Filomat 31:8 (2017), 2449–2458
DOI 10.2298/FIL1708449C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Subclasses of Meromorphically Multivalent Functions Associated with the Dziok-Srivastava Operator

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Abstract. We introduce two new subclasses $H_{p,k}(\lambda, A, B)$ and $Q_{p,k}(\lambda, A, B)$ of meromorphically multivalent functions associated with the Dziok-Srivastava operator which is a special case of the Srivastava-Wright operator. Distortion inequalities, partial sums and convolutional theorems for $H_{p,k}(\lambda, A, B)$ and $Q_{p,k}(\lambda, A, B)$ are obtained.

1. Introduction

Let Σ_p be the class of functions

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \quad (p \in N := \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic in the punctured open unit disk $U_0 = \{z : 0 < |z| < 1\}$. For functions f and g analytic in the open unit disk $U = \{z : |z| < 1\}$, we say that f is subordinate to g in U and write $f \prec g$, if there exists an analytic function in U such that

$$|w(z)| \le |z| \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in U).$$

Let

$$f_j(z) = z^{-p} + \sum_{n=p}^{\infty} a_{n,j} z^n \in \Sigma_p \quad (j = 1, 2).$$

Then the Hadamard product (or convolution) of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = z^{-p} + \sum_{n=p}^{\infty} a_{n,1} a_{n,2} z^n = (f_2 * f_1)(z).$$

2010 *Mathematics Subject Classification*. Primary 30C45.
Keywords. Meromorphically multivalent functions; subordination; Hadamard product (or convolution); Generalized hypergeometric function; Srivastava-Wright operator; Dziok-Srivastava operator; distortion; partial sums.
Received: 19 February 2016; Accepted: 11 July 2016
Communicated by Hari M. Srivastava
This work is supported by National Natural Science Foundation of China(Grant No.11571299) and Natural Science Foundation of Jiangsu Province(Grant No. BK20151304).
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For complex parameters

$$\alpha_1, \cdots, \alpha_q \text{ and } \beta_1, \cdots, \beta_s \quad (\beta_j \neq Z_0^- := \{0, -1, -2, \cdots\}; j = 1, \cdots, s),$$

we now define the generalized hypergeometric function

$${}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$$

as follows:

$${}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!}$$

$$(q \leq s + 1; q, s \in N_0 = N \cup \{0\}; z \in U),$$

where $(\lambda)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1, & (\nu = 0; \lambda \in C \setminus \{0\}), \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1), & (\nu \in n \in N; \lambda \in C). \end{cases}$$

By convoluting the generalized hypergeometric function $z^{-p} \cdot_q F_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$ with the function $f \in \Sigma_p$, Dziok and Srivastava [10] introduced the Dziok-Srivastava linear operator

$$H_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s)$$

which is defined as follows:

$$\begin{aligned} H_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s) f(z) &:= z^{-p} \cdot_q F_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) * f(z) \\ &= z^{-p} + \sum_{n=p}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{a^n}{n!} z^{n-p}. \end{aligned}$$

For convenience, we write

$$H_{p,q,s}(\alpha_1) := H_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s)$$

and

$$\Gamma_n(\alpha_1) := \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{n!(\beta_1)n \cdots (\beta_s)_n} \quad (n \in N).$$

It should be remarked in passing that the existing literature on Geometric Function Theory also contains systematic investigations of various analytic function classes associated with a further generalization of the Dziok-Srivastava operator, which is known as the Srivastava-Wright operator defined by using the Fox-Wright generalized hypergeometric function (see, for details, [14] and [20]; see also [26] and the references cited therein including [14] and [20]).

In the present paper, we assume that

$$N = \{1, 2, 3, \cdots\}, k \in N \setminus \{1\}, -1 \leq B < 0, B < A \leq 1, \tag{1.2}$$

$$\alpha_j > 0 \ (j = 1, 2, \cdots, q) \text{ and } \beta_j > 0 \ (j = 1, \cdots, s).$$

Lemma 1. Let $\frac{1}{2} \leq \lambda \leq 1$. Also let $f \in \Sigma_p$ defined by (1.1) satisfy

$$\sum_{n=p}^{\infty} [\lambda n - p(1 - \lambda)\delta_{n,p,k}] \Gamma_n(\alpha_1) |a_n| \leq \frac{p(A - B)}{1 - B}. \tag{1.3}$$

Then

$$(1 - \lambda) z^p f_{p,k}(z) - \frac{\lambda}{p} z^{p+1} (H_{p,q,s}(\alpha_1) f(z))' \prec \frac{1 + Az}{1 + Bz} \quad (z \in U), \tag{1.4}$$

where

$$f_{p,k}(z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{jp} H_{p,q,s}(\alpha_1) f(\varepsilon_k^j z), \quad \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right) \tag{1.5}$$

and

$$\delta_{n,p,k} = \begin{cases} 0 & \left(\frac{n+p}{k} \notin N\right), \\ 1 & \left(\frac{n+p}{k} \in N\right). \end{cases} \tag{1.6}$$

Proof. In terms of (1.2) and (1.6), we see that

$$\lambda n - p(1-\lambda)\delta_{n,p,k} \geq p(2\lambda - 1) \geq 0 \tag{1.7}$$

for $n \geq p$ and $\lambda \geq \frac{1}{2}$.

Let the inequality (1.3) hold. We deduce that

$$\begin{aligned} \left| \frac{\left[(1-\lambda)z^p f_{p,k}(z) - \frac{\lambda}{p} z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'\right] - 1}{A - B\left[(1-\lambda)z^p f_{p,k}(z) - \frac{\lambda}{p} z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'\right]} \right| &= \left| \frac{\sum_{n=p}^{\infty}[\lambda n - p(1-\lambda)\delta_{n,p,k}]\Gamma_n(\alpha_1)a_n z^{n+p}}{p(A-B) + B\sum_{n=p}^{\infty}[\lambda n - p(1-\lambda)\delta_{n,p,k}]\Gamma_n(\alpha_1)a_n z^{n+p}} \right| \\ &\leq \frac{\sum_{n=p}^{\infty}[\lambda n - p(1-\lambda)\delta_{n,p,k}]\Gamma_n(\alpha_1)|a_n|}{p(A-B) + B\sum_{n=p}^{\infty}[\lambda n - p(1-\lambda)\delta_{n,p,k}]\Gamma_n(\alpha_1)|a_n|} \\ &\leq 1 \quad (|z| = 1). \end{aligned}$$

Hence, by the maximum modulus theorem, we arrive at (1.4).

Definition 1. A function $f \in \Sigma_p$ is said to be in the class $H_{p,k}(\lambda, A, B)$ if and only if it satisfies the coefficient inequality (1.3).

It follows from Lemma 1 that, if $f \in H_{p,k}(\lambda, A, B)$, then the subordination relation (1.4) holds.

Definition 2. A function $f \in \Sigma_p$ is said to be in the class $Q_{p,k}(\lambda, A, B)$ if and only if it satisfies

$$\sum_{n=p}^{\infty} n[\lambda n - p(1-\lambda)\delta_{n,p,k}]\Gamma_n(\alpha_1)|a_n| \leq \frac{p^2(A-B)}{1-B}. \tag{1.8}$$

From the Definitions 1 and 2 one can see that $Q_{p,k}(\lambda, A, B) \subset H_{p,k}(\lambda, A, B)$. For $f \in \Sigma_p$ defined by (1.1), we have

$$2z^{-p} + \frac{zf'(z)}{p} = z^{-p} + \sum_{n=p}^{\infty} \frac{n}{p} a_n z^n,$$

which implies that

$$f \in Q_{p,k}(\lambda, A, B) \text{ if and only if } 2z^{-p} + \frac{zf'(z)}{p} \in H_{p,k}(\lambda, A, B). \tag{1.9}$$

Many interesting classes of meromorphically multivalent functions were considered by earlier authors (see, e.g., [1-27] and the references therein). Motivated essentially by some recent works of Srivastava et al. [7, 20-26], the main object of the present paper is to derive some distortion inequalities of functions in the classes $H_{p,k}(\lambda, A, B)$ and $Q_{p,k}(\lambda, A, B)$. In particular some results of partial sums and convolution of functions in these classes are also given.

2. Main Results

Our first theorem is given by the following.

Theorem 1. Let

$$\frac{2p}{k} \in N \text{ and } \frac{1+A-2B}{2(1-B)} \le \lambda \le 1.$$

Suppose that the sequence $\{\Gamma_n(\alpha_1)\}$ $(n \ge p)$ is nondecreasing.

(i) If $f \in H_{p,k}(\lambda, A, B)$, then for $z \in U_0$,

$$|z|^{-p} - \frac{A-B}{(2\lambda-1)(1-B)\Gamma_p(\alpha_1)}|z|^p \le |f(z)| \le |z|^{-p} + \frac{A-B}{(2\lambda-1)(1-B)\Gamma_p(\alpha_1)}|z|^p. \tag{2.1}$$

(ii) If $f \in Q_{p,k}(\lambda, A, B)$, then for $z \in U_0$,

$$p\left(|z|^{-p-1} - \frac{A-B}{(2\lambda-1)(1-B)\Gamma_p(\alpha_1)}|z|^{p-1}\right) \le |f'(z)| \le p\left(|z|^{-p-1} + \frac{A-B}{(2\lambda-1)(1-B)\Gamma_p(\alpha_1)}|z|^{p-1}\right). \tag{2.2}$$

The bounds in (2.1) and (2.2) are sharp.

Proof. From (1.2) one can see easily that $\frac{1}{2} < \frac{1+A-2B}{2(1-B)} \le 1$. Let $\frac{2p}{k} \in N$. For $n \ge p$ and $\frac{n+p}{k} \in N$, we have $n = p + k(l-1)$ $(l \in N)$, $\delta_{n,p,k} = \delta_{p,p,k} = 1$, and so

$$\frac{(1-B)[\lambda n - p(1-\lambda)\delta_{n,p,k}]\Gamma_n(\alpha_1)}{p(A-B)} \ge \frac{(2\lambda-1)(1-B)\Gamma_p(\alpha_1)}{A-B}. \tag{2.3}$$

For $n \ge p$ and $\frac{n+p}{k} \notin N$, we have $\delta_{n,p,k} = \delta_{p+1,p,k} = 0$ and

$$\begin{aligned}\frac{(1-B)[\lambda n - p(1-\lambda)\delta_{n,p,k}]\Gamma_n(\alpha_1)}{p(A-B)} &\ge \frac{\lambda(p+1)(1-B)\Gamma_p(\alpha_1)}{p(A-B)} \\ &\ge \frac{(2\lambda-1)(1-B)\Gamma_p(\alpha_1)}{A-B}\end{aligned} \tag{2.4}$$

for $\frac{1}{2} < \lambda \le 1$.

(i) If $f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \in H_{p,k}(\lambda, A, B)$, then it follows from (2.3) and (2.4) that

$$\frac{(2\lambda-1)(1-B)\Gamma_p(\alpha_1)}{A-B}\sum_{n=p}^{\infty}|a_n| \le 1,$$

which yields (2.1).

(ii) If $f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \in Q_{p,k}(\lambda, A, B)$, then (2.3) and (2.4) yield

$$\frac{(2\lambda-1)(1-B)\Gamma_p(\alpha_1)}{p(A-B)}\sum_{n=p}^{\infty}n|a_n| \le 1.$$

This leads to (2.2).

Further, the bounds in (2.1) and (2.2) are best possible which can be seen from the function f defined by

$$f(z) = z^{-p} + \frac{A-B}{(2\lambda-1)(1-B)\Gamma_p(\alpha_1)}z^p \in Q_{p,k}(\lambda, A, B) \subset H_{p,k}(\lambda, A, B). \tag{2.5}$$

Theorem 2. Let

$$\frac{2p}{k} \notin N \text{ and } \frac{p}{p+1} \le \lambda \le 1.$$

Suppose that the sequence $\{\Gamma_n(\alpha_1)\}$ $(n \geq p)$ is nondecreasing.

(i) If $f \in H_{p,k}(\lambda, A, B)$, then for $z \in U_0$,

$$|z|^{-p} - \frac{A-B}{\lambda(1-B)\Gamma_p(\alpha_1)}|z|^p \leq |f(z)| \leq |z|^{-p} + \frac{A-B}{\lambda(1-B)\Gamma_p(\alpha_1)}|z|^p. \tag{2.6}$$

(ii) If $f \in Q_{p,k}(\lambda, A, B)$, then for $z \in U_0$,

$$p\left(|z|^{-p-1} - \frac{A-B}{\lambda(1-B)\Gamma_p(\alpha_1)}|z|^{p-1}\right) \leq |f'(z)| \leq p\left(|z|^{-p-1} + \frac{A-B}{\lambda(1-B)\Gamma_p(\alpha_1)}|z|^{p-1}\right). \tag{2.7}$$

The bounds in (2.6) and (2.7) are sharp.

Proof. Let $\frac{2p}{k} \notin N$. For $n \geq p$ and $\frac{n+p}{k} \notin N$, we have $\delta_{n,p,k} = \delta_{p,p,k} = 0$ and

$$\frac{(1-B)[\lambda n - p(1-\lambda)\delta_{n,p,k}]\Gamma_n(\alpha_1)}{p(A-B)} \geq \frac{\lambda(1-B)\Gamma_p(\alpha_1)}{A-B}. \tag{2.8}$$

For $n \geq p$ and $\frac{n+p}{k} \in N$, we have

$$\delta_{n,p,k} = 1, n = k\left(\left[\frac{2p}{k}\right] + l\right) - p > p \quad (l \in N),$$

and so for $\frac{p}{p+1} \leq \lambda \leq 1$,

$$\frac{(1-B)[\lambda n - p(1-\lambda)\delta_{n,p,k}]\Gamma_n(\alpha_1)}{p(A-B)} \geq \frac{(1-B)[\lambda(p+1) - p(1-\lambda)]\Gamma_p(\alpha_1)}{p(A-B)} \geq \frac{\lambda(1-B)\Gamma_p(\alpha_1)}{A-B}. \tag{2.9}$$

(i) If $f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \in H_{p,k}(\lambda, A, B)$, then it follows from (2.8) and (2.9) that

$$\frac{\lambda(1-B)\Gamma_p(\alpha_1)}{A-B}\sum_{n=p}^{\infty}|a_n| \leq 1,$$

which leads to (2.6).

(ii) If $f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \in Q_{p,k}(\lambda, A, B)$, then (2.8) and (2.9) give

$$\frac{\lambda(1-B)\Gamma_p(\alpha_1)}{p(A-B)}\sum_{n=p}^{\infty} n|a_n| \leq 1,$$

which yields (2.7).

Furthermore, the function f defined by

$$f(z) = z^{-p} + \frac{A-B}{\lambda(1-B)\Gamma_p(\alpha_1)}z^p \in Q_{p,k}(\lambda, A, B) \subset H_{p,k}(\lambda, A, B) \tag{2.10}$$

shows that the bounds in (2.6) and (2.7) are best possible.

Next, we derive several results of the partial sums of functions in the classes $H_{p,k}(\lambda, A, B)$ and $Q_{p,k}(\lambda, A, B)$.

Let $f \in \Sigma_p$ be given by (1.1) and define the partial sums $s_1(z)$ and $s_m(z)$ by

$$s_1(z) = z^{-p} \text{ and } s_m(z) = z^{-p} + \sum_{n=p}^{p+m-2} a_n z^n \quad (m \in N \setminus \{1\}). \tag{2.11}$$

For simplicity we use the notation γ_n $(n \geq p)$ as following:

$$\gamma_n = \frac{(1-B)[\lambda n - p(1-\lambda)\delta_{n,p,k}]\Gamma_n(\alpha_1)}{p(A-B)}. \tag{2.12}$$

Theorem 3. Suppose that the sequence $\{\Gamma_n(\alpha_1)\}$ $(n \geq p)$ is nondecreasing and $\Gamma_p(\alpha_1) \geq 1$. Let $f \in H_{p,k}(\lambda, A, B)$ and let

$$\max\left\{\frac{1+A-2B}{2(1-B)}, \frac{p}{p+1}\right\} \leq \lambda \leq 1. \tag{2.13}$$

Then for $m \in N$, we have

$$\mathrm{Re}\left(\frac{f(z)}{s_m(z)}\right) > 1 - \frac{1}{\gamma_{p+m-1}} \quad (z \in U) \tag{2.14}$$

and

$$\mathrm{Re}\left(\frac{s_m(z)}{f(z)}\right) > \frac{\gamma_{p+m-1}}{1+\gamma_{p+m-1}} \quad (z \in U). \tag{2.15}$$

The bounds in (2.14) and (2.15) are sharp for each m.

Proof. For $n \geq p$, we have from (2.12) and (2.13) that

$$\gamma_n = \frac{(1-B)[\lambda n - p(1-\lambda)\delta_{n,p,k}]\Gamma_n(\alpha_1)}{p(A-B)} \geq \frac{(1-B)(2\lambda-1)}{A-B} \geq 1 \tag{2.16}$$

and

$$\begin{aligned}\gamma_{n+1} &\geq \gamma_n + \frac{(1-B)[\lambda - p(1-\lambda)]}{p(A-B)} \\ &\geq \gamma_n.\end{aligned} \tag{2.17}$$

Let $f \in H_{p,k}(\lambda, A, B)$. Then it follows from (2.16) and (2.17) that

$$\sum_{n=p}^{p+m-2} |a_n| + \gamma_{p+m-1} \sum_{n=p+m-1}^{\infty} |a_n| \leq \sum_{n=p}^{\infty} \gamma_n \Gamma_n(\alpha_1)|a_n| \leq 1 \quad (m \in N \setminus \{1\}). \tag{2.18}$$

If we put

$$p_1(z) = 1 + \gamma_{p+m-1}\left(\frac{f(z)}{s_m(z)} - 1\right)$$

for $z \in U$ and $m \in N \setminus \{1\}$, then $p_1(0) = 1$ and we deduce from (2.18) that

$$\begin{aligned}\left|\frac{p_1(z)-1}{p_1(z)+1}\right| &= \left|\frac{\gamma_{p+m-1}\sum_{n=p+m-1}^{\infty} a_n z^{n+p}}{2\left(1+\sum_{n=p}^{p+m-2} a_n z^{n+p}\right) + \gamma_{p+m-1}\sum_{n=p+m-1}^{\infty} a_n z^{n+p}}\right| \\ &\leq \frac{\gamma_{p+m-1}\sum_{n=p+m-1}^{\infty} |a_n|}{2 - 2\sum_{n=p}^{p+m-2}|a_n| - \gamma_{p+m-1}\sum_{n=p+m-1}^{\infty}|a_n|} \\ &\leq 1 \quad (z \in U;\ m \in N \setminus \{1\}).\end{aligned}$$

This implies that $\mathrm{Re}(p_1(z)) > 0$ for $z \in U$, and so (2.14) holds for $m \in N \setminus \{1\}$.

Similarly, by setting

$$p_2(z) = (1 + \gamma_{p+m-1})\frac{s_m(z)}{f(z)} - \gamma_{p+m-1},$$

it follows from (2.18) that

$$\begin{aligned}\left|\frac{p_2(z)-1}{p_2(z)+1}\right| &= \left|\frac{-(1+\gamma_{p+m-1})\sum_{n=p+m-1}^{\infty} a_n z^{n+p}}{2\left(1+\sum_{n=p}^{p+m-2} a_n z^{n+p}\right) + (1-\gamma_{p+m-1})\sum_{n=p+m-1}^{\infty} a_n z^{n+p}}\right| \\ &\leq \frac{(1+\gamma_{p+m-1})\sum_{n=p+m-1}^{\infty} |a_n|}{2 - 2\sum_{n=p}^{p+m-2} |a_n| - (\gamma_{p+m-1}-1)\sum_{n=p+m-1}^{\infty} |a_n|} \\ &\leq 1 \quad (z \in U;\ m \in N \setminus \{1\}).\end{aligned}$$

Hence we have (2.15) for $m \in N \setminus \{1\}$.

For $m = 1$, replacing (2.18) by

$$\gamma_p \sum_{n=p}^{\infty} |a_n| \leq \sum_{n=p}^{\infty} \gamma_n |a_n| \leq 1$$

and proceeding as the above, we see that (2.14) and (2.15) are also true.

Furthermore, taking the function f defined by

$$f(z) = z^{-p} + \frac{z^{p+m-1}}{\gamma_{p+m-1}} \in H_{p,k}(\lambda, A, B),$$

we have $s_m(z) = z^{-p}$,

$$\operatorname{Re}\left(\frac{f(z)}{s_m(z)}\right) \to 1 - \frac{1}{\gamma_{p+m-1}} \quad \text{as} \quad z \to \exp\left(\frac{\pi i}{2p+m-1}\right)$$

and

$$\operatorname{Re}\left(\frac{s_m(z)}{f(z)}\right) \to \frac{\gamma_{p+m-1}}{1+\gamma_{p+m-1}} \quad \text{as} \quad z \to 1.$$

The proof of the theorem is thus completed.

Corollary 1. Let the assumptions of Theorem 3 hold. Then, for $z \in U$, we have

$$\operatorname{Re}\left(z^p f(z)\right) > \begin{cases} \frac{(2\lambda-1)(1-B)-(A-B)}{(2\lambda-1)(1-B)} & \left(\frac{2p}{k} \in N\right), \\ \frac{\lambda(1-B)-(A-B)}{\lambda(1-B)} & \left(\frac{2p}{k} \notin N\right). \end{cases}$$

and

$$\operatorname{Re}\left(\frac{1}{z^p f(z)}\right) > \begin{cases} \frac{(2\lambda-1)(1-B)}{A-B+(2\lambda-1)(1-B)} & \left(\frac{2p}{k} \in N\right), \\ \frac{\lambda(1-B)}{A-B+\lambda(1-B)} & \left(\frac{2p}{k} \notin N\right). \end{cases}$$

The results are sharp.

Replacing $H_{p,k}(\lambda, A, B)$ by $Q_{p,k}(\lambda, A, B)$, it follows from Theorem 3 that the inequalities (2.14) and (2.15) are true. In Theorem 4 below we improve the bounds in (2.14) and (2.15) for $f \in Q_{p,k}(\lambda, A, B)$.

Theorem 4. Suppose that the sequence $\{\Gamma_n(\alpha_1)\}$ $(n \geq p)$ is nondecreasing and $\Gamma_p(\alpha_1) \geq 1$. Let $f \in Q_{p,k}(\lambda, A, B)$ and let the condition (2.13) in Theorem 3 be satisfied. Then for $m \in N$, we have

$$\operatorname{Re}\left(\frac{f(z)}{s_m(z)}\right) > 1 - \frac{p}{(p+m-1)\gamma_{p+m-1}} \quad (z \in U) \tag{2.19}$$

and

$$\mathrm{Re}\left(\frac{s_m(z)}{f(z)}\right) > \frac{(p+m-1)\gamma_{p+m-1}}{p+(p+m-1)\gamma_{p+m-1}} \quad (z \in U). \tag{2.20}$$

The bounds in (2.19) and (2.20) are sharp for the function f defined by

$$f(z) = z^{-p} + \frac{pz^{p+m-1}}{(p+m-1)\gamma_{p+m-1}} \in Q_{p,k}(\lambda, A, B).$$

Proof. By virtue of the assumptions of Theorem 4, it follows from (2.16) and (2.17) that

$$\sum_{n=p}^{p+m-2} |a_n| + \frac{(p+m-1)\gamma_{p+m-1}}{p} \sum_{n=p+m-1}^{\infty} |a_n| \leq \sum_{n=p}^{\infty} \frac{n}{p}\gamma_n |a_n| \leq 1 \ \ (m \in N \setminus \{1\}) \tag{2.21}$$

and

$$\gamma_p \sum_{n=p}^{\infty} |a_n| \leq \sum_{n=p}^{\infty} \frac{n}{p}\gamma_n |a_n| \leq 1. \tag{2.22}$$

If we put

$$p_1(z) = 1 + \frac{(p+m-1)\gamma_{p+m-1}}{p}\left(\frac{f(z)}{s_m(z)} - 1\right)$$

and

$$p_2(z) = \left(1 + \frac{(p+m-1)\gamma_{p+m-1}}{p}\right)\frac{s_m(z)}{f(z)} - \frac{(p+m-1)\gamma_{p+m-1}}{p},$$

then (2.21) and (2.22) lead to $\mathrm{Re}(p_j(z)) > 0$ ($z \in U$; $m \in N$; $j = 1, 2$). Hence we have (2.19) and (2.20). Sharpness can be verified easily.

Finally, we derive certain convolution properties of functions in the classes $H_{p,k}(\lambda, A, B)$ and $Q_{p,k}(\lambda, A, B)$.

Theorem 5. Let $f \in H_{p,k}(\lambda, A, B)$. Then

$$(H_{p,q,s}(\alpha_1)f * h_\sigma)(z) \neq 0 \quad (z \in U_0;\ \sigma \in C,\ |\sigma| = 1), \tag{2.23}$$

where

$$h_\sigma(z) = z^{-p} + \frac{\lambda(1+B\sigma)}{\sigma(A-B)} \cdot \frac{z^p}{1-z} + \frac{\lambda(1+B\sigma)}{p\sigma(A-B)} \cdot \frac{z^{p+1}}{(1-z)^2} - \frac{(1-\lambda)(1+B\sigma)}{\sigma(A-B)} g_{p,k}(z)$$

and

$$g_{p,k}(z) = \begin{cases} \frac{z^p}{1-z^k} & \left(\frac{2p}{k} \in N\right), \\ \frac{z^{k\left(\left[\frac{2p}{k}\right]+1\right)-p}}{1-z^k} & \left(\frac{2p}{k} \notin N\right). \end{cases}$$

Proof. For $f \in H_{p,k}(\lambda, A, B)$, from Lemma 1 we have (1.4), which is equivalent to

$$(1-\lambda)z^p f_{p,k}(z) - \frac{\lambda}{p} z^{p+1}(H_{p,q,s}(\alpha_1)f(z))' \neq \frac{1+A\sigma}{1+B\sigma} \quad (z \in U;\ \sigma \in C,\ |\sigma| = 1,\ 1+B\sigma \neq 0),$$

or to

$$(1+B\sigma)[(1-\lambda)f_{p,k}(z) - \frac{\lambda}{p} z(H_{p,q,s}(\alpha_1)f(z))'] - (1+A\sigma)z^{-p} \neq 0 \quad (z \in U_0;\ \sigma \in C,\ |\sigma| = 1). \tag{2.24}$$

Obviously

$$z^{-p} = H_{p,q,s}(\alpha_1)f(z) * z^{-p} \tag{2.25}$$

and

$$\begin{aligned} -\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{p} &= H_{p,q,s}(\alpha_1)f(z) * \left(z^{-p} - \frac{1}{p}\sum_{n=p}^{\infty} nz^n\right) \\ &= H_{p,q,s}(\alpha_1)f(z) * \left(z^{-p} - \frac{z^p}{1-z} - \frac{z^{p+1}}{p(1-z)^2}\right). \end{aligned} \tag{2.26}$$

If we put

$$f_{p,k}(z) = H_{p,q,s}(\alpha_1)f(z) * (z^{-p} + g_{p,k}(z)), \tag{2.27}$$

then for $\frac{2p}{k} \in N$,

$$g_{p,k}(z) = \sum_{n=p}^{\infty} \delta_{n,p,k} z^n = \sum_{l=0}^{\infty} z^{p+lk} = \frac{z^p}{1-z^k}, \tag{2.28}$$

and for $\frac{2p}{k} \notin N$,

$$g_{p,k}(z) = \sum_{l=1}^{\infty} z^{k\left(\left[\frac{2p}{k}\right]+l\right)-p} = \frac{z^{k\left(\left[\frac{2p}{k}\right]+1\right)-p}}{1-z^k}. \tag{2.29}$$

Now, making use of (2.24) to (2.29), we arrive at

$$H_{p,q,s}(\alpha_1)f(z) * \left\{(1+B\sigma)\left[(1-\lambda)(z^{-p} + g_{p,k}(z)) + \lambda\left(z^{-p} - \frac{z^p}{1-z} - \frac{z^{p+1}}{p(1-z)^2}\right)\right] - (1+A\sigma)z^{-p}\right\} \neq 0$$

for $z \in U_0$, $\sigma \in C$ and $|\sigma| = 1$. This gives the desired result (2.23).

Corollary 2. Let $f \in Q_{p,k}(\lambda, A, B)$. Then

$$H_{p,q,s}(\alpha_1)f(z) * \left(2z^{-p} + \frac{zh'_\sigma(z)}{p}\right) \neq 0 \quad (z \in U_0;\ \sigma \in C,\ |\sigma| = 1), \tag{2.30}$$

where $h_\sigma(z)$ is the same as in Theorem 5.

Proof. From (1.10) and Theorem 5 we have

$$H_{p,q,s}(\alpha_1)f(z) * \left(2z^{-p} + \frac{zh'_\sigma(z)}{p}\right) = \left(2z^{-p} + \frac{z(H_{p,q,s}(\alpha_1)f(z))'}{p}\right) * h_\sigma(z) \neq 0 \quad (z \in U_0;\ \sigma \in C,\ |\sigma| = 1).$$

ACKNOWLEDGEMENT

The authors would like to express sincere thanks to the referees for careful reading and suggestions which helped us to improve the paper.

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