) - y(t)| \text{ for all } x, y \in V.$$

Let $\{x_n\}$ be a sequence in V and $x \in V$ such that

(a) there exists a subsequence $\{x_{m_k}\}$ of $\{x_n\}$ such that $x_{m_k}(t) \le x_{m_{k+1}}(t)$, holds for all $k \in \mathbb{N}$ and $t \in [0, 1]$,

(b) $d(x_n, x) \to 0$ as $n \to \infty$,

then $(x_{m_k}, x) \in E$ for all $k \in \mathbb{N}$. Thus, (G, d) is regular.

Let $\mathbb{R}_+ = [0, \infty)$ and let Ψ represent the class of functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(i) ψ is increasing;

(ii) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$.

Remark 1.5. *Since each function $\psi \in \Psi$ satisfies condition (ii) above by definition, then we have that $\lim_{n\to\infty} \psi^n(t) = 0$, for each $t > 0$. Also, notice that conditions (i) and (ii) imply that $\psi(t) < t$, for each $t > 0$.*

Lemma 1.6. *[14] Let (X, d) be a metric space and $A, B \in CB(X)$ such that $H(A, B) > 0$. Then, for each $q > 1$ and for each $a \in A$ there exists $b \in B$ such that $d(a, b) < qH(A, B)$.*

2. Main Results

We start this section by introducing the notion of G-monotone multivalued mapping.

Definition 2.1. *Let (V, d) be a metric space endowed with a finite number of graphs $\{G_i\}_{i=1}^q$. A multivalued mapping $f : V \to CB(V)$ is said to be G-monotone, if $(x, y) \in E_i$ implies $(u, v) \in E_{i+1}$ for all $v \in fx$ and $w \in fy$, for each $i \in \{1, 2, \cdots, q\}$, with $E_{q+1} = E_1$.*

Example 2.2. *Consider the sets* $V = \mathbb{N}, E_1 = \{(2k-1, 2k+1) : k \in \mathbb{N}\} \cup \{(1,1)\}$ *and* $E_2 = \{(2k, 2k+2) : k \in \mathbb{N}\} \cup \{(2k, 2k+4) : k \in \mathbb{N}\} \cup \{(2k, 2k) : k \in \mathbb{N}\} \cup \{(4,2)\}$. *Let* $G = \{G_i : i = 1,2\}$ *be the family of graphs* $G_i = (V, E_i), i \in \{1,2\}$. *Define* $f : \mathbb{N} \to CB(\mathbb{N})$ *by:*

$$fx = \begin{cases} \{2n, 2n+2\} & \text{if } x = 2n-1, \\ \{1\} & \text{if } x = 2n, \end{cases}$$

where $n \in \mathbb{N}$. *Let* $(x,y) \in E_1$. *If* $(x,y) = (2n-1, 2n+1)$, *then* $fx = \{2n, 2n+2\}, fy = \{2n+2, 2n+4\}$ *and* $(2n, 2n+2) \in E_2, (2n, 2n+4) \in E_2, (2n+2, 2n+2) \in E_2, (2n+2, 2n+4) \in E_2$.
Also if $(x,y) = (1,1)$, *then* $f1 = \{2,4\}$ *and* $(2,2), (2,4), (4,2), (4,4) \in E_2$.
Now consider $(x,y) \in E_2$. *If* $(x,y) = (2n, 2n+2)$ *or* $(x,y) = (2n, 2n+4)$ *or* $(x,y) = (2n, 2n)$ *or* $(x,y) = (4,2)$, *then* $fx = fy = \{1\}$ *and* $(1,1) \in E_1$. *Hence* f *is G-monotone.*

Now, we state our result.

Theorem 2.3. *Let* (V,d) *be a complete metric space endowed with a finite number of graphs* $\{G_i\}_{i=1}^{q}$ *and let* $f : V \to CB(V)$ *be a G-monotone multivalued mapping. Suppose that the following conditions hold:*

(a) there exist $x_0 \in V$ *and* $u \in fx_0$ *such that* $(x_0, u) \in E_1$;

(b) (G,d) *is regular;*

(c) there exist $\psi \in \Psi$ *and a lower semi-continuous function* $\varphi : V \to [0,\infty)$ *such that*

$$H(fx, fy) + \varphi(u) + \varphi(w) \leqslant \psi(d(x,y) + \varphi(x) + \varphi(y)), \tag{1}$$

for all $u \in fx$ *and for all* $w \in fy$, *whenever* $(x,y) \in E_i$, *for any* $i \in \{1, 2, \cdots, q\}$, *with* $E_{q+1} = E_1$.

Then f *has a fixed point.*

Proof. By hypothesis of theorem, we have $x_0 \in V$ and $x_1 \in fx_0$ with $(x_0, x_1) \in E_1$. If $x_1 = x_0$, then x_0 is a fixed point of f. Therefore, suppose that $x_1 \neq x_0$ and denote

$$t_0 = d(x_0, x_1) + \varphi(x_0) + \varphi(x_1).$$

By Lemma 1.6, it follows that for $q > 1$, there exists $x_2 \in fx_1$ such that

$$\begin{aligned} d(x_1, x_2) + \varphi(x_1) + \varphi(x_2) &< q(H(fx_0, fx_1) + \varphi(x_1) + \varphi(x_2)) \\ &\leq q(\psi(d(x_0, x_1) + \varphi(x_0) + \varphi(x_1))) \\ &= q\psi(t_0). \end{aligned}$$

Since $\psi \in \Psi$ is increasing, we have

$$\psi(d(x_1, x_2) + \varphi(x_1) + \varphi(x_2)) < \psi(q\psi(t_0)).$$

Let $q_1 = \frac{\psi(q\psi(t_0))}{\psi(d(x_1,x_2)+\varphi(x_1)+\varphi(x_2))} > 1$. If $x_1 = x_2$, then x_1 is a fixed point. Suppose $x_1 \neq x_2$. Since f is G-monotone, we have $(x_1, x_2) \in E_2$. Again by using (1) and Lemma 1.6 there exists $x_3 \in fx_2$ such that

$$\begin{aligned} d(x_2, x_3) + \varphi(x_2) + \varphi(x_3) &< q_1(H(fx_1, fx_2) + \varphi(x_2) + \varphi(x_3)) \\ &\leq q_1\psi(d(x_1, x_2) + \varphi(x_1) + \varphi(x_2)) \\ &= \psi(q\psi(t_0)). \end{aligned}$$

Continuing in the same manner, we obtain a sequence $\{x_n\} \subset V$ such that $x_{n+1} \in fx_n$, and also $x_{n+1} \neq x_n$ for each $n \in \{0\} \cup \mathbb{N}$. Since f is G-monotone, for each $n \in \{0\} \cup \mathbb{N}$, there exists $i = i(n) \in \{1, 2, \cdots, q\}$ such that $(x_n, x_{n+1}) \in E_i$ and

$$d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) < \psi^{n-1}(q\psi(t_0)).$$

Letting $n \to \infty$ in above inequality and by using property (ii) of ψ and Remark 1.5, we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) = 0.$$

Consequently,

$$d(x_n, x_{n+1}) \to 0 \text{ and } \varphi(x_n) \to 0 \text{ as } n \to \infty.$$

From

$$\begin{aligned}\sum_{n=1}^{\infty} d(x_n, x_{n+1}) &\leq \sum_{n=1}^{\infty}[d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})]\\ &\leq \sum_{n=1}^{\infty} \psi^{n-1}(q\psi(t_0)) < \infty,\end{aligned}$$

we get that $\{x_n\}$ is a Cauchy sequence. Completeness of (V, d) ensures the existence of $x^* \in V$ such that $x_n \to x^*$ as $n \to \infty$. Since (G, d) is regular, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $j \in \{1, 2, \cdots, q\}$ such that $(x_{n_k}, x^*) \in E_j$ for all k. Since φ is lower semi-continuous, we get

$$\varphi(x^*) \leq \liminf_{n\to\infty} \varphi(x_n) = 0$$

and hence $\varphi(x^*) = 0$. As $(x_{n_k}, x^*) \in E_j$ for all k, by using triangular inequality, (1) and $\psi(t) < t$ for all $t > 0$, we get

$$\begin{aligned}d(x^*, fx^*) &\leq d(x^*, x_{n_{k+1}}) + d(x_{n_{k+1}}, fx^*)\\ &\leq d(x^*, x_{n_{k+1}}) + H(fx_{n_k}, fx^*)\\ &\leq d(x^*, x_{n_{k+1}}) + \psi(d(x_{n_k}, x^*) + \varphi(x_{n_k}) + \varphi(x^*))\\ &< d(x^*, x_{n_{k+1}}) + d(x_{n_k}, x^*) + \varphi(x_{n_k}) + \varphi(x^*),\end{aligned}$$

for all k. Letting $k \to \infty$ in above inequality, we get $d(x^*, fx^*) = 0$. □

Letting $\psi(t) = \alpha t$ for each $t \geq 0$, where $\alpha \in [0, 1)$. Theorem 2.3 reduces to following corollary.

Corollary 2.4. *Let (V, d) be a complete metric space endowed with a finite number of graphs $\{G_i\}_{i=1}^q$ and let $f : V \to CB(V)$ be a G-monotone multivalued mapping. Suppose that the following conditions hold:*

(a) there exist $x_0 \in V$ and $u \in fx_0$ such that $(x_0, u) \in E_1$;

(b) (G, d) is regular;

(c) there exist $\alpha \in [0, 1)$ and a lower semi-continuous function $\varphi : V \to [0, \infty)$ such that

$$H(fx, fy) + \varphi(u) + \varphi(w) \leqslant \alpha[(d(x, y) + \varphi(x) + \varphi(y)],$$

for all $u \in fx$ and for all $w \in fy$, whenever $(x, y) \in E_i$ for any $i \in \{1, 2, \cdots, q\}$, with $E_{q+1} = E_1$.

Then f has a fixed point.

Corollary 2.4, reduces to following corollary by taking $V = X$, $q = 1$ and $E_1 = X \times X$.

Corollary 2.5. *Let (X, d) be a complete metric space and let $f : X \to CB(X)$ be a multivalued mapping. Also suppose that there exist $\alpha \in [0, 1)$ and a lower semi-continuous function $\varphi : X \to [0, \infty)$ such that*

$$H(fx, fy) + \varphi(u) + \varphi(w) \leqslant \alpha[(d(x, y) + \varphi(x) + \varphi(y)], \tag{2}$$

for all $x, y \in X$, $u \in fx$ and $w \in fy$. Then f has a fixed point.

Remark 2.6. *From Corollary 2.5, we obtain Nadler's fixed point theorem, by taking $\varphi(x) = 0$ for each $x \in X$.*

3. Example and Application

3.1. Example

We give an example in which Nadler's fixed point theorem is not applicable, but Corollary 2.5 works.

Example 3.1. *Let $X = [0, 1]$ endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define $f : X \to CB(X)$ by*

$$fx = \begin{cases} \{0\} & \text{if } x \in [0, \frac{1}{2}), \\ \{0, \frac{1}{4}\} & \text{if } x \in [\frac{1}{2}, 1), \\ \{0, \frac{1}{2}\} & \text{if } x = 1. \end{cases}$$

Take $\alpha = \frac{1}{2}$ and observe that for $x \in (\frac{3}{4}, 1)$ and $y = 1$, we have $|x - y| < \frac{1}{4} = H(fx, fy)$. This implies that f is not a multivalued contraction. Thus Nadler's fixed point theorem is not applicable in this case.
Now, consider the mapping $\varphi : X \to [0, \infty)$ defined by $\varphi(x) = x$, for each $x \in X$. By following cases, we show that (2) holds true.

Case 1 *If $x = y = 1$, then $H(fx, fy) = 0$. For each $w \in fx$ and for each $v \in fy$, $H(fx, fy) + \varphi(w) + \varphi(v)$ attains the values $0, \frac{1}{2}, 1$. On the other hand, $d(x, y) + \varphi(x) + \varphi(y) = 1 + 1 = 2$.*

Case 2 *If $x \in [\frac{1}{2}, 1)$ and $y = 1$, then $H(fx, fy) = \frac{1}{4}$. For each $w \in fx$ and for each $v \in fy$, $H(fx, fy) + \varphi(w) + \varphi(v)$ attains the values $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. On the other hand, $d(x, y) + \varphi(x) + \varphi(y) = |x - 1| + x + 1 = 2$.*

Case 3 *If $x = 1$ and $y \in [0, \frac{1}{2})$, then $H(fx, fy) = \frac{1}{2}$. For each $w \in fx$ and for each $v \in fy$, $H(fx, fy) + \varphi(w) + \varphi(v)$ attains the values $\frac{1}{2}, 1$. On other hand, $d(x, y) + \varphi(x) + \varphi(y) = |1 - y| + 1 + y = 2$.*

Case 4 *If $x \in [\frac{1}{2}, 1)$ and $y \in [0, \frac{1}{2})$, then $H(fx, fy) = \frac{1}{4}$. For each $w \in fx$ and for each $v \in fy$, $H(fx, fy) + \varphi(w) + \varphi(v)$ attains the values $\frac{1}{4}, \frac{1}{2}$. On other hand, $1 \leq d(x, y) + \varphi(x) + \varphi(y) < 2$.*

Case 5 *If $x, y \in [\frac{1}{2}, 1)$, then $H(fx, fy) = 0$. For each $w \in fx$ and for each $v \in fy$, $H(fx, fy) + \varphi(w) + \varphi(v)$ attains the values $0, \frac{1}{4}, \frac{1}{2}$. On the other hand, $1 \leq d(x, y) + \varphi(x) + \varphi(y) < 2$.*

Case 6 *If $x, y \in [0, \frac{1}{2})$, then $H(fx, fy) = 0$. For each $w \in fx$ and for each $v \in fy$, $H(fx, fy) + \varphi(w) + \varphi(v) = 0$.*

From above six cases, we deduce that

$$H(fx, fy) + \varphi(w) + \varphi(v) \leq \frac{1}{2}[d(x, y) + \varphi(x) + \varphi(y)],$$

for all $x, y \in X$, $w \in fx$ and $v \in fy$. Since all the conditions of Corollary 2.5 holds, therefore f has a fixed point.

3.2. Application to nonlinear integral equation

We prove the existence of solution for certain nonlinear integral equations, by using an equivalent fixed point problem. We consider the set of all closed bounded real continuous functions on $[0, 1]$, say $X = CB([0, 1], \mathbb{R})$, endowed with the metric $d : X \times X \to \mathbb{R}$ given by

$$d(x, y) = \|x - y\|_\infty = \sup_{t \in [0,1]} |x(t) - y(t)|, \text{ for all } x, y \in X.$$

Clearly, (X, d) is a complete metric space, which can be equipped with the graph $G(V, E)$ with $V = X$ and $E \subseteq X \times X$ given by

$$(x, y) \in E \Leftrightarrow x(t) \leq y(t) \text{ for all } t \in [0, 1].$$

Thus, (G, d) is regular, see also Example 1.4.
It is well-known that nonlinear integral equations play a crucial role in modelling and solving many control

problems, arising in various sciences, see [7]. Here, we will prove the existence of solution for the following type of integral equation:

$$x(t) = p(t) + \int_0^t S(t,u)g(u,x(u))du, \quad t \in [0,1] \tag{3}$$

where $g : [0,1]\times\mathbb{R} \to \mathbb{R}$, $p : [0,1] \to \mathbb{R}$ are two closed bounded continuous functions and $S : [0,1]\times[0,1] \to [0,\infty)$ is a function such that $S(t,\cdot) \in L^1([0,1])$ for all $t \in [0,1]$.

Now, we consider the operator $f : X \to X$ defined by

$$f(x)(t) = p(t) + \int_0^t S(t,u)g(u,x(u))du \tag{4}$$

and observe that each fixed point of f is a solution of integral equation (3). Of course, f is well-defined since g and p are two closed bounded continuous functions. Thus, we state and prove the following result of existence of fixed point for (4), which reduces to a result of existence of solution for (3).

Theorem 3.2. *Let $f : X \to X$ be the integral operator given by* (4). *Suppose that the following conditions hold:*

(a) there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in E$;

(b) $g(u,\cdot) : \mathbb{R} \to \mathbb{R}$ is increasing, for every $u \in [0,1]$;

(c) for $x, y \in X$ with $(x,y) \in E$, we have

$$0 \leq g(u,y(u)) - g(u,x(u)) \leq \frac{1}{2}\ln(1 + |x(u) - y(u)|)$$

for every $u \in [0,1]$;

(d) for every $u \in [0,1]$, we have

$$\left\| \int_0^1 S(t,u)du \right\|_\infty < 1.$$

Then f has a fixed point.

Proof. By condition (b), the operator f is nondecreasing and hence f is trivially G-monotone. Also, by condition (c), for $x, y \in X$ with $(x,y) \in E(G)$, we write

$$\begin{aligned} |f(x)(t) - f(y)(t)| &= \left| \int_0^t S(t,u)[g(u,x(u)) - g(u,y(u))]du \right| \\ &\leq \int_0^t S(t,u)|g(u,x(u)) - g(u,y(u))|du \\ &\leq \int_0^t S(t,u)\frac{\ln(1 + |x(u) - y(u)|)}{2}du \\ &\leq \frac{1}{2}\ln(1 + \|x - y\|_\infty). \end{aligned}$$

Then we have

$$\left\| f(x) - f(y) \right\|_\infty \leq \frac{1}{2}\ln(1 + \|x - y\|_\infty)$$

and hence, for $x, y \in X$ with $(x,y) \in E$, we deduce that

$$d(f(x), f(y)) \leq \psi(d(x,y)),$$

where $\psi(r) = \frac{1}{2}\ln(1 + r)$ is such that $\psi \in \Psi$. Thus, all the conditions of Theorem 2.3 are immediately satisfied with $\varphi(x) = 0$ for each $x \in X$ and hence the operator f has a fixed point, which is a solution of the integral equation (3). □

As special case of Theorem 3.2, we give the following result for a fractional-order integral equation.

Corollary 3.3. *Let $f : X \to X$ be the integral operator defined by*

$$f(x)(t) = p(t) + \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} g(u, x(u))du, \quad t \in [0,1], \quad \alpha \in (0,1),$$

where Γ is the Euler gamma function given by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$. Suppose that the following conditions hold:

(a) there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in E$;

(b) $g(u, \cdot) : \mathbb{R} \to \mathbb{R}$ is increasing, for every $u \in [0,1]$;

(c) for $x, y \in X$ with $(x, y) \in E$, we have

$$0 \le g(u, y(u)) - g(u, x(u)) \le \frac{\Gamma(\alpha+1)}{2} \ln(1 + |x(u) - y(u)|)$$

for every $u \in [0,1]$.

Then f has a fixed point.

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