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On the Digitally Quasi Comultiplications of Digital Images

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Abstract. In this article we study the digitally quasi comultiplications of the digital wedge products of pointed digital images. After defining a digitally quasi co-H-space and a digital Whitehead product, we develop a method of how to calculate the cardinal number of digital homotopy classes based on the digitally quasi comultiplications of a pointed digital image as a particular case. We also construct a digitally quasi co-H-space as a digital retract of a given digitally quasi co-H-space.

1. Introduction

Kong [16] introduced the digital fundamental group of a discrete object. Boxer [6] showed how classical methods of basic algebraic topology might be used to construct the digital fundamental group based on the notions of digitally continuous functions and digital homotopy. Boxer's digital fundamental groups are defined for digital images of all dimensions with arbitrary adjacency relations, while the Kong's digital fundamental groups are defined only in dimension 2 and 3, and only for certain choices of adjacency relations. The digital fundamental group is basically derived from a classical notion of homotopy classes of based loops in the pointed homotopy category of pointed topological spaces or pointed CW-spaces.

The fundamental idea of algebraic topology is to associate to each topological space Y a group $F(Y)$ and to each continuous function $f : Y \to Z$ a homomorphism $F(f) : F(Y) \to F(Z)$ such that if Y and Z have the same homotopy type, then $F(Y)$ is isomorphic to $F(Z)$; F is called a functor from the category of topological spaces and continuous functions to the one of groups and homomorphisms. The characteristic of modern mathematics is to find out the properties of the covariant (or contravariant) functors. The covariant functor $\pi_1^k : \mathcal{D} \to \mathcal{G}$ from the category $\mathcal{D}$ of pointed digital images and pointed digitally continuous functions to the category $\mathcal{G}$ of (not necessarily abelian) groups and homomorphisms is one of them (see [6, Theorem 4.14]). Recently, the use of the whole-sample symmetric boundary conditions in image restoration was considered in [19], and the foundations of a homology-based heuristic for finding optimal discrete gradient vector fields on a general finite cell complex were introduced in [21] based on classification of cycles, cohomology algebra, homology $A(\infty)$-coalgebra, cohomology operations, homotopy groups and so on

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(see also [17]). The paper [12] presents cohomology in the context of structural pattern recognition and introduces an algorithm to compute efficiently representative cocycles (dual of cycles in homology) using a graph pyramid. Moreover, a set of tools to compute topological information of simplicial complexes, and tools that are applicable to extract topological information from digital pictures were presented in [13].

The co-H-spaces [1], also called spaces with a comultiplication, play a fundamental role in algebraic topology. One reason for this is that any two homotopy classes of maps from a co-H-space Z to a space Z' can be added. One then obtains a natural binary operation with identity on the set of homotopy classes. If the comultiplication is homotopy associative, then the set with this operation is a group, with the group operation depending on the comultiplication of Z. An important class of co-H-spaces consists of all n-spheres, $n \geq 1$. It is the associative comultiplication on the n-sphere S^n that induces group structure on the set of homotopy classes of S^n into a space Z', the nth homotopy group of Z'.

It is easily seen that the wedge of two co-H-spaces is a co-H-space, and therefore it is natural to ask about the comultiplications on a wedge of spheres. It turns out that the set of comultiplications is complicated - there are usually many comultiplications (sometimes infinitely many) with many different properties. Some indication of this complexity appeared in an early paper of Ganea [11, pp. 194-196] who gave an intricate argument to show that $S^3 \vee S^{15}$ has at least 72 associative comultiplications and at most 56 homotopy classes of suspension comultiplications. We can find the results for the calculations of the cardinality of comultiplications, associative comultiplications and commutative comultiplications based on the wedge of two spheres in [3] (see also [2] and [18]). For example, $S^2 \vee S^5$ has infinitely many homotopy classes of comultiplications and commutative comultiplications. However, it has only 2 homotopy associative comultiplications.

Motivated from the statement above, it is desirable for us to reformulate the digital version of co-H-spaces in a fashion that parallels the important approach of pointed homotopy category for the realm of computer science.

In this paper we work on the category of (pointed) digital images and (pointed) digitally continuous functions. We sometimes omit the base point of a digital image. The paper is organized as follows: In Section 2 we introduce the general notions of digital images. In Section 3 we define certain digitally quasi comultiplications and the digital Whitehead products on a wedge product of digital images, and then compute the cardinality of the set of digital homotopy classes based on digitally quasi comultiplications of the wedge products of pointed digital images. We also investigate a method to construct a digitally quasi co-H-space as a digital retract of a given digital image. In Section 4 a summary and a further work will be made. The list of notations will be described at the end of this paper.

2. Preliminaries

Let $\mathbb{Z}$ be the set of integers and $\mathbb{Z}^n$ the set of lattice points in the n-dimensional Euclidean space $\mathbb{R}^n$. A (*binary*) *digital image* is a pair (Y, k), where Y is a finite subset of $\mathbb{Z}^n$ and $k = k(u, n)$ indicates some adjacency relation for the members of Y. The k-adjacency relations are used in the study of digital images in $\mathbb{Z}^n$. For a positive integer u with $1 \leq u \leq n$, we define an adjacency relation of a digital image in $\mathbb{Z}^n$ as follows. Two distinct points $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ in $\mathbb{Z}^n$ are $k(u, n)$-*adjacent* [8, 9] if

- there are at most u distinct indices i such that $|p_i - q_i| = 1$; and
- for all indices j, if $|p_j - q_j| \neq 1$, then $p_j = q_j$.

A $k(u, n)$-adjacency relation on $\mathbb{Z}^n$ may be denoted by the number of points that are $k(u, n)$-adjacent to a point $p \in \mathbb{Z}^n$. Moreover,

- the $k(1, 1)$-adjacent points of $\mathbb{Z}$ are called 2-adjacent;
- the $k(1, 2)$-adjacent points of $\mathbb{Z}^2$ are called 4-adjacent, and the $k(2, 2)$-adjacent points in $\mathbb{Z}^2$ are called 8-adjacent;

- the $k(1,3)$-adjacent points of $\mathbb{Z}^3$ are called 6-adjacent, the $k(2,3)$-adjacent points of $\mathbb{Z}^3$ are called 18-adjacent, and the $k(3,3)$-adjacent points of $\mathbb{Z}^3$ are called 26-adjacent;
- the $k(1,4), k(2,4), k(3,4)$, and $k(4,4)$-adjacent points of $\mathbb{Z}^4$ are called 8-adjacent, 32-adjacent, 64-adjacent, and 80-adjacent, respectively; and so on.

We note that the number above is just the cardinality of the set of lattice points which have the $k(u,n)$-adjacency relations centered at p in $\mathbb{Z}^n$. A $k(u,n)$-*neighbor* of a lattice point $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is $k(u,n)$-adjacent to p. The above number $k(u,n)$ is the number of points $q \in \mathbb{Z}^n$ that are adjacent to a given point $p \in \mathbb{Z}^n$ according to the above relationship. For example, $k(1,1) = 2, k(1,2) = 4, k(2,2) = 8, k(1,3) = 6, k(2,3) = 18, k(3,3) = 26, k(1,4) = 8, k(2,4) = 32, k(3,4) = 64, k(4,4) = 80, k(2,6) = 72, k(2,12) = 288$, and so on.

Definition 2.1. ([23]) Let k be an adjacency relation defined on $\mathbb{Z}^n$. A digital image $Y \subset \mathbb{Z}^n$ is said to be *k-connected* if and only if for every pair of points $\{x, y\} \subset Y$ with $x \neq y$, there exists a set $P = \{x_0, x_1, \ldots, x_s\} \subset Y$ of $s+1$ distinct points such that $x = x_0, x_s = y$, and x_i and x_{i+1} are k-adjacent for $i = 0, 1, \ldots, s-1$. The *length* of the set P is the number s.

The following generalizes an earlier definition of digital continuity given in [23].

Definition 2.2. ([6]) Let $Y \subset \mathbb{Z}^{n_1}$ and $Z \subset \mathbb{Z}^{n_2}$ be the digital images with k_1-adjacency and k_2-adjacency, respectively. A function $f : Y \to Z$ is said to be (k_1, k_2)-*continuous* if the image under f of every k_1-connected subset of Y is a k_2-connected subset of Z.

The following is a consequence of the definition above: Let Y and Z be digital images with k_1-adjacency and k_2-adjacency, respectively. Then the function $f : Y \to Z$ is a (k_1, k_2)-continuous function if and only if for every $\{x_1, x_2\} \subset Y$ such that x_1 and x_2 are k_1-adjacent in Y, either $f(x_1) = f(x_2)$ or $f(x_1)$ and $f(x_2)$ are k_2-adjacent in Z.

It is easy to see that if $f : Y_1 \to Y_2$ is (k_1, k_2)-continuous and if $g : Y_2 \to Y_3$ is (k_2, k_3)-continuous, then the composite $g \circ f : Y_1 \to Y_3$ is (k_1, k_3)-continuous (see [5]).

Definition 2.3. ([5]) Two digital images (Y, k_1) and (Z, k_2) with adjacency relations k_1 and k_2, respectively, are (k_1, k_2)-*homeomorphic* if there is a bijective function $f : Y \to Z$ that is (k_1, k_2)-continuous such that the inverse function $f^{-1} : Z \to Y$ is (k_2, k_1)-continuous. In this case, we call the function $f : Y \to Z$ a *digital* (k_1, k_2)-*homeomorphism*, and denote it by $Y \approx_{(k_1,k_2)} Z$.

Definition 2.4. Let $a, b \in \mathbb{Z}, a < b$. A *digital interval* [5] is a set of the form

$$[a,b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$$

in which 2-adjacency is assumed.

Definition 2.5. ([6, 7]) Let $Y \subset \mathbb{Z}^n$ have an adjacency relation k. We say Y is a *digital simple closed k-curve* if there is an integer $m > 3$ and a $(2,k)$-continuous function $f : [0, m-1]_{\mathbb{Z}} \to Y$ such that

- f is a bijective function;
- $f(0)$ and $f(m-1)$ are k-adjacent; and
- for all $t \in [0, m-1]_{\mathbb{Z}}$, the only k-neighbors of $f(t)$ in $f([0, m-1]_{\mathbb{Z}})$ are $f((t-1) \bmod m)$ and $f((t+1) \bmod m)$.

There is a fundamental difference between a Euclidean simple closed curve in topology and a digital simple closed k-curve in digital image in that all Euclidean simple closed curves are homeomorphic, but digital simple closed k-curves of different cardinalities are not even of the same digital homotopy types (see [7] and below for digital homotopy).

Definition 2.6. ([6, 7, 15]) A *digital k-path* in a digital image Y is a (2,k)-continuous function $f : [0, m]_{\mathbb{Z}} \to Y$. If $f(0) = f(m)$, we call f a *digital k-loop*. If f is a constant function, it is called a *trivial loop.*

Definition 2.7. ([6, 8]) Let Y and Z be digital images with k_1-adjacency and k_2-adjacency, respectively, and let $f, g : Y \to Z$ be (k_1, k_2)-continuous functions. Suppose that there is a positive integer m and a function $F : Y \times [0, m]_{\mathbb{Z}} \to Z$ such that

- for all $y \in Y, F(y, 0) = f(y)$ and $F(y, m) = g(y)$;
- for all $y \in Y$, the induced function $F_y : [0, m]_{\mathbb{Z}} \to Z$ defined by $F_y(t) = F(y, t)$ for all $t \in [0, m]_{\mathbb{Z}}$ is $(2, k_2)$-continuous; and
- for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $F_t : Y \to Z$ defined by $F_t(y) = F(y, t)$ for all $y \in Y$ is (k_1, k_2)-continuous.

Then F is called a *digital* (k_1, k_2)*-homotopy* between f and g, written $f \simeq_{(k_1,k_2)} g$, and f and g are said to be *digitally* (k_1, k_2)*-homotopic* in Z.

We use $[f]$ to denote the digital homotopy class of a (k_1, k_2)-continuous function $f : Y \to Z$, i.e.,

$$[f] = \{g : Y \to Z \mid g \text{ is } (k_1, k_2) - \text{continuous, and } f \simeq_{(k_1,k_2)} g\}.$$

Similarly, we denote by $[f]$ the k-loop class of a digital k-loop $f : [0, m]_{\mathbb{Z}} \to Y$ in a digital image Y with k-adjacency.

A *pointed digital image* is a pair (Y, y_0), where Y is a digital image and $y_0 \in Y$; y_0 is called the *base point* of (Y, y_0). A *pointed digitally continuous function* $f : (Y, y_0) \to (Z, z_0)$ is a digitally continuous function from Y to Z such that $f(y_0) = z_0$. A digital homotopy $F : Y \times [0, m]_{\mathbb{Z}} \to Z$ between f and g is said to be *pointed digital homotopy* between f and g if for all $t \in [0, m]_{\mathbb{Z}}$, $F(y_0, t) = z_0$. If a pointed digital homotopy between f and g exists, we say f and g belong to the same pointed digital homotopy class. It is not difficult to see that the (pointed) digital homotopy is an equivalence relation among the (pointed) digital homotopy classes of digitally continuous functions (see [6] and [7]). We sometimes omit the base point for convenience.

We now consider the digital version of products just as in the case of products of paths (or loops) of homotopy classes in homotopy theory. If $f : [0, m_1]_{\mathbb{Z}} \to Y$ and $g : [0, m_2]_{\mathbb{Z}} \to Y$ are digital k-paths in Y with $f(m_1) = g(0)$, the *product* $(f * g) : [0, m_1 + m_2]_{\mathbb{Z}} \to Y$ (see [15], [6] and [8]) of f and g is the digital k-path in Y defined by

$$(f * g)(t) = \begin{cases} f(t) & \text{if } t \in [0, m_1]_{\mathbb{Z}}; \\ g(t - m_1) & \text{if } t \in [m_1, m_1 + m_2]_{\mathbb{Z}}. \end{cases}$$

The following result will be used to show that the product operation of digital loop classes is well-defined.

Proposition 2.8. ([6, 15]) *Suppose* f_1, f_2, g_1 *and* g_2 *are digital loops in a pointed digital image* (Y, y_0) *with* $f_2 \in [f_1]$ *and* $g_2 \in [g_1]$. *Then* $f_2 * g_2 \in [f_1 * g_1]$.

We now discuss the digital k-fundamental group originally derived from a classical notion of homotopy theory (see [24, 26]). Let (Y, y_0) be a pointed digital image with k-adjacency. Consider the set $\pi_1^k(Y, y_0)$ of k-loop classes $[f]$ in (Y, y_0) with base point y_0. By Proposition 2.8, the product operation

$$[f] + [g] = [f * g]$$

is well-defined on $\pi_1^k(Y, y_0)$. One can see that $\pi_1^k(Y, y_0)$ becomes a group under the $*$ product operation which is called the *digital k-fundamental group* of (Y, y_0). As in the case of basic notions in algebraic topology, it is well known in [6, Theorem 4.14] that π_1^k is a covariant functor from the category of pointed digital images and pointed digitally continuous functions to the category of groups and group homomorphisms.

We now describe the notion of trivial extension which is used to allow a loop to stretch and remain in the same pointed homotopy class.

Definition 2.9. ([10]) Let f and f' be digital k-loops in a pointed digital image (Y, y_0). We say that f' is a *trivial extension* of f if there are sets of k-paths $\{f_1, f_2, \ldots, f_s\}$ and $\{F_1, F_2, \ldots, F_t\}$ in Y such that

1. $s \leq t$;
2. $f = f_1 * f_2 * \cdots * f_s$;
3. $f' = F_1 * F_2 * \cdots * F_t$; and
4. there are indices $1 \leq i_1 < i_2 < \cdots < i_s \leq t$ such that
   - $F_{i_j} = f_j, 1 \leq j \leq s$; and
   - $i \notin \{i_1, i_2, \ldots, i_s\}$ implies F_i is a trivial loop.

Example 2.10. If $f_1, f_2, f_3 : [0, 1]_{\mathbb{Z}} \to Y$ are digital k-paths defined by

$$\begin{cases} f_1(0) = y_0; \\ f_1(1) = y_1 = f_2(0); \\ f_2(1) = y_2 = f_3(0); \\ f_3(1) = y_0; \end{cases}$$

and if $F_1, F_2, F_3, F_4 : [0, 1]_{\mathbb{Z}} \to Y$ are digital k-paths defined by

$$\begin{cases} F_1(0) = y_0 = F_1(1) = F_2(0); \\ F_2(1) = y_1 = F_3(0); \\ F_3(1) = y_2 = F_4(0); \\ F_4(1) = y_0, \end{cases}$$

then the digital k-loop $f' : [0, 4]_{\mathbb{Z}} \to Y$ defined by

$$f' = F_1 * F_2 * F_3 * F_4$$

is a trivial extension (see Figure 1) of the digital k-loop $f : [0, 3]_{\mathbb{Z}} \to Y$ defined by

$$f = f_1 * f_2 * f_3.$$

In a homotopical point of view,

$$\begin{aligned} [f'] &= [F_1 * F_2 * F_3 * F_4] \\ &= [F_1] + [F_2] + [F_3] + [F_4] \\ &= [e_1] + [f_1] + [f_2] + [f_3] \\ &= [e_1] + [f_1 * f_2 * f_3] \\ &= [e_1] + [f], \end{aligned}$$

where $e_1 : [0, 1]_{\mathbb{Z}} \to Y$ is a constant function at y_0.

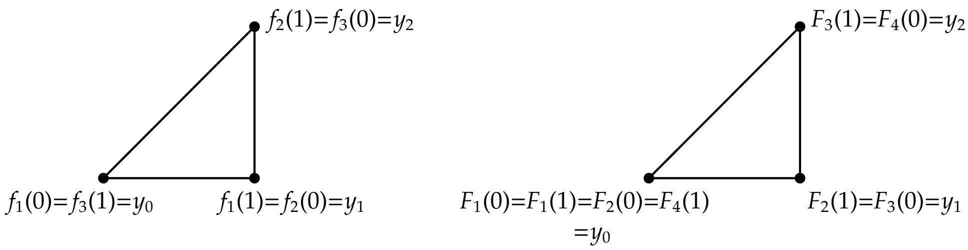

Figure 1: The image of f on the left, and the image of f' on the right

We end this section with digital notions of homotopy equivalence and nullhomotopy just like those of classical homotopy theory: Let $f : Y \to Z$ be a (k_1, k_2)-continuous function and $g : Z \to Y$ be a (k_2, k_1)-continuous function such that

$$g \circ f \simeq_{(k_1,k_1)} 1_Y \text{ and } f \circ g \simeq_{(k_2,k_2)} 1_Z.$$

Then $f : Y \to Z$ is said to be a *(k_1, k_2)-homotopy equivalence* [7]. Moreover, we say Y and Z have the *same (k_1, k_2)-homotopy type* and that Y and Z are *(k_1, k_2)-homotopy equivalent*. A digital continuous function $f : Y \to Z$ is *digitally nullhomotopic* in Z if f is digitally homotopic in Z to a constant function [6]. We can also consider the pointed digital homotopy equivalences between pointed digitally continuous functions.

It is well known that if $f : (Y, y_0) \to (Z, z_0)$ is a (k_1, k_2)-homotopy equivalence between pointed digital images with k_1- and k_2-adjacency relations, respectively, then f induces an isomorphism $\pi_1(f) : \pi_1^{k_1}(Y, y_0) \to \pi_1^{k_2}(Z, z_0)$ between digital fundamental groups (see [6, Theorem 4.14] and [7, Theorem 4.1]).

3. Digital Wedges and Digitally Quasi Comultiplications

Let $Z_\alpha, \alpha \in \Gamma$ be a collection of (disjoint) spaces with base point $z_\alpha \in Z_\alpha$. The *wedge product* (or *one-point union*) $\bigvee_{\alpha\in\Gamma} Z_\alpha$ is defined to be the quotient space Z/Z_0, where Z is the disjoint union of the spaces Z_α, and Z_0 is the subspace consisting of all the base points z_α; the base point of $\bigvee_{\alpha\in\Gamma} Z_\alpha$ is the point corresponding to Z_0. In other words, $\bigvee_{\alpha\in\Gamma} Z_\alpha$ is the space obtained from Z by identifying together the base points $z_\alpha, \alpha \in \Gamma$ in algebraic topology.

A graph product is a certain kind of binary operation on graphs such as the cartesian product, tensor product, lexicographical product, normal product, conormal product and rooted product. Recall that the *cartesian product* [4] of simple graphs G and H is the graph $G \square H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2$. In this section we use the cartesian product as graph products.

We now consider the digital version of the wedge product of digital images with adjacency relations as follows (see [9] and [24, 26] for the original definition):

Definition 3.1. Let (Y, y_0) and (Z, z_0) be the pointed digital images with $k(u, n)$-adjacency relations in $\mathbb{Z}^n$. The *digital wedge product* $Y \vee Z$ is defined by

$$Y \vee Z = Y \times \{z_0\} \cup \{y_0\} \times Z \subset Y \times Z$$

which is the pointed digital image with $k(u, 2n)$-adjacency and base point (y_0, z_0) in $\mathbb{Z}^{2n}$. Here the cartesian product in graph theory is assumed.

Let $(Y, y_0), (Y_1, \bar{y}_0)$ and $(Y_2, \bar{y}_0)$ be the pointed digital images with $k(u, n)$-adjacency relations in $\mathbb{Z}^n$ such that

(1) $Y_1 \cap Y_2 = \{\bar{y}_0\}$, a single point set; and
(2) any element of $Y_1 - \{\bar{y}_0\}$ is not a $k(u, n)$-neighbor of any element of $Y_2 - \{\bar{y}_0\}$.

If $\psi_1 : (Y, y_0) \to (Y_1, \bar{y}_0)$ and $\psi_2 : (Y, y_0) \to (Y_2, \bar{y}_0)$ are the base point preserving digital $(k(u, n), k(u, n))$-homeomorphisms in $\mathbb{Z}^n$, then the function $\psi : Y \vee Y \to Y_1 \vee Y_2$ defined by $\psi = \psi_1 \vee \psi_2$, explicitly,

$$\begin{cases} \psi(y, y_0) = (\psi_1 \vee \psi_2)(y, y_0) = (\psi_1(y), \bar{y}_0) \in Y_1 \times \{\bar{y}_0\}; \text{ and} \\ \psi(y_0, y) = (\psi_1 \vee \psi_2)(y_0, y) = (\bar{y}_0, \psi_2(y)) \in \{\bar{y}_0\} \times Y_2, \end{cases}$$

are $(k(u, 2n), k(u, 2n))$-homeomorphism. Since the functions $\eta_1 : Y_1 \times \{\bar{y}_0\} \to Y_1$ and $\eta_2 : \{\bar{y}_0\} \times Y_2 \to Y_2$ defined by

$$\begin{cases} \eta_1(y_1, \bar{y}_0) = y_1; \text{ and} \\ \eta_2(\bar{y}_0, y_2) = y_2 \end{cases}$$

are a $(k(u,2n),k(u,n))$-homeomorphisms, the function $\eta : Y_1 \vee Y_2 \to Y_1 \cup Y_2$ defined by

$$\begin{cases} \eta(y_1,\bar{y}_0) = \eta_1(y_1,\bar{y}_0) = y_1; \text{ and} \\ \eta(\bar{y}_0,y_2) = \eta_2(\bar{y}_0,y_2) = y_2 \end{cases}$$

is well-defined and a $(k(u,2n),k(u,n))$-homeomorphism. By using those homeomorphisms, we can identify the digital wedge product $Y \vee Y$ with $Y_1 \cup Y_2$ as the digital image with $k(u,n)$-adjacency and base point $\bar{y}_0$ in $\mathbb{Z}^n$.

We now give examples of the digital simple closed 18-curves in $\mathbb{Z}^3$ and the digital homeomorphisms which will be used in this paper.

Example 3.2. The following are some examples of the digital simple closed 18-curves in $\mathbb{Z}^3$.

(1) $X_1 = \{x_i^1 \mid i = 0,1,2,\ldots,7\} \subset \mathbb{Z}^3$, where $x_0^1 = (0,0,0), x_1^1 = (1,1,0), x_2^1 = (2,2,0), x_3^1 = (1,3,0), x_4^1 = (0,4,0), x_5^1 = (-1,3,0), x_6^1 = (-2,2,0), x_7^1 = (-1,1,0)$ (see Figure 2);
(2) $X_2 = \{x_i^2 \mid i = 0,1,2,\ldots,7\} \subset \mathbb{Z}^3$, where $x_0^2 = (0,0,0), x_1^2 = (-1,-1,0), x_2^2 = (-2,-2,0), x_3^2 = (-1,-3,0), x_4^2 = (0,-4,0), x_5^2 = (1,-3,0), x_6^2 = (2,-2,0), x_7^2 = (1,-1,0)$;
(3) $X_3 = \{x_i^3 \mid i = 0,1,2,\ldots,7\} \subset \mathbb{Z}^3$, where $x_0^3 = (0,0,0), x_1^3 = (1,0,1), x_2^3 = (2,0,2), x_3^3 = (1,0,3), x_4^3 = (0,0,4), x_5^3 = (-1,0,3), x_6^3 = (-2,0,2), x_7^3 = (-1,0,1)$; and
(4) $X_4 = \{x_i^4 \mid i = 0,1,2,\ldots,7\} \subset \mathbb{Z}^3$, where $x_0^4 = (0,0,0), x_1^4 = (-1,0,-1), x_2^4 = (-2,0,-2), x_3^4 = (-1,0,-3), x_4^4 = (0,0,-4), x_5^4 = (1,0,-3), x_6^4 = (2,0,-2), x_7^4 = (1,0,-1)$.

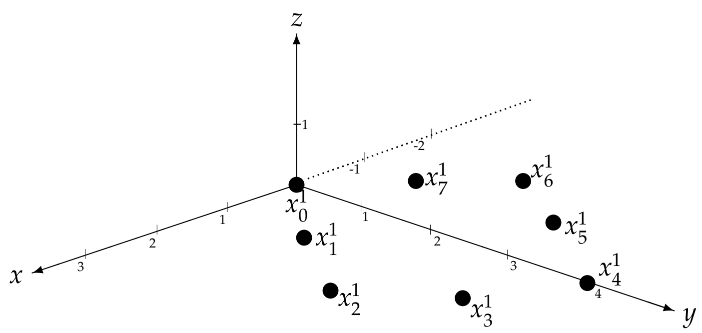

Figure 2: Digital simple closed 18-curve X_1 in $\mathbb{Z}^3$

Convention We work on the digital images with 4-adjacency relation on $\mathbb{Z}^2$ and the 18-adjacency relation on $\mathbb{Z}^3$ in the rest of the paper. The point $x_0^i = (0,0,0)$ of $\mathbb{Z}^3$ in Example 3.2 will be denoted by x_0 as the base point of X_i for $i = 1,2,3,4$. And we will make use of the notations listed at the end of the article.

We remark that $X_i \approx_{(18,18)} X_j$ for each $i,j = 1,2,3,4$. We also note that

$$X_u \vee X_u = X_u \times \{x_0\} \cup \{x_0\} \times X_u = \{(x_i^u,x_0),(x_0,x_i^u) \mid i = 0,1,\ldots,7\}$$

with $k(2,6)$-adjacency for $u = 1,2,3,4$.

Example 3.3. Let

$$\begin{cases} \alpha : X_u \vee X_u \to X_s \vee X_t, \ \ (s,t,u = 1,2,3,4); \\ \beta : X_s \vee X_t \to X_s \cup X_t, \ \ (s,t = 1,2,3,4); \text{ and} \\ \gamma : X_1 \vee X_2 \vee X_3 \vee X_4 \to X_1 \cup X_2 \cup X_3 \cup X_4 \end{cases}$$

be the functions defined by

$$\begin{cases} \alpha(x_i^u, x_0) = (x_i^s, x_0); \;\; \alpha(x_0, x_i^u) = (x_0, x_i^t); \\ \beta(x_i^s, x_0) = x_i^s; \;\; \beta(x_0, x_i^t) = x_i^t; \\ \gamma(x_i^1, x_0, x_0, x_0) = x_i^1; \\ \gamma(x_0, x_i^2, x_0, x_0) = x_i^2; \\ \gamma(x_0, x_0, x_i^3, x_0) = x_i^3; \;\; \text{and} \\ \gamma(x_0, x_0, x_0, x_i^4,) = x_i^4. \end{cases}$$

Then it is easy to see that α, β and γ are well-defined and that they are bijective functions. Furthermore, α is a $(k(2,6), k(2,6))$-homeomorphism, β is a $(k(2,6), 18)$-homeomorphism, and γ is a $(k(2,12), 18)$-homeomorphism. Similarly

$$(X_1 \vee X_2) \vee (X_1 \vee X_2) \approx_{(k(2,12),k(2,12))} X_1 \vee X_2 \vee X_3 \vee X_4.$$

We now describe the basic notions in algebraic topology. In the category of pointed and connected CW-spaces, a pair $((Z, z_0), \varphi)$ consisting of a pointed space (Z, z_0) and a function $\varphi : Z \to Z \vee Z$ is called a *co-H-space* if $p_1\varphi = 1$ and $p_2\varphi = 1$, where p_1 and p_2 are the projections $Z \vee Z \to Z$ onto the first and second summands of the wedge product and 1 is the identity map of Z. In this case, the map $\varphi : Z \to Z \vee Z$ is called a *comultiplication*. Equivalently, (Z, φ) is a co-H-space if $J\varphi = \Delta : Z \to Z \times Z$, where Δ is the diagonal map and $J : Z \vee Z \to Z \times Z$ is the inclusion.

It is well known that the Whitehead products in algebraic topology have the properties of biadditivity and anticommutativity. In addition, there is the Jacobi identity for the Whitehead products (see [14, Theorem 5.3] and [26, pp. 472-478]) as follows: If $a \in \pi_p(Y)$, $b \in \pi_q(Y)$ and $c \in \pi_r(Y)$, then

$$(-1)^{p(r-1)}[a, [b, c]] + (-1)^{q(p-1)}[b, [c, a]] + (-1)^{r(q-1)}[c, [a, b]] = 0.$$

If $a, b, c \in \pi_p(Y)$, then

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$$

Can we construct the digital versions of the notions above? The following gives an answer to this question.

Definition 3.4. A pair $((Y, y_0), \varphi_Y)$ consisting of a pointed digital image (Y, y_0) with k-adjacency and a (k, k)-continuous function $\varphi_Y : Y \to Y \vee Y$ is called a *digitally quasi co-H-space* if for a given digital k-loop $f : [0, m]_{\mathbb{Z}} \to (Y, y_0)$, there exists a digital k-loop $f' : [0, n]_{\mathbb{Z}} \to (Y, y_0)$ such that

- $p_1 \circ \varphi_Y \circ f'$ is the trivial extension of f; or
- $p_2 \circ \varphi_Y \circ f'$ is the trivial extension of f.

Here, $p_1 : Y \vee Y \to Y$ and $p_2 : Y \vee Y \to Y$ are the first and second projections, respectively. The above (k, k)-continuous function $\varphi_Y : Y \to Y \vee Y$ is called a *digitally quasi comultiplication* of Y.

Example 3.5. The $(18, 18)$-continuous functions

$$\iota_1 : X(= X_{1\vee 2}) \to X \vee X(= X_{1\vee 2\vee 3\vee 4})$$

and

$$\iota_2 : X(= X_{1\vee 2}) \to X \vee X(= X_{1\vee 2\vee 3\vee 4})$$

defined by $\iota_1(x_i^j) = x_i^j$ and $\iota_2(x_i^j) = x_i^{j+2}$, respectively, are the digitally quasi comultiplications of X, where $i = 0, 1, \ldots, 7$ and $j = 1, 2$ (see the list of notations at the end of this paper).

Let $f_1, f_2 : [0, 8]_{\mathbb{Z}} \to X_{1\vee 2}$ be the $(2, 18)$-continuous functions defined by

- $f_1(\{0,8\}) = x_0 = f_2(\{0,8\})$;
- $f_1(i) = x_i^1$ for $i = 1, 2, \ldots, 7$; and
- $f_2(i) = x_i^2$ for $i = 1, 2, \ldots, 7$;

that is, f_1 and f_2 are digital 18-loops going around X_1 and X_2 just once, respectively. We note that $\pi_1^{18}(X_{1\vee 2}, x_0)$ is a free group on two generators $[f_1]$ and $[f_2]$ (cf. [9] and see also [22, Theorem 71.1] for detail in the case of algebraic topology).

Definition 3.6. ([6]) Let $f : [0,m]_{\mathbb{Z}} \to Y$ be the digital k-loop in the digital image Y with the k-adjacency. Then the digital k-loop $\bar{f} : [0,m]_{\mathbb{Z}} \to Y$ defined by $\bar{f}(i) = f(m-i)$ is called the *reverse* of f

The k-loop class $[\bar{f}]$ of the reverse $\bar{f}$ plays a role of the inverse of $[f]$ in the digital fundamental group just like the classical homotopy theory.

Let $(S, s_0) = \{s_i \mid i = 0, 1, 2, \ldots, 32\} \subset \mathbb{Z}^2$ be the pointed digital image with the 4-adjacency relation and base point s_0 as a digital simple closed 4-curve, where $s_0 = (0,0), s_1 = (1,0), s_2 = (2,0), s_3 = (3,0), s_4 = (4,0), s_5 = (5,0), s_6 = (6,0), s_7 = (7,0), s_8 = (8,0), s_9 = (8,1), s_{10} = (8,2), s_{11} = (8,3), s_{12} = (8,4), s_{13} = (8,5), s_{14} = (8,6), s_{15} = (8,7), s_{16} = (8,8), s_{17} = (7,8), s_{18} = (6,8), s_{19} = (5,8), s_{20} = (4,8), s_{21} = (3,8), s_{22} = (2,8), s_{23} = (1,8), s_{24} = (0,8), s_{25} = (0,7), s_{26} = (0,6), s_{27} = (0,5), s_{28} = (0,4), s_{29} = (0,3), s_{30} = (0,2), s_{31} = (0,1), s_{32} = (0,0)$. Then we can think of this finite sequence in set theory as the image of the the digital 4-loop $s : [0,32]_{\mathbb{Z}} \to S$ defined by $s(i) = s_i$ for $i = 0, 1, 2, \ldots, 32$, where $s(32) = s_{32} = s_0$. We now define the following:

Definition 3.7. The *digital Whitehead product* (see Figure 3) denoted by $[f_1, f_2]_{\mathrm{dW}}$ of f_1 and f_2 is the $(4,18)$-continuous map

$$[f_1, f_2]_{\mathrm{dW}} : S \to X_{1\vee 2}$$

defined by

$$[f_1, f_2]_{\mathrm{dW}}(s_i) = \begin{cases} f_1(i) & \text{if } 0 \le i \le 8; \\ f_2(i \bmod 8) & \text{if } 8 \le i \le 16; \\ \bar{f}_1(i \bmod 8) & \text{if } 16 \le i \le 24; \\ \bar{f}_2(i \bmod 8) & \text{if } 24 \le i \le 32. \end{cases}$$

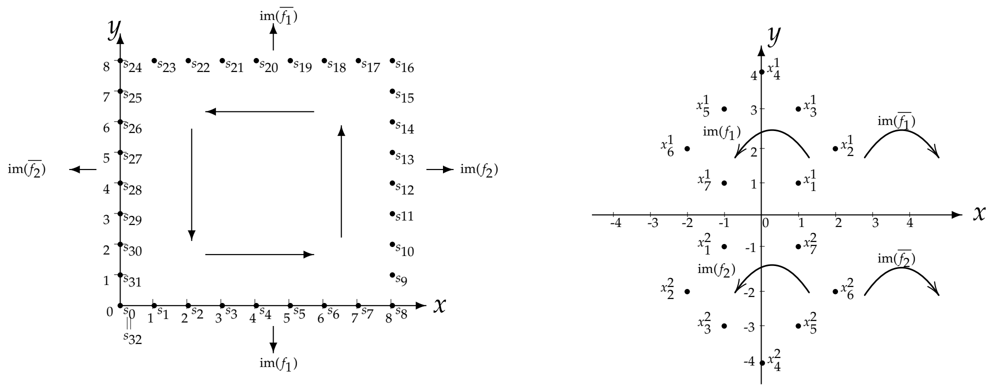

Figure 3: Images of the digital Whitehead product $[f_1,f_2]_{dW}$: The digital interval $[0,8]_{\mathbb{Z}}$ gets wrapped by f_1 and f_2 (or $\bar{f}_1$ and $\bar{f}_2$) in a counterclockwise (or clockwise) fashion around X_1 and X_2, respectively.

Indeed, $[f_1, f_2]_{\mathrm{dW}}$ is well-defined and it is not difficult to see that it is a pointed digitally $(4,18)$-continuous function.

We note that the notion of Whitehead products, an Eckmann-Hilton dual of the Samelson products, of homotopy classes in homotopy groups plays an important role in algebraic topology in that the graded homotopy groups with the Whitehead products has the graded quasi-Lie algebra structure which is called the *Whitehead algebra* [26]. Moreover, the digital Whitehead product $[f_1, f_2]_{\mathrm{dW}}$ is the commutator of the digital k-loops f_1 and f_2 in the pointed digital images $(X_{1\vee 2}, x_0)$, i.e., $[f_1, f_2]_{\mathrm{dW}} = f_1 * f_2 * \bar{f_1} * \bar{f_2}$. Explicitly,

$$[f_1, f_2]_{\mathrm{dW}}(s_i) = \begin{cases} x_0 & \text{if } i = 0, 8, 16, 24, 32; \\ x_i^1 & \text{if } 1 \leq i \leq 7; \\ x_{i-8}^2 & \text{if } 9 \leq i \leq 15; \\ x_{24-i}^1 & \text{if } 17 \leq i \leq 23; \\ x_{32-i}^2 & \text{if } 25 \leq i \leq 31. \end{cases}$$

Given pointed digital images (Y_1, y_1) and (Y_2, y_2) with k-adjacency relations and base points y_1 and y_2, respectively, in $\mathbb{Z}^n$, we denote by $[(Y_1, y_1), (Y_2, y_2)]$ the set of pointed digital homotopy classes of pointed (k, k)-continuous functions $f : (Y_1, y_1) \to (Y_2, y_2)$ with $f(y_1) = y_2$.

Definition 3.8. The pointed digitally continuous function $\nabla : (Y, y_0) \vee (Y, y_0) \to (Y, y_0)$ defined by

$$\nabla(y, y_0) = y = \nabla(y_0, y)$$

is said to be the *digital folding map*.

We note that the digital folding map has a cross section. We also remark that if U and V are digital images with k-adjacency relations in $\mathbb{Z}^n$ such that $U \approx_{(k,k)} U'$ and $V \approx_{(k,k)} V'$ with $U' \cap V' = \{\bar{u}\}$, a single point set, and such that any element of $U' - \{\bar{u}\}$ is not a k-neighbor of any element of $V' - \{\bar{u}\}$, then $U \vee V$ can be considered as the pointed digital image $(U' \cup V', \bar{u})$ with k-adjacency and base point $\bar{u}$ in $\mathbb{Z}^n$ via digital homeomorphism, as described earlier. We need the following lemma to prove Theorem 3.10.

Lemma 3.9. *For any pointed digital images $(U, u_0), (V, v_0)$ and (W, w_0) with k-adjacency relations in $\mathbb{Z}^n, n \geq 1$, the inclusions $i : U \hookrightarrow U \vee V$ and $j : V \hookrightarrow U \vee V$ as pointed (k, k)-continuous functions induce a bijection of $[U \vee V, W]$ with the cartesian product $[U, W] \times [V, W]$.*

Proof. Define a map

$$\tau : [U \vee V, W] \to [U, W] \times [V, W]$$

by

$$\tau([f]) = ([f \circ i], [f \circ j])$$

for $[f] \in [U \vee V, W]$. To prove τ is a bijection, we define

$$\sigma : [U, W] \times [V, W] \to [U \vee V, W]$$

by

$$\sigma([g], [h]) = [\nabla \circ (g \vee h)],$$

where $\nabla : W \vee W \to W$ is the digital folding map. We note that if $g : U \to W$ and $h : V \to W$ are pointed digitally (k, k)-continuous functions, then the function

$$g \vee h : U \vee V \to W \vee W$$

defined by

$$\begin{cases} (g \vee h)(u, v_0) = (g(u), w_0), \\ (g \vee h)(u_0, v) = (w_0, h(v)) \end{cases}$$

is also a pointed digitally (k, k)-continuous function via digital homeomorphisms. Thus, we have

$$\begin{aligned} \sigma \circ \tau([f]) &= \sigma([f \circ i], [f \circ j]) \\ &= [\nabla \circ (f \circ i \vee f \circ j)] \\ &= [f]. \end{aligned}$$

The last equality is obtained from the following facts:

$$\begin{aligned} \nabla \circ (f \circ i \vee f \circ j)(u, v_0) &= \nabla \circ (f \vee f) \circ (i \vee j)(u, v_0) \\ &= \nabla \circ (f \vee f)((u, v_0), (u_0, v_0)) \\ &= \nabla(f(u, v_0), w_0) \\ &= f(u, v_0), \end{aligned}$$

and

$$\begin{aligned} \nabla \circ (f \circ i \vee f \circ j)(u_0, v) &= \nabla \circ (f \vee f) \circ (i \vee j)(u_0, v) \\ &= \nabla \circ (f \vee f)((u_0, v_0), (u_0, v)) \\ &= \nabla(w_0, f(u_0, v)) \\ &= f(u_0, v). \end{aligned}$$

On the other hand, since

$$\begin{aligned} (\nabla \circ (g \vee h) \circ i)(u) &= \nabla \circ (g \vee h)(u, v_0) \\ &= \nabla(g(u), h(v_0)) \\ &= \nabla(g(u), w_0) \\ &= g(u), \end{aligned}$$

and

$$\begin{aligned} (\nabla \circ (g \vee h) \circ j)(v) &= \nabla \circ (g \vee h)(u_0, v) \\ &= \nabla(g(u_0), h(v)) \\ &= \nabla(w_0, h(v)) \\ &= h(v), \end{aligned}$$

we get

$$\begin{aligned} \tau \circ \sigma([g], [h]) &= \tau([\nabla \circ (g \vee h)]) \\ &= ([\nabla \circ (g \vee h) \circ i], [\nabla \circ (g \vee h) \circ j]) \\ &= ([g], [h]). \end{aligned}$$

Thus τ is a bijection as required. □

Even though algebraic or topological devices have been incredibly developed since the 1930s (originally H. Poincaré in the 1890s), there are no general solutions for computing the unstable (even stable) homotopy groups of a space [25]. As previously mentioned in the introduction, it is worth noting how the cardinality of the set of homotopy classes satisfying certain conditions could be calculated. It is, however, difficult for us to compute the cardinality of homotopy groups of a pointed topological space (or a pointed digital image) except for very special cases. Indeed, only a few results have been known so far. Motivated by this, we now calculate the cardinality of the set of digital homotopy classes based on digitally quasi comultiplications of the digital wedge products as follows:

Let $\varphi : X \to X \vee X$ be a digitally quasi comultiplication of X. Then for a given digital 18-loop $\omega : [0, m]_{\mathbb{Z}} \to (X, x_0)$, there exists at least one digital 18-loop $\omega' : [0, n]_{\mathbb{Z}} \to (X, x_0)$ such that the composite of $\varphi \circ \omega'$ with the first projection or the second projection is a trivial extension of ω. It raises the following question. How many digital homotopy classes $[\varphi \circ \omega'] \in \pi_1^{18}(X \vee X, x_0)$ making $\varphi : X \to X \vee X$ into the digitally quasi comultiplication are there? We are mainly interested in these digital homotopy classes $[\varphi \circ \omega']$ because we can construct a digitally quasi co-H-space $((X, x_0), \varphi)$ depending on the digital homotopy class. So we let

$$HDC_\omega = \{\, [\varphi \circ \omega'] \in \pi_1^{18}(X \vee X, x_0) | \pi_{1\vee 2} \circ \varphi \circ \omega' \text{ or } \pi_{3\vee 4} \circ \varphi \circ \omega' \text{ is a trivial extension of } \omega\}.$$

Then we have an answer to this query in a particular case as follows:

Theorem 3.10. *Let $\varphi : X \to X \vee X$ be an $(18, 18)$-continuous function, and let $f_1 : [0, 8]_{\mathbb{Z}} \to X$ and $f_2 : [0, 8]_{\mathbb{Z}} \to X$ be the digital 18-loops defined by $f_1(i) = x_i^1$ and $f_2(i) = x_i^2$ for $i = 0, 1, 2, \ldots, 8$ in the pointed digital images (X_1, x_0) and (X_2, x_0), respectively. Let $N(HDC_{f_i}), i = 1, 2$ be the cardinal number of the set $HDC_{f_i}, i = 1, 2$ of digital homotopy classes $[\varphi \circ f_i'], i = 1, 2$. Then $N(HDC_{f_i}) = \aleph_0$ for $i = 1, 2$.*

Proof. Since $X = X_{1\vee 2} \approx_{(18,k(2,6))} X_1 \vee X_2$, by Lemma 3.9, there is a bijection of $[X, X \vee X]$ with $[X_1, X \vee X] \times [X_2, X \vee X]$, i.e., any map defined on the wedge $X(\approx_{(18,k(2,6))} X_1 \vee X_2)$ can be expressed by each factor as a kind of coordinate functions. Thus, by considering

$$X \vee X = X_{1\vee 2\vee 3\vee 4} \approx_{(18,k(2,12))} X_1 \vee X_2 \vee X_3 \vee X_4,$$

we now consider the composite functions $\varphi \circ f_i'$ of $f_i, i = 1, 2$ and the $(18, 18)$-continuous function $\varphi : X \to X \vee X$ defined by

$$\begin{cases} \varphi \circ f_1' &= \iota_1 \circ f_1 * \iota_2 \circ f_1 * (h_1 \vee h_2) \circ [f_1, f_2]_{\mathrm{dW}} \circ a_n; \text{and} \\ \varphi \circ f_2' &= \iota_1 \circ f_2 * \iota_2 \circ f_2 * (h_1 \vee h_2) \circ [f_1, f_2]_{\mathrm{dW}} \circ b_m. \end{cases} \qquad (\star)$$

Here

(1) $f_1' : [0, 16 + 32n]_{\mathbb{Z}} \to X$ and $f_2' : [0, 16 + 32m]_{\mathbb{Z}} \to X$ are digital 18-loops;
(2) $a_n : [0, 32n]_{\mathbb{Z}} \to S$ and $b_m : [0, 32m]_{\mathbb{Z}} \to S$ are the digital 18-loops in the pointed digital image $(S, s_0) \subset \mathbb{Z}^2$ based at $s_0 = (0, 0)$, i.e., $[a_n], [b_n] \in \pi_1^4(S, s_0)$; and
(3) the composite functions are obtained, via digital homeomorphisms, as follows:

$$[0, 32n]_{\mathbb{Z}} \xrightarrow[(2,4)-\text{conti}]{a_n} S \xrightarrow[(4,18)-\text{conti}]{[f_1,f_2]_{dW}} X_{1\vee 2} \approx_{(18,k(2,6))} X_1 \vee X_2$$

$$\xrightarrow[(k(2,6),k(2,6))-\text{conti}]{h_1 \vee h_2} X_{1\vee 2} \vee X_{3\vee 4} \approx_{(k(2,6),18)} X_{1\vee 2\vee 3\vee 4} = X \vee X,$$

and similarly for the second equation.

From the constructions above, the followings are straightforward:

(1) $\pi_{1\vee 2} \circ \iota_1 = 1_{X_{1\vee 2}}$;
(2) $\pi_{3\vee 4} \circ \iota_1 = c_{x_0}$ (a constant function at x_0);
(3) $\pi_{1\vee 2} \circ \iota_2 = c_{x_0}$ (a constant function at x_0);
(4) $(\pi_{3\vee 4} \circ \iota_2)(x_i^1) = x_i^3$ for $i = 0, 1, 2, \ldots, 7$; and
(5) $(\pi_{3\vee 4} \circ \iota_2)(x_i^2) = x_i^4$ for $i = 0, 1, 2, \ldots, 7$.

Moreover, by using the same notations of digitally continuous functions, via digital homeomorphisms, $X_{1\vee 2} \approx_{(18,k(2,6))} X_1 \vee X_2$, $X_{3\vee 4} \approx_{(18,k(2,6))} X_3 \vee X_4$, and $X_{1\vee 2\vee 3\vee 4} \approx_{(18,k(2,12))} X_1 \vee X_2 \vee X_3 \vee X_4$, we have the following commutative diagrams:

$$\begin{tikzcd} {[0,8]_{\mathbb{Z}}} \arrow[r, "f_1"] \arrow[d, "f_1"] & (X_1 \vee X_2, x_0) \arrow[d, "h_1 \vee h_2"] \\ (X_1 \vee X_2, x_0) \arrow[r, "\iota_1"] & (X_1 \vee X_2 \vee X_3 \vee X_4, x_0) \end{tikzcd}$$

and

$$\begin{tikzcd} {[0,8]_{\mathbb{Z}}} \arrow[r, "f_2"] \arrow[d, "f_2"] & (X_1 \vee X_2, x_0) \arrow[d, "h_1 \vee h_2"] \\ (X_1 \vee X_2, x_0) \arrow[r, "\iota_2"] & (X_1 \vee X_2 \vee X_3 \vee X_4, x_0). \end{tikzcd}$$

Since the digital fundamental group construction induces a covariant functor, it can be seen that if $F : (A, a_0) \to (B, b_0)$ is a pointed digitally (k, k)-continuous function, then the map $\pi_1^k(F) : \pi_1^k(A, a_0) \to \pi_1^k(B, b_0)$, defined by $\pi_1^k(F)([f]) = [F \circ f]$, where $[f] \in \pi_1^k(A, a_0)$ is a group homomorphism. Indeed,

$$\begin{aligned} \pi_1^k([f * g]) &= [F \circ (f * g)] \\ &= [(F \circ f) * (F \circ g)] \\ &= [(F \circ f)] + [(F \circ g)] \\ &= \pi_1^k([f]) + \pi_1^k([g]). \end{aligned}$$

Thus, by applying the projection $\pi_{1\vee 2}$ on the above equation ($\star$), we have

$$\begin{aligned} \pi_{1\vee 2} \circ \varphi \circ f_1' &= \pi_{1\vee 2} \circ \iota_1 \circ f_1 \; * \; \pi_{1\vee 2} \circ \iota_2 \circ f_1 \; * \; \pi_{1\vee 2} \circ (h_1 \vee h_2) \circ [f_1, f_2]_{\mathrm{dW}} \circ a_n \\ &= f_1 \; * \; c_{x_0} \circ f_1 \; * \; \pi_{1\vee 2} \circ [\iota_1 \circ f_1, \iota_2 \circ f_2]_{\mathrm{dW}} \circ a_n \\ &= f_1 \; * \; e_8 \; * \; [\pi_{1\vee 2} \circ \iota_1 \circ f_1, \pi_{1\vee 2} \circ \iota_2 \circ f_2]_{\mathrm{dW}} \circ a_n \\ &= f_1 \; * \; e_8 \; * \; [1_{X_{1\vee 2}} \circ f_1, c_{x_0} \circ f_2]_{\mathrm{dW}} \circ a_n \\ &= f_1 \; * \; e_8 \; * \; [f_1, e_8]_{\mathrm{dW}} \circ a_n \\ &= f_1 \; * \; e_8 \; * \; e_{32n}, \end{aligned}$$

that is,

$$\pi_{1\vee 2} \circ \varphi \circ f_1'(x) = \begin{cases} f_1(x) & \text{for } 0 \le i \le 8; \\ x_0 & \text{for } 8 \le i \le 16; \\ x_0 & \text{for } 16 \le i \le 32n. \end{cases}$$

Thus $\pi_{1\vee 2} \circ \varphi \circ f_1'$ is a trivial extension of f_1. Indeed, the composite

$$\pi_1^{18}(X, x_0) \xrightarrow{\iota_{2*}} \pi_1^{18}(X \vee X, x_0) \xrightarrow{\pi_{1\vee 2*}} \pi_1^{18}(X, x_0)$$

is trivial, where ι_{2*} and $\pi_{1\vee 2*}$ are the induced homomorphisms induced by ι_2 and $\pi_{1\vee 2}$, respectively, between digital 18-fundamental groups, $X = X_{1\vee 2}$ and $X \vee X = X_{1\vee 2\vee 3\vee 4}$ with 18-adjacency and base point x_0 considered in $\mathbb{Z}^3$ via digital homeomorphisms. The constant function e_{32n} in the above equation is derived from the fact that the digital Whitehead product $[f_1, e_8]_{\mathrm{dW}}$ of f_1 and e_8 is also a constant function, because

$$\begin{aligned} [f_1, e_8]_{\mathrm{dW}} &= [f_1 * e_8 * \bar{f_1} * \bar{e}_8] \\ &= [f_1 * \bar{f_1}] \\ &= [c_{x_0}], \end{aligned}$$

so $c_{x_0} \circ a_n = e_{32n}$. Similarly, we have

(1) $\pi_{3\vee 4} \circ \varphi \circ f_1'$ is a trivial extension of f_1;
(2) $\pi_{1\vee 2} \circ \varphi \circ f_2'$ is a trivial extension of f_2; and
(3) $\pi_{3\vee 4} \circ \varphi \circ f_2'$ is a trivial extension of f_2.

In order to calculate the cardinality of the set of digital homotopy classes $[\varphi \circ f_1'] \in \pi_1^{18}(X \vee X, x_0)$ making $\varphi : X \to X \vee X$ into a digitally quasi comultiplication, it suffices to check that the conditions of digitally quasi comultiplications are satisfied only for the free generators, $[f_1]$ and $[f_2]$, of the digital fundamental group $\pi_1^{18}(X, x_0)$ which is the free product of two infinite cyclic groups; that is, a free group on these two generators. Indeed, the digital wedge product $X = X_s \vee X_t$ $(s, t = 1, 2, 3, 4)$ plays a role of the figure-eight space, i.e., a bouquet of two circles (see [20, pp. 123-124]).

Let $f : [0, 32]_{\mathbb{Z}} \to (S, s_0)$ be a digital 4-loop in the pointed digital image (S, s_0) with 4-adjacency in $\mathbb{Z}^2$ defined by

$$f(i) = \begin{cases} s_0 & \text{for } i = 0, 32; \\ s_i & \text{for } 1 \le i \le 31. \end{cases}$$

Then by using the same methods as in [22, p. 345], we can see that the digital 4-fundamental group of (S, s_0) becomes the infinite cyclic group generated by the digital 4-loop class $[f]$ (see also [8]), i.e.,

$$\pi_1^4(S, s_0) \;\cong\; <[f]> \;\cong\; 32\mathbb{Z}.$$

We note that

(1) the identity $[e]$ of $\pi_1^4(S, s_0)$ is the class of a constant function;
(2) the inverse $[f]^{-1}$ of the digital 4-loop class $[f]$ in $\pi_1^4(S, s_0)$ is the class $[\bar{f}]$ of the digital 4-loop $\bar{f} : [0, 32]_{\mathbb{Z}} \to (S, s_0)$ defined by $\bar{f}(i) = f(32 - i) = s_{32-i}$; and
(3) the digital 4-loop class $n[f]$ means the class of the $(2, 4)$-continuous function

$$nf = \underbrace{f * f * \cdots * f}_{(n \text{ times})} : [0, 32n]_{\mathbb{Z}} \longrightarrow (S, s_0)$$

defined by $(nf)(i \bmod 32) = f(i)$.

Finally, we also note that the set of digitally quasi comultiplications of $X(= X_{1\vee 2})$ is nonempty and finite, and we can identify a_n with nf, and b_m with mf from $\pi_1^4(S, s_0) = 32\mathbb{Z}$ so that the cardinal number of a set of digital homotopy classes $[\varphi \circ f_1']$ based on the digitally quasi comultiplication $\varphi : X \to X \vee X$ has the cardinality $\aleph_0$, and similarly for $[\varphi \circ f_2']$, as required. $\square$

We remark that the digital image (S, s_0) with 4-adjacency in $\mathbb{Z}^2$ plays a role of the unit circle $(S^1, (1, 0))$ in the 2-dimensional Euclidean space. Indeed, the fundamental group $\pi_1(S^1, (1, 0))$ of the unit circle S^1 is isomorphic to the additive group of integers $\mathbb{Z}$, but for $n \geq 2$, $\pi_1(S^n, (1, 0, \ldots, 0))$ is the trivial group.

Let (Y, y_0) be a digital image with k-adjacency in $\mathbb{Z}^n$. Let $A \subset Y$ and let $r : Y \to A$ be a digitally (k, k)-continuous function such that $r(a) = a$ for all $a \in A$, i.e., the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "r"] & A \\ A, \arrow[u, hook, "i"] \arrow[ur, "1_A"'] & \end{tikzcd}$$

where $1_A : A \to A$ is the identity map on A. Such a map $r : Y \to A$ is called a *digital retraction* [5], and A is said to be a *digital retract* of Y.

Example 3.11. Let $A = X_1$ and $Y = A \cup \{u\}$ with the 18-adjacency relation on $\mathbb{Z}^3$, where $u = (-1, 0, 0) \in \mathbb{Z}^3$. We define a $(18, 18)$-continuous function $r : Y \to A$ by

$$r(y) = \begin{cases} y & \text{for } y \in A; \\ x_0 = (0, 0, 0) & \text{for } y = u. \end{cases}$$

Then $r : Y \to A$ is a digital retraction (see Figure 4).

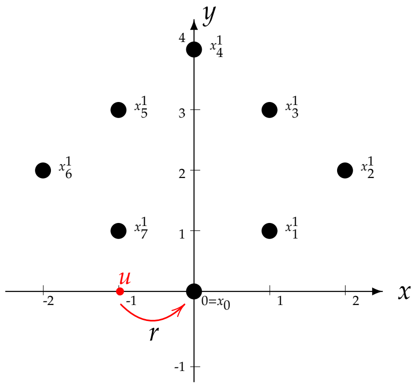

Figure 4: Image of the digital retraction $r : Y \to A$ in the xy-plane with $z = 0$: The point u goes to $x_0 = (0, 0, 0)$ under the digital retraction $r : Y \to A$, and the elements of A are the fixed points of r.

It is well known that the (digital) retraction induces an epimorphism between (digital) fundamental groups [7], as well as many kinds of algebraic tools in algebraic topology such as higher homotopy groups, homology groups and more generally all homology theories. This raises the basic question concerning with a digital homotopical viewpoint: For a given digitally quasi co-H-space $((Y, y_0), \varphi_Y)$ with $y_0 \in A \subsetneqq Y$, can we construct a digitally quasi comultiplication of A? The following answers to this question.

Theorem 3.12. *Let $((Y, y_0), \varphi_Y)$ be a digitally quasi co-H-space consisting of a pointed digital image (Y, y_0) with k-adjacency and a (k, k)-continuous function $\varphi_Y : Y \to Y \vee Y$. If $r : Y \to A$ is a digital retraction, then (A, φ_A) is a digital quasi co-H-space with a (k, k)-continuous function $\varphi_A : A \to A \vee A$ as a digitally quasi comultiplication of A.*

Proof. Let $f : [0, m]_{\mathbb{Z}} \to (Y, y_0)$ be any digital k-loop in (Y, y_0). Since $\varphi_Y : Y \to Y \vee Y$ is a digitally quasi comultiplication, there is a digital k-loop $f' : [0, n]_{\mathbb{Z}} \to (Y, y_0)$ such that $p_1 \circ \varphi_Y \circ f'$ or $p_2 \circ \varphi_Y \circ f'$ is a trivial extension of f, where $p_1 : Y \vee Y \to Y$ and $p_2 : Y \vee Y \to Y$ are the first and second projections, respectively. Thus we have the following commutative diagrams

$$\begin{tikzcd}
{[0,n]_{\mathbb{Z}}} \arrow[r, "f'"] & Y \arrow[r, "\varphi_Y"] \arrow[d, "p_1\circ\varphi_Y"'] & Y \vee Y \arrow[dl, "p_1"] \\
{[0,m]_{\mathbb{Z}}} \arrow[r, "f"] & Y &
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
{[0,n]_{\mathbb{Z}}} \arrow[r, "f'"] & Y \arrow[r, "\varphi_Y"] \arrow[d, "p_2\circ\varphi_Y"'] & Y \vee Y \arrow[dl, "p_2"] \\
{[0,m]_{\mathbb{Z}}} \arrow[r, "f"] & Y. &
\end{tikzcd}$$

Let $g : [0, m]_{\mathbb{Z}} \to A$ be a digital k-loop in A, and let $q_1 : A \vee A \to A$ and $q_2 : A \vee A \to A$ be the first and second projections, respectively. Then by restricting the digital image Y to A with the k-adjacency and by considering the hypotheses, we have the following commutative diagram (similarly, for the second projections $p_2 : Y \vee Y \to Y$ and $q_2 : A \vee A \to A$)

$$\begin{tikzcd}
{[0,n]_{\mathbb{Z}}} \arrow[r, "\exists g'"] & A \arrow[r, hook, "i"] & Y \arrow[r, "\varphi_Y"] \arrow[d, "p_1\circ\varphi_Y"'] & Y \vee Y \arrow[r, "r\vee r"] \arrow[dl, "p_1"] & A \vee A \arrow[dl, "q_1"] \\
{[0,m]_{\mathbb{Z}}} \arrow[r, "g"] & A \arrow[r, hook, "i"] & Y \arrow[r, "r"] & A. &
\end{tikzcd}$$

Indeed,

(1) $q_1 \circ (r \vee r)(y, y_0) = q_1(r(y), y_0) = r(y) = r \circ p_1(y, y_0)$;
(2) $q_1 \circ (r \vee r)(y_0, y) = q_1(y_0, r(y)) = y_0 = r(y_0) = r \circ p_1(y_0, y)$;

(3) $q_2 \circ (r \vee r)(y, y_0) = q_2(r(y), y_0)) = y_0 = r(y_0) = r \circ p_2(y, y_0)$;
(4) $q_2 \circ (r \vee r)(y_0, y) = q_2(y_0, r(y)) = r(y) = r \circ p_2(y_0, y)$; and
(5) the existence of a digital k-loop $g' : [0, n]_{\mathbb{Z}} \to A$ can be guaranteed because $\varphi_Y : Y \to Y \vee Y$ is a digitally quasi comultiplication of Y with $A \subset Y$.

Since $((Y, y_0), \varphi_Y)$ is a digitally quasi co-H-space, the composite $p_1 \circ \varphi_Y \circ i \circ g'$, via the inclusion $i : A \hookrightarrow Y$, is a trivial extension of $i \circ g$, or $p_2 \circ \varphi_Y \circ i \circ g'$ is a trivial extension of $i \circ g$.

We now define $\varphi_A : A \to A \vee A$ by

$$\varphi_A = (r \vee r) \circ \varphi_Y \circ i.$$

Then the following diagrams are commutative:

$$\begin{tikzcd} A \arrow[r, "\varphi_A"] \arrow[d, hook, "i"] & A \vee A \arrow[r, "q_1"] & A \\ Y \arrow[r, "\varphi_Y"] & Y \vee Y \arrow[r, "p_1"] \arrow[u, "r \vee r"] & Y \arrow[u, "r"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "\varphi_A"] \arrow[d, hook, "i"] & A \vee A \arrow[r, "q_2"] & A \\ Y \arrow[r, "\varphi_Y"] & Y \vee Y \arrow[r, "p_2"] \arrow[u, "r \vee r"] & Y. \arrow[u, "r"] \end{tikzcd}$$

Moreover, we have

$$\begin{aligned} q_1 \circ \varphi_A \circ g' &= q_1 \circ (r \vee r) \circ \varphi_Y \circ i \circ g' \\ &= r \circ p_1 \circ \varphi_Y \circ i \circ g'. \end{aligned}$$

Since $p_1 \circ \varphi_Y \circ i \circ g'$ is a trivial extension of $i \circ g$, we can see that $q_1 \circ \varphi_A \circ g'$ is a trivial extension of $g(= r \circ i \circ g)$ which shows that $\varphi_A : A \to A \vee A$ is a digitally quasi comultiplication of A. Similarly for the projections $p_2 : Y \vee Y \to Y$ and $q_2 : A \vee A \to A$, as required. $\square$

Example 3.13. Let $\varphi : X \to X \vee X$ be a digitally quasi comultiplication of $X = X_{1\vee 2}$ and let $r : X \to X_1$ be a digital retraction (see Figure 5) defined by

$$r(x_s^t) = \begin{cases} x_s^t & \text{for } s = 0, 1, \ldots, 7 \text{ and } t = 1; \\ x_0 = (0, 0, 0) & \text{for } s = 0, 1, \ldots, 7 \text{ and } t = 2. \end{cases}$$

Then $\varphi_{X_1} = (r \vee r) \circ \varphi \circ i : X_1 \to X_1 \vee X_1$ becomes a digitally quasi comultiplication of X_1, where $i : X_1 \hookrightarrow X$ is the inclusion.

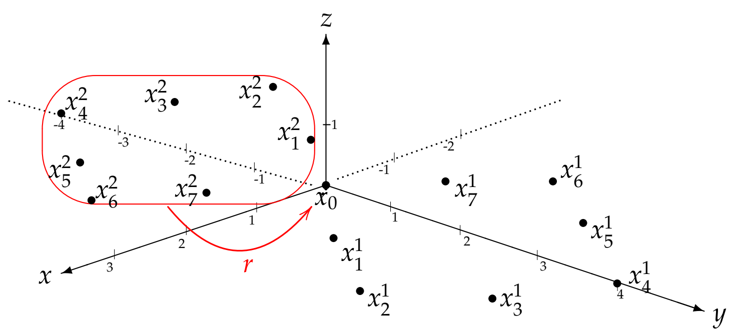

Figure 5: Image of the digital retraction $r : X \to X_1$: The points $x_s^2, s = 0, 1, 2, \ldots, 7$ go to $x_0 = (0, 0, 0)$, and the points $x_s^1, s = 0, 1, 2, \ldots, 7$ have remained fixed under the digital retraction $r : X \to X_1$.

4. Summary and further work

By using the basic properties of digital images and the digital Whitehead products, we have constructed the fundamental concepts of digitally quasi co-H-spaces and developed a method of calculating the cardinal number of digital homotopy classes based on the digitally quasi comultiplications of the digital wedge products of pointed digital images as a particular case. We have also introduced a new method for constructing a digitally quasi co-H-space as a digital retract of a given digitally quasi co-H-space.

As a further work and a subsequent paper, using the singular (or simplicial) homology in algebraic topology, we can consider a matter in all its aspects of the corresponding digital versions of homology theory and (rational) homotopy theory from the computer science theoretical and digital topological points of view.

List of Notations

We finally present some of the basic notations used in this paper as follows:

- For $s, t = 1, 2, 3, 4$ with $s \neq t$, we denote $X_{s\vee t}$ by the digital image $X_s \cup X_t$ with 18-adjacency and base point x_0 in $\mathbb{Z}^3$ which is $(18, k(2, 6))$-homeomorphic to $X_s \vee X_t$, and similarly denote $X_{1\vee 2\vee 3\vee 4}$ by the digital image $X_1 \cup X_2 \cup X_3 \cup X_4$ with 18-adjacency and base point x_0 in $\mathbb{Z}^3$ (see Example 3.2).
- $\pi_{s\vee t} : X_{1\vee 2\vee 3\vee 4} \to X_{s\vee t}$ is the projection to the (s, t)th factor among digital wedge products for $s, t = 1, 2, 3, 4$.
- $h_1 : X_1 \to X_{1\vee 2}$ is the $(18, 18)$-continuous function defined by $h_1(x_i^1) = x_i^1$ for $i = 0, 1, 2, \ldots, 7$.
- $h_2 : X_2 \to X_{3\vee 4}$ is the $(18, 18)$-continuous function defined by $h_2(x_i^2) = x_i^4$ for $i = 0, 1, 2, \ldots, 7$.
- $\iota_1 : X_{1\vee 2} \to X_{1\vee 2\vee 3\vee 4}$ is the $(18, 18)$-continuous function defined by $\iota_1(x_i^j) = x_i^j$ for $i = 0, 1, \ldots, 7$ and $j = 1, 2$.
- $\iota_2 : X_{1\vee 2} \to X_{1\vee 2\vee 3\vee 4}$ is the $(18, 18)$-continuous function defined by $\iota_2(x_i^j) = x_i^{j+2}$ for $i = 0, 1, \ldots, 7$ and $j = 1, 2$.
- $e_n : [0, n]_{\mathbb{Z}} \to X_{1\vee 2}$ is a constant function at $x_0 = (0, 0, 0)$.
- Since $X_{s\vee t} \approx_{(18,k(2,6))} X_s \vee X_t$ for $s, t = 1, 2, 3, 4$, we denote any one of them by X as the digital image with 18-adjacency and base point x_0 in $\mathbb{Z}^3$. Since $X_1 \vee X_2 \vee X_1 \vee X_2 \approx_{(k(2,12),k(2,12))} X_1 \vee X_2 \vee X_3 \vee X_4 \approx_{(k(2,12),18)} X_{1\vee 2\vee 3\vee 4}$, we denote any one of them by $X \vee X$ as the digital image with 18-adjacency and base point x_0 in $\mathbb{Z}^3$.
- More generally, the digital wedge product $Y \vee Y$ denotes the pointed digital image $(Y_1 \cup Y_2, \bar{y}_0)$ with $k(u, n)$-adjacency in $\mathbb{Z}^n$. Here, as previously mentioned, $Y_1 \approx_{(k(u,n),k(u,n))} Y \approx_{(k(u,n),k(u,n))} Y_2$, $Y_1 \cap Y_2$ is a single point set $\{\bar{y}_0\}$, and any element of $Y_1 - \{\bar{y}_0\}$ is not a $k(u, n)$-neighbor of any element of $Y_2 - \{\bar{y}_0\}$.
- $\aleph_0$ means the aleph-naught; that is, the cardinality of the set of all natural numbers.

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