Filomat 31:8 (2017), 2433–2439
DOI 10.2298/FIL1708433S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the $H_{\hat{N}}$-Integration of Spatial (Integral) Derivatives of Multivector Fields with Singularities in $\mathbb{R}^N$

Branko Sarić[a]

[a]*Faculty of Sciences, University of Novi Sad. Trg Dositeja Obradovića 2, 21000 Novi Sad, Serbia;*
College of Technical Engineering Professional Studies, Svetog Save 65, 32 000 Čačak, Serbia

Abstract. A method of spatial (integral) differentiation of multivector fields in an $\hat{N}$ -dimensional manifold M, into which a hyper-rectangle $[a, b]$ is mapped by a bijective smooth map $\boldsymbol{r} : [a, b] \to M$, has been introduced. For a class of discontinuous multivector fields a new concept of a residual field as well as the concept of total $H_{\hat{N}}$ integrability have been defined. Finally, this led naturally to an extension of *Cauchy's* residue theorem in M.

1. Introduction

The *Kurzweil-Henstock* integral [2, 5], defined by means of *Riemann* sums, and with certain modification of the fineness of a partition of the interval $[a, b] \subset \mathbb{R}$, was the first generalized *Riemann* integral. This integral is equivalent to the integrals of *Denjoy* and *Peron*, [5]. The *McShane* integral [3], which is equivalent to the *Lebesque* integral [5], was the second generalized *Riemann* integral. In contrast to the one-dimensional case, the *Kurzweil-Henstock* integral in $\mathbb{R}^N$ does not integrate all derivatives (see *Pfeffer* [13]). In order to remove this flow *Mawhin* [12] added a condition restricting the class of admissible partitions of an N-dimensional interval. This led to another *Riemann* type integral named regular partition integral, that would integrate all derivatives in $\mathbb{R}^N$. *Macdonald* [11] used the regular partition integral to overcome the deficiency in *Hestenes'* proof of *Stokes'* theorem, [1, 6]. *Sarić* [15] defined a new integral named total H_1-integral. This integral solves the problem in formulating the fundamental theorem of calculus in $\mathbb{R}$ whenever a primitive F is defined at the end points of $[a, b] \subset \mathbb{R}$. Accordingly, in what follows, we will try to extend *Cauchy's* integral formula to an $\hat{N}$-dimensional manifold M immersed in $\mathbb{R}^N$, for a large scale class of multivector fields F, and in the spirit of *Hestenes'* appealing proof. To do this, we must firstly define a so-called spatial (integral) derivative of F in M. After that it remains to define an integral that would integrate this derivative.

During the last few decades, many researchers focused their attention on the study and generalizations of the *Montgomery* identity. In 2015., using the total value of the *Riemann* integral, *Sarić* and *Jakupović* [14] establish a generalized *Montgomery* identity. In the same year, *Sarikaya*, *Filiz* and *Kiris* [17] used a generalized *Montogomery* identity for the *Riemann-Liouville* fractional integrals to establish some new *Ostrowski* type integral inequalities, and *Hussain* and *Rashad* [9] generalized n-dimensional *Montgomery* identity, which

2010 *Mathematics Subject Classification*. Primary 26B20; Secondary 26A39
Keywords. Spatial (integral) derivative, total $H_{\hat{N}}$ integrability
Received: 17 January 2016; Accepted: 09 July 2016
Communicated by Hari M. Srivastava
The author's research is supported by the Ministry of Science, Technology and Development, Republic of Serbia (Project ON 174024)
Email address: saric.b@mts.rs (Branko Sarić)

they used to subsequently find an *Ostrowski* type integral inequalities unifying the continuous, discrete and quantum cases. Accordingly, a further generalization of the n-dimensional *Montgomery* identity may be established on the total value of the $H_{\hat{N}}$-directed integral defined in what follows.

2. Preliminaries

The ambient space of this note is the N-dimensional *Euclidean* space $\mathbb{R}^N$. By $\mathbb{N}$ we denote the set of natural numbers. The measure $|E|$ of a set E in $\mathbb{R}^N$ is the *Lebesgue* outer measure. Let $(\boldsymbol{e}_i)_{i=1}^N = (\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_N)$ be the standard orthonormal basis for $\mathbb{R}^N$. With the *Cartesian* coordinate system every point x in $\mathbb{R}^N$ has an ordered set $\left(x^i\right)_{i=1}^N$ associated with it. We work in $\mathbb{R}^N$ with the usual inner (dot) product $x \cdot y = \sum_{i=1}^N x^i y^i$ and the associated *Euclidean* norm $\|\cdot\|$. Given a positive integer $\hat{N} \leq N$, by a brick $[a, b]$ in $\mathbb{R}^{\hat{N}}$ we mean an $\hat{N}$-dimensional hyper-rectangle (also called an orthotope) formally defined as the *Cartesian* product of $\hat{N}$ non-degenerate compact intervals $\left[a^{\hat{n}}, b^{\hat{n}}\right]$ ($\hat{n} = 1, 2, ..., \hat{N}$). In mathematical symbols, $[a, b] := \Pi_{\hat{n}=1}^{\hat{N}} \left[a^{\hat{n}}, b^{\hat{n}}\right] = \left[a^1, b^1\right] \times \left[a^2, b^2\right] \times ... \times [a^{\hat{N}}, b^{\hat{N}}]$. The collection $\mathcal{I}([a, b])$ is a family of all non-degenerate compact hyper-rectangles $I = \Pi_{\hat{n}=1}^{\hat{N}} \left[u^{\hat{n}}, v^{\hat{n}}\right]$ such that $\left[u^{\hat{n}}, v^{\hat{n}}\right] \subseteq \left[a^{\hat{n}}, b^{\hat{n}}\right]$, [10]. By $\mathfrak{L}(\hat{N})$ we denote the set of all multi-indices $\hat{\boldsymbol{\imath}} = (i_{\hat{n}})_{\hat{n}=1}^{\hat{N}} = (i_1, i_2, ..., i_{\hat{N}})$ with $i_{\hat{n}} = 1, 2, ..., \nu_{\hat{n}}$ for each $\hat{n} = 1, 2, ..., \hat{N}$. The product partition of $[a, b]$, denoted by $P[a, b]$, is a finite collection of all orthotope-point pairs $([a_{\hat{\imath}}, b_{\hat{\imath}}], x_{\hat{\imath}})$ such that $[a_{\hat{\imath}}, b_{\hat{\imath}}] = \Pi_{\hat{n}=1}^{\hat{N}} [a_{i_{\hat{n}}}^{\hat{n}}, b_{i_{\hat{n}}}^{\hat{n}}] = [a_{i_1}^1, b_{i_1}^1] \times [a_{i_2}^2, b_{i_2}^2] \times ... \times [a_{i_{\hat{N}}}^{\hat{N}}, b_{i_{\hat{N}}}^{\hat{N}}]$ and $x_{\hat{\imath}} = (x_{i_{\hat{n}}}^{\hat{n}})_{\hat{n}=1}^{\hat{N}} = (x_{i_1}^1, x_{i_2}^2, ..., x_{i_{\hat{N}}}^{\hat{N}})$, for each $\hat{\boldsymbol{\imath}} \in \mathfrak{L}(\hat{N})$. In addition, for each $\hat{n} = 1, 2, ..., \hat{N}$ the intervals $[a_{i_{\hat{n}}}^{\hat{n}}, b_{i_{\hat{n}}}^{\hat{n}}]$ are non-overlapping (they have pairwise disjoint interiors), $\cup_{i_{\hat{n}}=1}^{\nu_{\hat{n}}} [a_{i_{\hat{n}}}^{\hat{n}}, b_{i_{\hat{n}}}^{\hat{n}}] = \left[a^{\hat{n}}, b^{\hat{n}}\right]$ and $x_{i_{\hat{n}}}^{\hat{n}} \in [a_{i_{\hat{n}}}^{\hat{n}}, b_{i_{\hat{n}}}^{\hat{n}}]$. It is evident that a given product partition $P([a, b])$ of $[a, b]$ can be tagged in infinitely many ways by choosing different points as tags. If E is a set of points belonging to $[a, b]$, then the restriction of $P([a, b])$ to E is a finite collection of $([a_{\hat{\imath}}, b_{\hat{\imath}}], x_{\hat{\imath}}) \in P([a, b])$ such that each pair of $[a_{\hat{\imath}}, b_{\hat{\imath}}]$ and E intersects in at least one point and all $x_{\hat{\imath}}$ are tagged in E. In mathematical symbols, $P([a, b])|_E = \{([a_{\hat{\imath}}, b_{\hat{\imath}}], x_{\hat{\imath}}) \in P([a, b]) \mid x_{\hat{\imath}} \in [a_{\hat{\imath}}, b_{\hat{\imath}}] \cap E \neq \emptyset \text{ and } \hat{\boldsymbol{\imath}} \in \mathfrak{L}(\hat{N})\}$. The distance of the $[a_{\hat{\imath}}, b_{\hat{\imath}}]$, denoted by $diam([a_{\hat{\imath}}, b_{\hat{\imath}}])$, is defined as follows: $diam([a_{\hat{\imath}}, b_{\hat{\imath}}]) = \sup\{\left\|x - y\right\| \mid x, y \in [a_{\hat{\imath}}, b_{\hat{\imath}}]\}$, where $\left\|x - y\right\|$ is computed using the *Euclidean* norm of a vector in $\mathbb{R}^{\hat{N}}$. Given $\delta : [a, b] \to (0, 1)$, named a gauge, a product partition $P([a, b]) = \{([a_{\hat{\imath}}, b_{\hat{\imath}}], x_{\hat{\imath}}) \mid \hat{\boldsymbol{\imath}} \in \mathfrak{L}(\hat{N})\}$ is called *δ-fine* if $diam([a_{\hat{\imath}}, b_{\hat{\imath}}]) \leq \delta(x_{\hat{\imath}})$ for every $([a_{\hat{\imath}}, b_{\hat{\imath}}], x_{\hat{\imath}}) \in P([a, b])$. Let $\mathcal{P}([a, b])$ be the family of all product partitions $P([a, b])$ of $[a, b]$. For $E \subset [a, b]$ the family of all *δ-fine* product partitions $P([a, b]) \in \mathcal{P}([a, b])$ of $[a, b]$, such that $P([a, b])|_E \subset P([a, b])$, we denote by $\mathcal{P}_\delta([a, b])|_E$. For the infinite set of product partitions $P_n([a, b]) \in \mathcal{P}([a, b])$ of $[a, b]$, denoted by $\langle P_n([a, b])\rangle_{n=1}^{+\infty}$, we write $\langle P_n([a, b])\rangle_{n=1}^{+\infty} \in (\mathcal{P}([a, b]), \prec)$ if $P_n([a, b]) \prec P_{n+1}([a, b])$ for each $n \in \mathbb{N}$. The statement $P_n([a, b]) \prec P_{n+1}([a, b])$ means that for each orthotope-point pair $([a_{\hat{\imath}_{n+1}}, b_{\hat{\imath}_{n+1}}], x_{\hat{\imath}_{n+1}}) \in P_{n+1}([a, b])$ there exists a corresponding orthotope-point pair $([a_{\hat{\imath}_n}, b_{\hat{\imath}_n}], x_{\hat{\imath}_n}) \in P_n([a, b])$ such that $([a_{\hat{\imath}_{n+1}}, b_{\hat{\imath}_{n+1}}]) \subset ([a_{\hat{\imath}_n}, b_{\hat{\imath}_n}])$ and

$$\{x_{\hat{\imath}_n} \mid ([a_{\hat{\imath}_n}, b_{\hat{\imath}_n}], x_{\hat{\imath}_n}) \in P_n([a, b])\} \subset \{x_{\hat{\imath}_{n+1}} \mid ([a_{\hat{\imath}_{n+1}}, b_{\hat{\imath}_{n+1}}], x_{\hat{\imath}_{n+1}}) \in P_{n+1}([a, b])\}.$$

Then, $(\mathcal{P}([a, b]), \prec)$ is the family of directed sets, [4]. Clearly, for any $x \in [a, b]$ there exists a directed set $\langle P_n([a, b])\rangle_{n=1}^{+\infty} \in (\mathcal{P}([a, b]), \prec)$ so that x is a tag for it.

If by w we denote a set of independent variables $(w^{\hat{n}})_{\hat{n}=1}^{\hat{N}}$ each of which takes values within the corresponding interval $\left[a^{\hat{n}}, b^{\hat{n}}\right]$, then a bijective smooth map $\boldsymbol{r} : [a, b] \to M \subset \mathbb{R}^N$ that is a orientation-preserving diffeomorphism (has a smooth inverse), usually represented by an ordered set of *Cartesian* coordinates $x(w) = (x^{\hat{m}}((w^{\hat{n}})_{\hat{n}=1}^{\hat{N}})_{\hat{m}=1}^{\hat{N}}$ associated with it, is the parameterization of an $\hat{N}$-dimensional manifold denoted by M and immersed in $\mathbb{R}^N$. At each point $w \in M$ the set of $\hat{N}$ vectors $\{\partial_{w^{\hat{n}}} \boldsymbol{r}\}_{\hat{n}=1}^{\hat{N}}$ ($\partial_{w^{\hat{n}}}$ denotes the partial derivatives) constitute a basis for the tangent space $\mathfrak{I}_x(M)$ of M, [8]. Note that M is compact and has a boundary denoted by ∂M. Clearly, the partition $P(M)$ of M is a collection of all manifold-point pairs $(m_{\hat{\imath}}, w_{\hat{\imath}})$ into which the orthotope-point pairs $([a_{\hat{\imath}}, b_{\hat{\imath}}], x_{\hat{\imath}})$ belonging to $P[a, b] \in \mathcal{P}([a, b])$ are mapped by $\boldsymbol{r}$. Consequently, for a set $W \subset M$, which a image of $E \subset [a, b]$ induced by $\boldsymbol{r}$, the family $\mathcal{P}_\delta(M)|_W$ contains all

δ-fine partitions $P(M) = \{(m_{\hat{\imath}}, w_{\hat{\imath}}) \mid \hat{\boldsymbol{\imath}} \in \mathfrak{L}(\hat{N})\} \in \mathcal{P}(M)$ of M such that $P(M)|_W \subset P(M)$, and $(\mathcal{P}(M), \prec)$ is the family of directed sets $\langle P_n(M)\rangle_{n=1}^{+\infty}$.

In addition to the above mentioned inner product we work in $\mathbb{R}^{\hat{N}}$ with the geometric product too. The geometric product of vectors in $\mathbb{R}^{\hat{N}}$, which can be decomposed into the symmetric inner and anti-symmetic outer (wedge) product, has the following properties: associativity, distributivity and $\boldsymbol{v}\boldsymbol{v} = \boldsymbol{v}^2 = \|v\|^2$. Although the vector space $\mathbb{R}^{\hat{N}}$ is closed under vector addition, it is not closed under multiplication. Instead, by multiplication and addition the vectors of $\mathbb{R}^{\hat{N}}$ generate a larger linear space $\mathcal{G}(\mathbb{R}^{\hat{N}})$ called the geometric algebra of $\mathbb{R}^{\hat{N}}$. Given an integer $k \leq \hat{N}$, we denote by $\mathfrak{I}(\hat{N}, k)$ the set of all multi-indices $\boldsymbol{i}_k = (i_j)_{j=1}^k$ with $1 \leq i_1 < i_2 < \ldots < i_k \leq \hat{N}$, and for every $\boldsymbol{i}_k \in \mathfrak{I}(\hat{N}, k)$ the geometric product $\boldsymbol{e}_{\boldsymbol{i}_k} = \boldsymbol{e}_{i_1}\boldsymbol{e}_{i_2}\ldots\boldsymbol{e}_{i_k}$ of the orthonormal basis vectors $(\boldsymbol{e}_{i_j})_{j=1}^k$ is reduced to the outer product $\Lambda_{j=1}^k \boldsymbol{e}_{i_j} = \boldsymbol{e}_{i_1}\wedge\boldsymbol{e}_{i_2}\wedge\ldots\wedge\boldsymbol{e}_{i_k}$, since the inner product $\Theta_{j=1}^k \boldsymbol{e}_{i_j} = \boldsymbol{e}_{i_1} \cdot \boldsymbol{e}_{i_2} \cdot \ldots \cdot \boldsymbol{e}_{i_k}$ vanishes (there are no repeated factors in the product). The outer product is completely determined by the following properties: associativity, linearity in both arguments, $\boldsymbol{e}_i\wedge\boldsymbol{e}_j = -\boldsymbol{e}_j\wedge\boldsymbol{e}_i$ for every $i \neq j$ and $\boldsymbol{e}_i\wedge\boldsymbol{e}_i = 0$ for every i, [1]. A k-vector in $\mathbb{R}^{\hat{N}}$ is any formal linear combination $\sum_{\boldsymbol{i}_k\in\mathfrak{I}(\hat{N},k)} \alpha_{\boldsymbol{i}_k}\boldsymbol{e}_{\boldsymbol{i}_k}$ with $\alpha_{\boldsymbol{i}_k} \in \mathbb{R}$ for every $\boldsymbol{i}_k \in \mathfrak{I}(\hat{N}, k)$. The space of k-vectors is denoted by $\mathcal{G}_k(\mathbb{R}^{\hat{N}})$. In particular, $\mathcal{G}_1(\mathbb{R}^{\hat{N}}) = \mathbb{R}^{\hat{N}}$. For reasons of formal convenience, we set $\mathcal{G}_0(\mathbb{R}^{\hat{N}}) := \mathbb{R}$ and $\mathcal{G}_k(\mathbb{R}^{\hat{N}}) := \{0\}$ for $k > \hat{N}$. As $\mathcal{G}(\mathbb{R}^{\hat{N}}) = \sum_{k=0}^{\hat{N}} \mathcal{G}_k(\mathbb{R}^{\hat{N}})$, the elements $\mathcal{F}$ of $\mathcal{G}(\mathbb{R}^{\hat{N}})$ are called multivectors which can be expressed uniquely as a sum of its k-vector parts. The norm of $\mathcal{F} \in \mathcal{G}(\mathbb{R}^{\hat{N}})$ is defined by $\|\mathcal{F}\| = (\sum_{k=0}^{\hat{N}} \mathcal{F}_k^{\dagger}\mathcal{F}_k)^{1/2} \geq 0$, where $\mathcal{F}_k^{\dagger}$ is reverse (or adjoint) of $\mathcal{F}$, [6]. A simple k-vector is the outer product of k linearly independent 1-vectors, that is, $v = \boldsymbol{v}_1\wedge\boldsymbol{v}_2\wedge\ldots\wedge\boldsymbol{v}_k$, [1] . Since k linearly independent vectors also span a k-dimensional subspace V of $\mathbb{R}^{\hat{N}}$, it is apparent that to every simple k-vector there corresponds a unique k-dimensional subspace of $\mathbb{R}^{\hat{N}}$. In fact, every simple k-vector can be interpreted geometrically as an oriented volume of some k-dimensional subspace of $\mathbb{R}^{\hat{N}}$, [6]. Hence, the invertible map $\Lambda_{\hat{n}=1}^{\hat{N}}\partial_{w^{\hat{n}}}\boldsymbol{r} \leftrightarrow M$ $(\Lambda_{\hat{n}=1}^{\hat{N}}\partial_{w^{\hat{n}}}\boldsymbol{r} = \partial_{w^1}\boldsymbol{r} \wedge \partial_{w^2}\boldsymbol{r} \wedge \ldots \wedge \partial_{w^{\hat{N}}}\boldsymbol{r})$ is a one-to-one correspondence between the simple $\hat{N}$-vector and the oriented manifold M into which $[a, b]$ is mapped by $\boldsymbol{r}$, [1]. So, an orientation $\mathbb{I}_{\hat{N}}$ of M is a simple $\hat{N}$-vector with the norm $\left\|\mathbb{I}_{\hat{N}}\right\| = 1$ related to a positively oriented coordinate system by $\Lambda_{\hat{n}=1}^{\hat{N}}\partial_{w^{\hat{n}}}\boldsymbol{r} = \mathbb{I}_{\hat{N}}|\det g_{\hat{n}\hat{m}}|^{1/2}$, where $|\det g_{\hat{n}\hat{m}}|^{1/2} = \left\|\Lambda_{\hat{n}=1}^{\hat{N}}\partial_{w^{\hat{n}}}\boldsymbol{r}\right\|$ and $g_{\hat{n}\hat{m}} = \partial_{w^{\hat{n}}}\boldsymbol{r} \cdot \partial_{w^{\hat{m}}}\boldsymbol{r}$. The boundary ∂M of M is an $\hat{N} - 1$-dimensional manifold with induced orientation $\mathbb{I}_{\hat{N}-1}$ determined by the convention $\mathbb{I}_{\hat{N}} = \mathbb{I}_{\hat{N}-1}\aleph$, where $\aleph$ is the unit outward boundary normal at each point on ∂M. So, if $\mathbb{I}_1$ is a unit tangent vector on a curve in $\mathbb{R}^N$, then $\mathbb{I}_0 = \pm 1$ is a scalar-valued orientation assigned respectively to the end points of the curve segments. Clearly, the two end points of a curve segment have opposite orientation.

A multivector field $f : M \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ associates with each point $w \in M$ a multivector $f(w) \in \mathcal{G}(\mathbb{R}^{\hat{N}})$. By a manifold field on $\mathcal{M}(M)$ ($\mathcal{M}(M)$ is a image of $\mathcal{I}([a, b])$ induced by $\boldsymbol{r}$) we mean $\mathcal{F} : \mathcal{M}(M) \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ which associates with each manifold $m \in \mathcal{M}(M)$ a multivector $\mathcal{F}(m) \in \mathcal{G}(\mathbb{R}^{\hat{N}})$. Such a manifold field is called additive on $\mathcal{M}(M)$ if for any manifold $m \in \mathcal{M}(M)$ and any collection of non-overlapping manifolds $(m_i)_{i=1}^n$, whose union is m, the equality $\mathcal{F}(m) = \sum_{i=1}^n \mathcal{F}(m_i)$ holds. There are a number of different ways to define the limit of a manifold field. The definition given below comes from the definition of the *Moore-Smith* limit, [4].

Definition 2.1. *Let $\mathcal{F} : \mathcal{M}(M) \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ be a manifold field and $W \subset M$. A multivctor field f is the Moore-Smith limit of $\mathcal{F}$ on $M\backslash W$ if there exists a gauge δ on M such that for each $\langle P_n(M)\rangle_{n=1}^{+\infty} \in (\mathcal{P}_\delta(M)|_W, \prec)$ and for every $\varepsilon > 0$ there exists a partition $P_{n_\varepsilon}(M) \in \langle P_n(M)\rangle_{n=1}^{+\infty}$ such that*

$$\left\|\mathcal{F}(m_{\hat{\imath}_n}) - f(w_{\hat{\imath}_n})\right\| < \varepsilon, \tag{1}$$

whenever $(m_{\hat{\imath}_n}, w_{\hat{\imath}_n}) \in P_n(M)\backslash P_n(M)|_W$, $P_n(M) \in \langle P_n(M)\rangle_{n=1}^{+\infty}$ and $P_{n_\varepsilon}(M) \prec P_n(M)$. In mathematical symbols, $f(w) = \lim_{m\to w} \mathcal{F}(m)$.

If $\mathcal{F}$ converges to its limit f almost everywhere on M, that means for every $w \in M$ except for a set $W \subset M$ of *Lebesgue* outer measure zero, then the domain of f may not be all of M. If the set W is a countable set, then $\mathcal{F}$ is said to converge to f nearly everywhere on M.

Let $\triangle w = (\triangle w^{\hat{n}})_{\hat{n}=1}^{\hat{N}}$ be an ordered set of the line segments of the curvilinear coordinates $w^{\hat{n}}$ ($\hat{n} = 1, 2, ..., \hat{N}$), whose end points are obtained by mapping the end points of the coordinate intervals $\triangle x^{\hat{n}}$ ($\hat{n} = 1, 2, ..., \hat{N}$). The *Lebesgue* outer measure $|m|$ of $m \in \mathcal{M}(M)$ is the corresponding hyper-rectangle's volume. Hence, we set $|m| = \triangle^{\hat{N}} w = \left\| \Lambda_{\hat{n}=1}^{\hat{N}} \partial_{w^{\hat{n}}} \boldsymbol{r} \triangle w^{\hat{n}} \right\| = |\det g_{\hat{n}\hat{m}}|^{1/2} \triangle w^1 \triangle w^2 ... \triangle w^{\hat{N}}$. Now, we are in a position to define a differential $\hat{N}$-form $d^{\hat{N}}\mathcal{F} : M \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ on M, as follows.

Definition 2.2. *Let a multivector field f be the Moore-Smith limit of a manifold field $\mathcal{F} : \mathcal{M}(M) \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ on $M \backslash W$, where $W \subset M$. Then, the Moore-Smith limit of $\triangle^{\hat{N}}\mathcal{F} = \mathcal{F} \triangle^{\hat{N}} w$ on M is said to be a differential $\hat{N}$-form $d^{\hat{N}}\mathcal{F} = f d^{\hat{N}} w$ on M. In mathematical symbols, $d^{\hat{N}}\mathcal{F}(w) = \lim_{m \to w} \triangle^{\hat{N}}\mathcal{F}(m)$.*

In spite of the fact that the *Moore-Smith* limit of $\triangle^{\hat{N}} w$ vanishes identically on M, a differential $\hat{N}$-form $d^{\hat{N}}\mathcal{F} = f d^{\hat{N}} w$ could be a null multivector field on M (A multivector field $\mathcal{F} : M \to \mathcal{G}(R^{\hat{N}})$ is said to be a null multivector field on M if the set $\{w \in M \mid \mathcal{F}(w) \neq 0\}$ is a set of *Lebesgue* outer measure zero, see *2.4 Definition* in [2]). In what follows, we will use the notations $\sum_{\hat{\imath} \in \mathfrak{L}(\hat{N})} \mathbb{I}_{\hat{N}}(w_{\hat{\imath}}) \triangle^{\hat{N}}\mathcal{F}(m_{\hat{\imath}}) = \Xi_{\mathbb{I}\triangle\mathcal{F}}(P(M))$ and $\sum_{\hat{\imath} \in \mathfrak{L}(\hat{N})} \mathbb{I}_{\hat{N}}(w_{\hat{\imath}}) f(w_{\hat{\imath}}) |m_{\hat{\imath}}| = \Xi_{\mathbb{I}f\triangle w}(P(M))$, for each $(m_{\hat{\imath}}, w_{\hat{\imath}}) \in P(M)$. The following definition of the $\mathrm{H}_{\hat{N}}$-directed integral of $\mathbb{I}_{\hat{N}} d^{\hat{N}}\mathcal{F} = \mathbb{I}_{\hat{N}} f d^{\hat{N}} w$ comes from [4] and [6].

Definition 2.3. *For a multivector field $f : M \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ a multivector $\mathcal{L} \in \mathcal{G}(\mathbb{R}^{\hat{N}})$ is the $H_{\hat{N}}$-directed integral of $\mathbb{I}_{\hat{N}} f d^{\hat{N}} w$ over M, if there exists a gauge δ on M such that for each $\langle P_n(M) \rangle_{n=1}^{+\infty} \in (\mathcal{P}_\delta(M)|_W, \prec)$ and for every $\varepsilon > 0$ there exists a partition $P_{n_\varepsilon}(M) \in \langle P_n(M) \rangle_{n=1}^{+\infty}$ such that*

$$\|\Xi_{\mathbb{I}\triangle\mathcal{F}}(P(M) - \mathcal{L}\| < \varepsilon, \tag{2}$$

whenever $P_n(M) \in \langle P_n(M) \rangle_{n=1}^{+\infty}$ and $P_{n_\varepsilon}(M) \prec P_n(M)$. In mathematical symbols, $\mathcal{L} = H_{\hat{N}} - \int_M \mathbb{I}_{\hat{N}} f d^{\hat{N}} w$.

When working with multivector fields, which have a finite number of discontinuities on M, it does not really matter how these fields will be defined on the set of discontinuities $W \subset M$. The validity of this statement will be clarified as the theory unfolds. As this situation will arise frequently, we adopt the convention that, unless mentioned otherwise, such multivector fields are equal to 0 at all points at which they can take the infinite values or not be defined at all. Accordingly, we may define a multivector field $D_{ex} f : M \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ by extending f from $M \backslash W$ to W by $D_{ex} f(w) = 0$ for $w \in W$, so that

$$D_{ex} f(w) = \begin{cases} f(w), \text{ if } w \in M \backslash W \\ 0, \text{ if } w \in W \end{cases}. \tag{3}$$

3. Main Results

Let for $\mathcal{F} : \mathcal{M}(M) \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ the *Moore-Smith* limit f of $\mathcal{F}$ is defined on M. Then, it follows from *Definitions 2.1.* and *2.2.* that there exists a gauge δ on M such that for each $\langle P_n(M) \rangle_{n=1}^{+\infty} \in (\mathcal{P}_\delta(M)|_W, \prec)$ and for every $\varepsilon > 0$ there exists a partition $P_{n_\varepsilon}(M) \in \langle P_n(M) \rangle_{n=1}^{+\infty}$ such that

$$\left\| \Xi_{\mathbb{I}f\triangle w}(P_n(M)) - \Xi_{\mathbb{I}\triangle\mathcal{F}}(P_n(M)) \right\| < \varepsilon \left| m_{\hat{\imath}_n} \right|, \tag{4}$$

whenever $(m_{\hat{\imath}_n}, w_{\hat{\imath}_n}) \in P_n(M)$, $P_n(M) \in \langle P_n(M) \rangle_{n=1}^{+\infty}$ and $P_{n_\varepsilon}(M) \prec P_n(M)$. Consequently, in this case there holds $H_{\hat{N}} - \int_M \mathbb{I}_{\hat{N}} d^{\hat{N}}\mathcal{F} = H_{\hat{N}} - \int_M \mathbb{I}_{\hat{N}} f d^{\hat{N}} w$.

In the opposite case, when the *Moore-Smith* limit f of $\mathcal{F}$ on M, at some points of M, can take the infinite values or not be defined at all and hence the $\mathrm{H}_{\hat{N}}$-directed integrals of $\mathbb{I}_{\hat{N}} f d^{\hat{N}} w$ and $\mathbb{I}_{\hat{N}} d^{\hat{N}}\mathcal{F}$ over M can be distinguished from each other, it would be reasonable to make use of $\Xi_{\mathbb{I}\triangle\mathcal{F}}(P_n(M))$ instead of $\Xi_{\mathbb{I}D_{ex}f\triangle w}(P_n(M))$ to define an integral of both $\mathbb{I}_{\hat{N}} d^{\hat{N}}\mathcal{F}$ and $\mathbb{I}_{\hat{N}} f d^{\hat{N}} w$ over M. This is obviously our way of attempting to totalize the $\mathrm{H}_{\hat{N}}$-directed integral afore defined. The definition of the total $\mathrm{H}_{\hat{N}}$-directed integral which follows is more general one since it includes one more manifold field.

Definition 3.1. *Let $W \subset M$ and $G \subset M$ be disjoint sets of points at each of which, respectively, the Moore-Smith limits f and g of manifold fields $\mathcal{F} : \mathcal{M}(M) \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ and $\mathfrak{g} : \mathcal{M}(M) \to \mathcal{G}(\mathbb{R}^{\hat{N}})$, can take the infinite values or not be defined at all. A multivector $\mathcal{L} \in \mathcal{G}(\mathbb{R}^{\hat{N}})$ is the total $H_{\hat{N}}$-directed integral of $g\mathbb{I}_{\hat{N}} d^{\hat{N}}\mathcal{F} = g\mathbb{I}_{\hat{N}} f d^{\hat{N}} w$ over M if there exists a gauge δ on M such that for each $\langle P_n(M)\rangle_{n=1}^{+\infty} \in (\mathcal{P}_\delta(M)|_W, \prec)$ and for every $\varepsilon > 0$ there exists a partition $P_{n_\varepsilon}(M) \in \langle P_n(M)\rangle_{n=1}^{+\infty}$ such that $\left\|\Xi_{\mathfrak{g}\mathbb{I}\triangle\mathcal{F}}(P_n(M)) - \mathcal{L}\right\| < \varepsilon$, whenever $P_n(M) \in \langle P_n(M)\rangle_{n=1}^{+\infty}$ and $P_{n_\varepsilon}(M) \prec P_n(M)$. In mathematical symbols, $\mathcal{L} = H_{\hat{N}}\text{–}vt \int_M g\mathbb{I}_{\hat{N}} d^{\hat{N}}\mathcal{F}$.*

The crucial advantage of the integration process established by *Definition 3,1.* in comparison with any other integration process defined until now, including all the generalized *Riemann* approach to integration, lies in the fact that it is not necessary that g and $d^{\hat{N}}\mathcal{F}$, as the *Moore-Smith* limits of $\mathfrak{g}$ and $\triangle^{\hat{N}}\mathcal{F}$, respectively, to be defined at all points of M. This fact, upon which our theory is based in what follows, gives us the possibility to include the calculus of residues in the process of integration of multivector fields in M. Before that, we are prepared to prove the extended version of the fundamental theorem of calculus in M. As we will see, the proof becomes trivial if the definition of the total $H_{\hat{N}}$-directed integral is applied. In fact, in this way, we will attempt to put into a rigorous form *Hestenes'* proof based on the integral definition of the derivative and on the *Riemann* integral, [6]. A major motivation for the formulation of integration in this manuscript has been to achieve as simple and general a statement of the fundamental theorem as possible, just as was *Hestenes'* motivation too.

Let $\mathbb{I}_{\hat{N}-1} d^{\hat{N}-1}\sigma$ be a simple $\hat{N}-1$-vector describing a volume element of ∂M and let $\partial\mathcal{M}(M)$ be the family of the boundaries ∂m of $m \in \mathcal{M}(M)$. An oriented differential $\hat{N}-1$-form $\mathbb{I}_{\hat{N}-1} F d^{\hat{N}-1}\sigma$ on ∂M is said to be totally $H_{\hat{N}-1}$-directly integrable with respect to $\partial\mathcal{M}(M)$ if $\mathbb{I}_{\hat{N}-1} F d^{\hat{N}-1}\sigma$ is totally $H_{\hat{N}-1}$-directly integrable on every $\partial m \in \partial\mathcal{M}(M)$.

Definition 3.2. *Let for a multivector field F the oriented differential $\hat{N}-1$-form $\mathbb{I}_{\hat{N}-1} F d^{\hat{N}-1}\sigma$ be totally $H_{\hat{N}-1}$-directly integrable with respect to $\partial\mathcal{M}(M)$. Then, F is said to be spatially (integrally) differentiable to f on M, if f is the Moore-Smith limit on M of $\mathcal{F}$ defined by*

$$\mathcal{F}(m) = \frac{\mathbb{I}_{\hat{N}}^{\dagger}}{\triangle^{\hat{N}} w} H_{\hat{N}-1}\text{–}vt \oint_{\partial m} \mathbb{I}_{\hat{N}-1} F d^{\hat{N}-1}\sigma. \tag{5}$$

Theorem 3.3. *Let $W \subset M$ be a set at whose points the multivector field f, as the Moore-Smith limit on $M \backslash W$ of $\mathcal{F}$ defined by (5), can take the infinite values or not be defined at all. Then, $\mathbb{I}_{\hat{N}} f d^{\hat{N}} w = \mathbb{I}_{\hat{N}} d^{\hat{N}}\mathcal{F}$ is totally $H_{\hat{N}}$-directly integrable on M and*

$$H_{\hat{N}}\text{–}vt \int_M \mathbb{I}_{\hat{N}} f d^{\hat{N}} w = H_{\hat{N}}\text{–}vt \int_M \mathbb{I}_{\hat{N}} d^{\hat{N}}\mathcal{F} = H_{\hat{N}-1}\text{–}vt \oint_{\partial m} \mathbb{I}_{\hat{N}-1} F d^{\hat{N}-1}\sigma. \tag{6}$$

Proof. As the manifold field $\triangle^{\hat{N}}\mathcal{F} = \mathbb{I}_{\hat{N}}^{\dagger} H_{\hat{N}-1}\text{–}vt \oint_{\partial m} \mathbb{I}_{\hat{N}-1} F d^{\hat{N}-1}\sigma$ defined by (5) is additive and hence

$$\Xi_{\mathbb{I}\triangle\mathcal{F}}(P(M)) = H_{\hat{N}-1}\text{–}vt \oint_{\partial m} \mathbb{I}_{\hat{N}-1} F d^{\hat{N}-1}\sigma,$$

for each $P(M) \in \mathcal{P}(M)$, it follows from *Definition 3.1.* that

$$H_{\hat{N}}\text{–}vt \int_M \mathbb{I}_{\hat{N}} f d^{\hat{N}} w = H_{\hat{N}}\text{–}vt \int_M \mathbb{I}_{\hat{N}} d^{\hat{N}}\mathcal{F} = H_{\hat{N}-1}\text{–}vt \oint_{\partial m} \mathbb{I}_{\hat{N}-1} F d^{\hat{N}-1}\sigma.$$

□

In spite of the fact that the differential $\hat{N}$-form $d^{\hat{N}}\mathcal{F}$, as the *Moore-Smith* limit of $\triangle^{\hat{N}}\mathcal{F}$ defined by (5), can take the infinite values at some points of $W \subset M$ or be a null multivector field on M, in both cases, (6) is valid also. All this refer us to the following definitions.

Definition 3.4. *Let $W \subset M$. For an arbitrary multivector field F, such that the oriented differential $\hat{N}-1$-form $\mathbb{I}_{\hat{N}-1}Fd^{\hat{N}-1}\sigma$ is totally $H_{\hat{N}-1}$-directly integrable with respect to $\partial\mathcal{M}(M)$, the oriented differential $\hat{N}$-form $\mathbb{I}_{\hat{N}}d^{\hat{N}}\mathcal{F}$, as the Moore-Smith limit of $H_{\hat{N}-1}{-}vt\int_{\partial m}^{\circlearrowleft}\mathbb{I}_{\hat{N}-1}Fd^{\hat{N}-1}\sigma$ on M, is basically summable (BS_δ) in W to the sum $\mathfrak{R}$, if there exists a gauge δ on M such that for each $\langle P_n(M)\rangle_{n=1}^{+\infty} \in (\mathcal{P}_\delta(M)\,|_W\,,\prec)$ and for every $\varepsilon > 0$ there exists a partition $P_{n_\varepsilon}(M) \in \langle P_n(M)\rangle_{n=1}^{+\infty}$ such that $|\Xi_{\mathbb{I}\triangle\mathcal{F}}(P_n(M)\,|_W) - \mathfrak{R}| < \varepsilon$, whenever $P_n(M) \in \langle P_n(M)\rangle_{n=1}^{+\infty}$ and $P_{n_\varepsilon}(M) \prec P_n(M)$. If in addition W can be written as a countable union of sets on each of which $\mathbb{I}_{\hat{N}}d^{\hat{N}}\mathcal{F}$ is BS_δ, then $\mathbb{I}_{\hat{N}}d^{\hat{N}}\mathcal{F}$ is said to be BSG_δ in the set W. In mathematical symbols, $\mathfrak{R} := \sum_{w\in W}\mathbb{I}_{\hat{N}}d^{\hat{N}}\mathcal{F}\,(w)$.*

Definition 3.5. *Let F be an arbitrary multivector field such that the oriented differential $\hat{N}-1$-form $\mathbb{I}_{\hat{N}-1}Fd^{\hat{N}-1}\sigma$ is totally $H_{\hat{N}-1}$-directly integrable with respect to $\partial\mathcal{M}(M)$. Then, the Moore-Smith limit $\mathbb{I}_{\hat{N}}d^{\hat{N}}\mathcal{F}$ of $H_{\hat{N}-1}{-}vt\int_{\partial m}^{\circlearrowleft}\mathbb{I}_{\hat{N}-1}Fd^{\hat{N}-1}\sigma$ on M is said to be the residual field denoted by $\mathcal{R}$ of F. In mathematical symbols, $\mathcal{R} := \mathbb{I}_{\hat{N}}d^{\hat{N}}\mathcal{F}$.*

Comparing the two previous definitions with *Definition 3.1.* we may conclude that the sum of residues of F in M is the total $H_{\hat{N}}$-directed integral of $\mathbb{I}_{\hat{N}}d^{\hat{N}}\mathcal{F}$ over M, as follows

$$H_{\hat{N}}{-}vt\int_M \mathbb{I}_{\hat{N}}d^{\hat{N}}\mathcal{F} = \sum_{w\in M}\mathcal{R}\,(w)\,. \tag{7}$$

Let $W \subset M$ be a set of *Lebesgue* outer measure zero at whose points the spacial (integral) derivative f of a multivector field F, as the *Moore-Smith* limit on $M\backslash W$ of $\mathcal{F}$ defined by (5), can take the infinite values or not be defined at all. Since by (3) $\sum_{w\in W}\mathbb{I}_{\hat{N}}D_{ex}fd^{\hat{N}}w = 0$, this further implies that if $\mathbb{I}_{\hat{N}}fd^{\hat{N}}w$ is $H_{\hat{N}}$-directly integrable on M and hence $H_{\hat{N}}{-}\int_M \mathbb{I}_{\hat{N}}fd^{\hat{N}}w = \sum_{w\in M\backslash W}\mathcal{R}\,(w)$, then

$$H_{\hat{N}-1}{-}vt\int_{\partial m}^{\circlearrowleft}\mathbb{I}_{\hat{N}-1}Fd^{\hat{N}-1}\sigma = H_{\hat{N}}{-}vt\int_M \mathbb{I}_{\hat{N}}d^{\hat{N}}\mathcal{F} = H_{\hat{N}}{-}\int_M \mathbb{I}_{\hat{N}}fd^{\hat{N}}w + \sum_{w\in W}\mathcal{R}\,(w)\,. \tag{8}$$

In what follows we shall formulate the previous result as a theorem and prove it explicitly.

Theorem 3.6. *Let $W \subset M$ be a set of Lebesgue outer measure zero at whose points the spacial (integral) derivative f of a multivector field F, as the Moore-Smith limit on $M\backslash W$ of $\mathcal{F}$ defined by (5), can take the infinite values or not be defined at all. If $\mathbb{I}_{\hat{N}}d^{\hat{N}}\mathcal{F}$ is basically summable (BS_δ) in W to the sum $\mathfrak{R}$, then $\mathbb{I}_{\hat{N}}fd^{\hat{N}}w$ is $H_{\hat{N}}$-directly integrable on M and*

$$H_{\hat{N}}{-}\int_M \mathbb{I}_{\hat{N}}fd^{\hat{N}}w + \mathfrak{R} = H_{\hat{N}-1}{-}vt\int_{\partial m}^{\circlearrowleft}\mathbb{I}_{\hat{N}-1}Fd^{\hat{N}-1}\sigma. \tag{9}$$

Proof. Let the oriented differential form $\mathbb{I}_{\hat{N}-1}Fd^{\hat{N}-1}\sigma$ be totally $\mathrm{H}_{\hat{N}-1}$-directly integrable with respect to $\partial\mathcal{M}(M)$. Since $\mathbb{I}_{\hat{N}}d^{\hat{N}}\mathcal{F}$ is BS_δ in the set W to the sum $\mathfrak{R}$, it follows from *Definition 3.4* that there exists a gauge δ in M such that for each $\langle P_n(M)\rangle_{n=1}^{+\infty} \in (\mathcal{P}_\delta(M)\,|_W\,,\prec)$ and for every $\varepsilon > 0$ there exists a partition $P_{n_\varepsilon}(M) \in \langle P_n(M)\rangle_{n=1}^{+\infty}$ such that $\left\|\Xi_{\mathbb{I}\triangle\mathcal{F}}(P_n(M)\,|_W) - \mathfrak{R}\right\| < \varepsilon$, whenever $P_n(M) \in \langle P_n(M)\rangle_{n=1}^{+\infty}$ and $P_{n_\varepsilon}(M) \prec P_n(M)$. In addition, $D_{ex}f \equiv 0$ in W and $\Xi_{\mathbb{I}\triangle\mathcal{F}}(P(M)) = H_{\hat{N}-1}{-}vt\int_{\partial M}^{\circlearrowleft}\mathbb{I}_{\hat{N}-1}Fd^{\hat{N}-1}\sigma$, whenever $P(M) \in \mathcal{P}(M)\,|_W$. Take (4) into consideration it is readily seen that

$$\left\|\Xi_{\mathbb{I}D_{ex}f\triangle w}(P_n(M)) - [H_{\hat{N}-1}{-}vt\int_{\partial M}^{\circlearrowleft}\mathbb{I}_{\hat{N}-1}Fd^{\hat{N}-1}\sigma - \mathfrak{R}]\right\| \le$$

$$\le \left\|\Xi_{\mathbb{I}D_{ex}f\triangle w}(P_n(M)\backslash P_n(M)\,|_W) - \Xi_{\mathbb{I}\triangle\mathcal{F}}(P_n(M)\backslash P_n(M)\,|_W)\right\| + \left\|\Xi_{\mathbb{I}\triangle\mathcal{F}}(P_n(M)\,|_W) - \mathfrak{R}\right\| < \varepsilon(|M|+1),$$

whenever $P_n(M) \in \langle P_n(M)\rangle_{n=1}^{+\infty}$, $\langle P_n([a,b]_{i_k})\rangle_{n=1}^{+\infty} \in (\mathcal{P}_\delta(M)\,|_W\,,\prec)$ and $P_{n_\varepsilon}(M) \prec P_n(M)$. Therefore, $\mathbb{I}_{\hat{N}}fd^{\hat{N}}w$ is $H_{\hat{N}}$-directly integrable over M and

$$H_{\hat{N}}{-}\int_M \mathbb{I}_{\hat{N}}fd^{\hat{N}}w + \mathfrak{R} = H_{\hat{N}-1}{-}vt\int_{\partial M}^{\circlearrowleft}\mathbb{I}_{\hat{N}-1}Fd^{\hat{N}-1}\sigma.$$

□

By *Definition 3.5.* and (6), the result (9) of *Theorem 3.6.* can be rewritten as

$$H_{\hat{N}-1}\text{–}vt\oint_{\partial M}\mathbb{I}_{\hat{N}-1}Fd^{\hat{N}-1}\sigma = H_{\hat{N}}\text{–}vt\int_{M}\mathbb{I}_{\hat{N}}fd^{\hat{N}}w = H_{\hat{N}}\text{–}\int_{M}\mathbb{I}_{\hat{N}}fd^{\hat{N}}w + \sum_{w\in W}\mathcal{R}(w)\,. \tag{10}$$

If the spatial (integral) derivative f of F vanishes identically on $M\backslash W$, then it follows from (10) that

$$H_{\hat{N}-1}\text{–}vt\oint_{\partial M}\mathbb{I}_{\hat{N}-1}Fd^{\hat{N}-1}\sigma = \sum_{w\in W}\mathcal{R}(w)\,. \tag{11}$$

The obtained result provides an extension of *Cauchy's* integral formula from the calculus of residues in M (compare with results in [7, 16]).

References

[1] G. Alberti, Geometric Measure Theory. Encyclopedia of Mathematical Physics 2 (*J.P. Françoise* et al. eds.) Oxford, Elsevier (2006) 520–527.
[2] R.G. Bartle, A Modern Theory of Integration, Graduate Studies in Math. 32, Providence, AMS 2001.
[3] D.H. Fremlin, J. Mendoza, On the integration of vector-valued functions. Illinois Journal of Mathematics 38 (1994) 127–147.
[4] I.J.L. Garces, P.Y. Lee, Convergence theorems for the H_1-integral, Taiwanese Journal of Mathematics 4 (3) (2000) 439–445.
[5] R.A. Gordon, The Integrals of *Lebesgue*, *Denjoy*, *Perron* and *Henstock*. Graduate Studies in Mathematics 4, Providence, AMS, 1994.
[6] D. Hestenes, Multivector Calculus, Journal of Mathematical Analysis and Applications 24 (1968) 313–325.
[7] D. Hestene, Multivector Functions, Journal of Mathematical Analysis and Applications 24 (1968) 467–473.
[8] D. Hestenes, Differential Forms in Geometric Calculus. *Clifford* Algebras and their Applications in Mathematical Physics (Eds: F. Brackx et al.), 269–285, Dordrecht/Boston, Kluwer Academic Publishers, 1993.
[9] S. Hussain, M. S. Rashad, n-Dimensional *Ostrowski* Type Inequalities with Applications on Time Scales, Filomat 29 (7) (2015) 1623–1629.
[10] T.Y. Lee, A multidimensional integration by parts formula for the *Henstock-Kurzweil* integral, Mathematica bohemica 1 (2008) 63–74.
[11] A. Macdonald, *Stokes'* theorem, Real Analysis Exchange 27 (2002) 739–747.
[12] J. Mawhin, Generalized *Riemann* integrals and the divergence theorem for differentiable vector fields. In: *E.B. Christofel* (International *Christoffel* Symposium in Honour of *Christoffel* on the 150th Anniversary of his Birth, Aachen and Monschau 1979). The Influence of his Work on Mathematics and the Physical Sciences (Eds: *P.L. Butzer* and *F. Fehér*), Basel-Boston, Mass., Birkhäuser-Verlag (1981) 704–714.
[13] W. Pfeffer, The multidimensional fundamental theorem of calculus, Journal of the Australian Mathematical Society (Ser. A) 43 (1987) 143–170.
[14] B. Sarić, E. Jakupović, Totalization of the *Montgomery* identity, Novi Sad Journal of Mathematics 45 (2) (2015) 77–84.
[15] B. Sarić, On totalization of the H_1-integral, Taiwanese Journal of Mathematics 15 (4) (2011) 1691–1700.
[16] B. Sarić, *Cauchy's* residue theorem for a class of real valued functions, Czechoslovak Mathematical Journal 60 (4) (2010) 1043–1048.
[17] M. Z. Sarikaya, H. Filiz, M. E. Kiris, On Some Generalized Integral Inequalities for *Riemann-Liouville* Fractional Integrals, Filomat 29 (6) (2015) 1307–1314.