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On the Completeness of Eigenfunctions of a Discontinuous Dirac Operator with an Eigenparameter in the Boundary Condition

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Abstract. This paper deals with a discontinuous Dirac operator with an eigenparameter in the boundary condition. By using Lidskii's theorem, we investigate the completeness of the system of eigenfunctions of such an operator.

1. Introduction

It is well known that the Dirac equation is a cornerstone in the history of physics. The fundamental physics of relativistic quantum mechanics was formulated by the Dirac equation. It provides a natural description of the electron spin, predicts the existence of antimatter and is able to reproduce accurately the spectrum of the hydrogen atom (see [26, 27]). We refer to the monographs [21], [30]-[36] for background and further information about Dirac operators and their applications.

Discontinuous boundary-value problems arise in different areas of mathematics, radio science, electronics, geophysics and mechanics [5]. Boundary value problems with transmission conditions were investigated in [1, 3, 4, 7] and [14]-[19].

The completeness theorems are important for solving various problems in mathematical physics by the separation variables (Fourier method), and also for the spectral theory itself. In the literature there are a few analogous studies on completeness of the system of eigenvectors and associated vectors for such problems (see [6]-[9],[12]-[22], [28, 29]).

On the other hand, parameter dependent systems are of great interest to a lot of problems in physics and engineering. A boundary value problem with a spectral parameter in the boundary condition appears commonly in mathematical models of mechanics. There are a lot of studies about parameter dependent problems ([37]-[50]).

Krein and Nudelman [51] obtained the first general results on completeness property of non-homogeneous string with dissipative boundary condition. The questions of completeness and spectral synthesis for general $n\times n$ first order systems of ODE were handled by recent publications such as [52]-[55] (see also the references therein). It was shown in [52], [53] and [54] that the completeness property for some classes of

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boundary conditions essentially depends on the boundary values of the potential matrix, and explicit conditions of the completeness were found. In particular, in [52], an example of incomplete dissipative 2×2 Dirac operator was constructed. In [54] and [55], it was shown that, the resolvent of any complete dissipative Dirac type operator admits the spectral synthesis. Moreover, explicit conditions of the dissipativity and completeness of such operators were found. It is also worth to mention some recent papers such as [56]-[60], devoted to the Riesz basis property for 2×2 Dirac operator (see also the references therein).

In this paper, we study a discontinuous equation (3.1) with an eigenparameter in the boundary conditions. We shall answer whether all eigenfunctions and associated functions of this operator span the whole space or not. For this, we will use Lidskii's theorem. A similar way was employed earlier in the Sturm-Liouville operator case in [6]-[8], [12] and [13].

2. Preliminaries

In this section, we recall some of the basic concepts and results related to our subject matter, and we refer to [11],[36] for more details.

Let A denote the linear non-self-adjoint operator on the Hilbert space with domain $D(A)$. A complex number λ_0 is called an *eigenvalue* of the operator A if there exists a non-zero element $y_0 \in D(A)$ such that $Ay_0 = \lambda_0 y_0$; in this case, y_0 is called the *eigenvector* of A for λ_0. The eigenvectors for λ_0 span a subspace of $D(A)$, called the *eigenspace* for λ_0.

The element $y \in D(A)$, $y \neq 0$ is called a *root vector* of A corresponding to the eigenvalue λ_0 if $(T - \lambda_0 I)^n y = 0$ for some $n \in \mathbb{N}$. The root vectors for λ_0 span a linear subspace of $D(A)$, called the *root lineal* for λ_0. The algebraic multiplicity of λ_0 is the *dimension* of the root lineal. A root vector is called an *associated vector* if it is not an eigenvector. The completeness of the system of all eigenvectors and associated vectors of A is equivalent to the completeness of the system of all root vectors of this operator.

An operator A is called *dissipative* if $Im(Ax, x) \geq 0$ for all $x \in D(A)$. A bounded operator is dissipative if and only if $ImA = \frac{1}{2i}(A - A^*) \geq 0$.

Theorem 2.1. *Let A be an invertible operator. Then, $-A$ is dissipative if and only if the inverse operator A^{-1} of A is dissipative ([22]).*

A linear bounded operator A defined on the separable Hilbert space H is said to be of *trace class (nuclear)* if the series $\sum_j \left(Ae_j, e_j\right)$ converges and has the same value in any orthonormal basis $\left\{e_j\right\}$ of H. The sum $TrA = \sum_j \left(Ae_j, e_j\right)$ is called the *trace* of A.

The kernel $G(x, t)$ (where $x, t \in \mathbb{R}$) of the integral operator K on $L^2(\mathbb{R})$ defined by $Kf = \int_{\mathbb{R}} G(x, t) f(x)\, dx$, ($f \in L^2(\mathbb{R})$) is a *Hilbert-Schmidt kernel* if $|G(x, t)|^2$ is integrable on $\mathbb{R}^2$, i.e., $\int_{\mathbb{R}^2} |G(x, t)|^2 dxdt < \infty$. If $G(x, x)$ is measurable and summable, then it is called a *trace-class kernel* ([23], [24]). An integral operator with a trace class kernel is nuclear.

Theorem 2.2 (Lidskii [11],[25]). *If the dissipative operator A is the nuclear operator, then the system of root functions is complete in H.*

3. Statement of the Problem

In this section, we consider the Dirac systems. We know that Dirac systems describe a relativistic electron in the electrostatic field $V(x)$. If V is spherically symmetric, then a separation of variables, using spherical harmonics transforms this problem into of two dimensional systems of the form

$$l := -c\hbar J \frac{d}{dx} + \begin{pmatrix} V(x) - mc^2 & c\hbar k x^{-1} \\ c\hbar k x^{-1} & V(x) + mc^2 \end{pmatrix}, \ x \in I := I_1 \cup I_2,\ I_1 := (0, a),\ I_2 := (a, 1], \tag{1}$$

with singular point 0; where $c > 0$ is the velocity of light, $k \in \mathbb{Z} \setminus \{0\}$, $V(x)$ is a spherically symmetric potential, $m > 0$ is the mass of the particle and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (see [35]). To pass from the differential expression l to operators, we introduce the Hilbert space $H_1 := L^2(I; E)$ $\left(\text{where } E := \mathbb{C}^2\right)$ of vector valued functions with values in $\mathbb{C}^2$ and with the inner product $(y, z) = \int_0^1 (y(x), z(x))_E \, dx$.

Let us denote by D the linear set of all vectors $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \in H_1$ such that y_1 and y_2 are locally absolutely continuous functions on I and $l(y) \in H_1$. We define the operator L on D by the equality $Ly = ly$. We assume that L is in the limit circle case.

For two arbitrary vectors $y, z \in D$, we have the Green's formula given by

$$(Ly, z) - (y, Lz) = [y, z]_{a-} - [y, z]_0 + [y, z]_1 - [y, z]_{a+}, \tag{2}$$

where $[y, z]_x := W_x[y, \overline{z}] = y_1(x)\overline{z_2(x)} - y_2(x)\overline{z_1(x)}$, and the limit $[y, z]_0 = \lim_{x \to 0^+}[y, z]_x$ exists and is finite.

Let us denote by

$$\begin{aligned}
u(x, \lambda) &= \begin{pmatrix} u_1(x, \lambda) \\ u_2(x, \lambda) \end{pmatrix}, \quad v(x, \lambda) = \begin{pmatrix} v_1(x, \lambda) \\ v_2(x, \lambda) \end{pmatrix}, \\
u_1(x, \lambda) &= \begin{cases} u_{11}(x, \lambda), & x \in I_1 \\ u_{12}(x, \lambda), & x \in I_2 \end{cases}, \quad u_2(x, \lambda) = \begin{cases} u_{21}(x, \lambda), & x \in I_1 \\ u_{22}(x, \lambda), & x \in I_2 \end{cases} \\
v_1(x, \lambda) &= \begin{cases} v_{11}(x, \lambda), & x \in I_1 \\ v_{12}(x, \lambda), & x \in I_2 \end{cases}, \quad v_2(x, \lambda) = \begin{cases} v_{21}(x, \lambda), & x \in I_1 \\ v_{22}(x, \lambda), & x \in I_2 \end{cases}
\end{aligned}$$

the solutions of the equation $l(y) = \lambda y$, $x \in I$ satisfying the initial conditions

$$\begin{aligned}
&u_{12}(1, \lambda) = 1, \quad u_{22}(1, \lambda) = 0, \\
&v_{12}(1, \lambda) = 0, \quad v_{22}(1, \lambda) = 1,
\end{aligned}$$

and

$$\begin{aligned}
&u_{11}(a-, \lambda) = \delta u_{12}(a+, \lambda), \quad u_{21}(a-, \lambda) = \frac{1}{\delta} u_{22}(a+, \lambda), \\
&v_{11}(a-, \lambda) = \delta v_{12}(a+, \lambda), \quad v_{21}(a-, \lambda) = \frac{1}{\delta} v_{22}(a+, \lambda),
\end{aligned}$$

where δ and $\frac{1}{\delta}$ are some real numbers.

The Wronskian of the two solutions (4) does not depend on x, and the two solutions of this equation are linearly independent if and only if their Wronskian is nonzero. It is clear that their Wronskian is nonzero. Since L is in the limit circle case, we have $u, v \in H_1$ and moreover $u, v \in D$. The solutions $u(x, \lambda)$ and $v(x, \lambda)$ form a fundamental system of (4), and they are entire functions of λ (see [21]). Now let $u(x) = u(x, 0)$ and $v(x) = v(x, 0)$ be the solutions of the equation $l(y) = 0$ satisfying the initial conditions

$$\begin{aligned}
&u_{12}(1) = 1, \; u_{22}(1) = 0, \\
&v_{12}(1) = 0, \; v_{22}(1) = 1.
\end{aligned}$$

Lemma 3.1. *[[2]] Let $[u, v]_x = 1$ $(x \in I)$ for some real solutions $u(x)$ and $v(x)$ of Eq. $l(y) = 0$. Then, for arbitrary $y, s \in D$, one has the equality*

$$[y, s]_x = ([y, u]_x[\overline{s}, v]_x - [y, v]_x[\overline{s}, u]_x) \; (x \in I). \tag{3}$$

Now let us consider the boundary value problem

$$l(y) = \lambda y,\ y \in D,\ x \in I, \tag{4}$$

$$-(\beta_1 y_1(1) - \beta_2 y_2(1)) = \lambda\left(\beta_1' y_1(1) - \beta_2' y_2(1)\right), \tag{5}$$

$$[y,u]_0 + h[y,v]_0 = 0,\ Imh > 0, \tag{6}$$

$$y_1(a-) = \delta y_1(a+) \tag{7}$$

$$y_2(a-) = \frac{1}{\delta} y_2(a+), \tag{8}$$

where λ is a complex spectral parameter, $\beta_1,\ \beta_2, \beta_1', \beta_2' \in \mathbb{R}$, and $\beta := \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} > 0$. For convenience, we adopt the following notations:

$$\begin{aligned} R_1(y) &:= \beta_1 y_1(1) - \beta_2 y_2(1),\quad R_1'(y) := \beta_1' y_1(1) - \beta_2' y_2(1), \\ N_0^1(y) &:= [y,u]_0,\ N_0^2(y) := [y,v]_0,\quad R_0(y) := N_0^1(y) + hN_0^2(y). \\ N_1(y) &:= y_1(a-) - \delta y_1(a+),\quad N_2(y) := y_2(a-) - \frac{1}{\delta} y_2(a+). \end{aligned}$$

Lemma 3.2. *For any $y, z \in D$, we have $R_1(\overline{z}) = \overline{R_1(z)},\ R_1'(\overline{z}) = \overline{R_1'(z)}$.*

$$[y,z]_1 = \frac{1}{\beta}\left[R_1(y)\overline{R_1'(z)} - R_1'(y)\overline{R_1(z)}\right]. \tag{9}$$

Proof. We can write

$$\begin{aligned} \frac{1}{\beta}\left[R_1(y)\overline{R_1'(z)} - R_1'(y)\overline{R_1(z)}\right] &= \frac{1}{\beta}\left[\begin{array}{c} (\beta_1 y_1(1) - \beta_2 y_2(1))\overline{\left(\beta_1' z_1(1) - \beta_2' z_2(1)\right)} \\ -\left(\beta_1' y_1(1) - \beta_2' y_2(1)\right)\overline{(\beta_1 z_1(1) - \beta_2 z_2(1))} \end{array}\right] \\ &= \frac{1}{\beta}\left[\left(\beta_1' \beta_2 - \beta_1' \beta_2\right)\left(y_1(1)\overline{z_2(1)} - y_2(1)\overline{z_1(1)}\right)\right] \\ &= [y,z]_1. \end{aligned}$$

□

4. Completeness Theorem

In this section, we will define a Hilbert space $H := H_1 \oplus \mathbb{C}$ and an operator A whose root vectors coincide with those of the problem (4)-(8). Later, we will prove the completeness of the system of eigenfunctions of such an operator, by using Lidskii's theorem.

The inner product of H is defined by $< Y, Z >_H = \int_0^a (y(x), z(x))_E\, dx + \int_a^1 (y(x), z(x))_E\, dx + \frac{1}{\beta}\widetilde{y}\overline{\widetilde{z}}$, where $Y(x) = \begin{pmatrix} y(x) \\ \widetilde{y} \end{pmatrix},\ Z(x) = \begin{pmatrix} z(x) \\ \widetilde{z} \end{pmatrix},\ y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},\ z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix},\ y_i(.), z_i(.) \in H_1,\ i = 1,2,\ \widetilde{y},\ \widetilde{z} \in \mathbb{C}$.

Denote by $D(A)$ the linear set of all vectors $Y(x) = \begin{pmatrix} y(x) \\ \widetilde{y} \end{pmatrix} \in H$ such that $y(x) \in D,\ R_0(y) = 0,\ \widetilde{y} = R_1'(y)$, and the one sided limits $y(a\pm)$ exist and are finite, $N_1(y) = 0,\ N_2(y) = 0$. We construct the operator A on $D(A)$ as $AY = \widetilde{l}(Y) := \begin{pmatrix} l(y) \\ -R_1(y) \end{pmatrix}$.

Thus we can pose the boundary-value problems (4)-(8) in H as $AY = \lambda Y,\ Y \in D(A)$. It is clear that the eigenvalues and root lineals of A and boundary value problems (4)-(8) coincide. Now we will prove that the operator A is a dissipative operator.

Theorem 4.1. *The operator A is dissipative in H.*

Proof. Let $Y \in D(A)$. Then, by the Lagrange identity, we get

$$(AY, Y) - (Y, AY) = [y, y]_{a-} - [y, y]_0 + [y, y]_1 - [y, y]_{a+} + \frac{1}{\beta}\left[R_1'(y)\overline{R_1(y)} - R_1(y)\overline{R_1'(y)}\right]. \tag{10}$$

Since $Y \in D(A)$, we have

$$[y, y]_{a-} = [y, y]_{a+}. \tag{11}$$

From Lemma 3.1, we have

$$[y, y]_0 = ([y, u]_0[\overline{y}, v]_0 - [y, v]_0[\overline{y}, u]_0) = -2iImh([y, v]_0)^2. \tag{12}$$

From (11), (9) and (12) we get

$$Im(AY, Y)) = Imh([y, v]_0)^2, \tag{13}$$

and so A is dissipative in H. □

It follows from Theorem 4.1 that, all the eigenvalues of A lie in the closed upper half plane $Im\lambda \geq 0$.

Theorem 4.2. *The operator A has not any real eigenvalue.*

Proof. Suppose that the operator A has a real eigenvalue λ_0. Let $\eta_0(x) = v(x, \lambda_0)$ be the corresponding eigenfunction. Since $Im(A\eta_0, \eta_0)) = Im\left(\lambda_0 \left\|\eta_0\right\|^2\right)$, we get from (13) that $[\eta_0, v]_0 = 0$. By the boundary condition (6), we have $[\eta_0, u]_0 = 0$. Thus

$$[\eta_0(x, \lambda_0), u]_0 = [\eta_0(x, \lambda_0), v]_0 = 0. \tag{14}$$

From Lemma 3.1 with $\xi_0(x) = u(x, \lambda_0)$, we get $1 = [\eta_0, \xi_0]_0 = [\eta_0, u]_0[\xi_0, v]_0 - [\eta_0, v]_0[\xi_0, u]_0$. From (14), the right-hand side is equal to 0. This contradiction proves the theorem. □

Note that, in particular, zero is not an eigenvalue of A.

Now let us define the two solutions of Eq. (4) as $\varphi(x, \lambda) = \begin{pmatrix} \varphi_1(x, \lambda) \\ \varphi_2(x, \lambda) \end{pmatrix}$ and $\chi(x, \lambda) = \begin{pmatrix} \chi_1(x, \lambda) \\ \chi_2(x, \lambda) \end{pmatrix}$, where

$$\begin{aligned} \varphi_1(x, \lambda) &= \begin{cases} \varphi_{11}(x, \lambda), & x \in I_1 \\ \varphi_{12}(x, \lambda), & x \in I_2 \end{cases}, \ \varphi_2(x, \lambda) = \begin{cases} \varphi_{21}(x, \lambda), & x \in I_1 \\ \varphi_{22}(x, \lambda), & x \in I_2 \end{cases}, \\ \chi_1(x, \lambda) &= \begin{cases} \chi_{11}(x, \lambda), & x \in I_1 \\ \chi_{12}(x, \lambda), & x \in I_2 \end{cases}, \ \chi_2(x, \lambda) = \begin{cases} \chi_{21}(x, \lambda), & x \in I_1 \\ \chi_{22}(x, \lambda), & x \in I_2 \end{cases}, \end{aligned}$$

satisfying the initial and transmission conditions

$$\varphi_{12}(1, \lambda) = \beta_2'\lambda + \beta_2, \ \ \varphi_{22}(1, \lambda) = \beta_1'\lambda + \beta_1, \ \ N_0^1(\chi) = -h, \ N_0^2(\chi) = 1,$$

and

$$\begin{aligned} \varphi_{11}(a-, \lambda) &= \delta\varphi_{12}(a+, \lambda), \ \varphi_{21}(a-, \lambda) = \frac{1}{\delta}\varphi_{22}(a+, \lambda), \\ \chi_{11}(a-, \lambda) &= \delta\chi_{12}(a+, \lambda), \ \chi_{21}(a-, \lambda) = \frac{1}{\delta}\chi_{22}(a+, \lambda). \end{aligned}$$

Let us set $\omega_1(\lambda) := W_x[\varphi, \overline{\chi}]$ $(x \in I_1)$ and $\omega_2(\lambda) := W_x[\varphi, \overline{\chi}]$ $(x \in I_2)$ .It is clear that $\omega_1(\lambda) = \omega_2(\lambda)$, $\forall \lambda \in \mathbb{C}$. Therefore the zeros of $\omega_1(\lambda)$ and $\omega_2(\lambda)$ coincide. If we define $\omega(\lambda) = \omega_1(\lambda) = \omega_2(\lambda)$, then the function ω is an entire function. By Eq. (9), we have

$$\omega(\lambda) = W_x[\varphi, \overline{\chi}] = W_1[\varphi, \overline{\chi}] = \frac{1}{\beta}\left(R_1(\varphi)R_1'(\chi) - R_1'(\varphi)R_1(\chi)\right) = R_1(\chi) + \lambda R_1'(\chi).$$

Then, by Equality (3), we get

$$\omega(\lambda) = W_x[\varphi, \overline{\chi}] = W_0[\varphi, \overline{\chi}] = N_0^1(\varphi)N_0^2(\chi) - N_0^2(\varphi)N_0^1(\chi) = R_0(\varphi).$$

It is clear that the roots of the equation $\omega(\lambda) = 0$ coincides with the spectrum of the boundary value problems (4)-(8). The function $\omega(\lambda)$ is not identically zero and it has at most a countable number of isolated zeros with finite multiplicity and possible limit points at infinity.

Now we define

$$G(x,t,\lambda) = \begin{cases} \frac{\varphi(x)\chi^T(t)}{\omega(\lambda)}, & 0 \le t \le x \le 1,\ x \ne a,\ t \ne 0 \\ \frac{\varphi(t)\chi^T(x)}{\omega(\lambda)}, & 0 \le x \le t \le 1,\ x \ne a,\ t \ne 0 \end{cases} \tag{15}$$

where ^{T} denotes the matrix transpose. Then the general solution of the equation $AY - \lambda Y = F$ can be represented as

$$Y = (A - \lambda I)^{-1} F = \begin{pmatrix} \left(\widetilde{G}_{x,\lambda}, \overline{F}\right) \\ R_1'\left[\left(\widetilde{G}_{x,\lambda}, \overline{F}\right)\right] \end{pmatrix}, \quad \widetilde{G}_{x,\lambda} = \begin{pmatrix} G(x,t,\lambda) \\ R_1'[G(x,t,\lambda)] \end{pmatrix},$$

where $\overline{F(x)} = \begin{cases} \overline{f_1(x)}, & x \in I_1 \\ \overline{f_2(x)}, & x \in I_2 \end{cases}$ Then $K := (A - \lambda I)^{-1}$ is a compact linear operator in the space H. The root lineals of the operators $A - \lambda I$ and K coincide, and hence the completeness in H of the system of all eigenvectors and associated vectors of $A - \lambda I$ is equivalent to the completeness of those for K. Since the algebraic multiplicity of nonzero eigenvalues of a compact operator is finite, each eigenvector of $A - \lambda I$ may have only a finite number of linearly independent associated vectors. On the other hand, the root lineals of the operators $A - \lambda I$ and A also coincide([29]). Then the completeness in H of the system of all eigenvectors and associated vectors of A is equivalent to the completeness of those for K.

Since the defect index of the operator L is $(2, 2)$, we have $\varphi, \chi \in H$. Therefore, we obtain that $G(x,t,\lambda)$ is a Hilbert-Schmidth kernel. Furthermore, $G(x,t,\lambda)$ is measurable and integrable on $(0, 1]$. Hence K is of trace class. Since A is a dissipative operator, by Theorem 2.1, $-K$ is a dissipative operator. Thus, for the Lidskii 's theorem, all the conditions are satisfied. Hence we have the following

Theorem 4.3. *The system of all root functions of $-K$ (also K) is complete in H.*

From the conclusions above, we have the following

Theorem 4.4. *All eigenvalues of the operator A lie in the open upper half-plane and they are purely discrete. The limit points of these eigenvalues can only occur at infinity. The system of all eigenfunctions and associated functions of A is complete in H.*

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