Filomat 31:9 (2017), 2657–2673
https://doi.org/10.2298/FIL1709657A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Coupled Coincidence Point Theorems for (α-μ -ψ -H-$\mathcal{F}$)-Two Sided-Contractive Type Mappings in Partially Ordered Metric Spaces Using Compatible Mappings

Arslan Hojat Ansari[a], Diana Dolićanin-Djekić[b], Tatjana Došenović[c], Stojan Radenović[*d]

[a]*Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran*
[b]*Faculty of Technical Sciences, University of Priština-Kosovska Mitrovica, Serbia*
[c]*Faculty of Technology, Bulevar cara Lazara 1, University of Novi Sad, Novi Sad, Serbia*
[d]*Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia*

Abstract. Using the concepts of a pair upclass, $\alpha-$admissible and $\mu-$subadmissible mappings in this paper, are proven a coupled coincidence point results for mappings $F : X^2 \to X$ and $g : X \to X$. In this way, recent papers [27, 33] have been reformed and generalized. Two examples are given to support the theoretical results.

1. Introduction and Mathematical Preliminaries

The fixed point theory has drawn much attention in partially ordered metric spaces. Turinici [50] has expanded Banach contraction principle [8] in the setting of partially ordered sets, and laid the foundation for a new trend in the fixed point theory. Ran and Reurings [34] are impacted by development of the applications of Turinici's theorem to matrix equations and they have established some results in this direction. Their results are further extended by Nieto and Rodríguez-López [31, 32] for non-decreasing mappings. Lakshmikantham at al. [9, 15] introduced the new notion of coupled fixed points for the mappings satisfying the mixed monotone property in partially ordered spaces and discussed the existence and uniqueness of a solution for a periodic boundary value problem. Later on, Lakshmikantham and Ćirić [24] proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces. Further on, Choudhury and Kundu [10], proved the coupled coincidence point results for compatible mappings in the settings of partially ordered metric space. Samet et al. [45, 48] have recently introduced the notion of $(\alpha-\psi)-$contractive and $\alpha-$admissible mapping, and they proved fixed point theorems for such mappings in complete metric spaces. For more results regarding coupled fixed points in various metric spaces one can refer to ([1]-[49]).

The aim of the present paper is to generalize the results of Mursaleen et al. [27] and Kumar [33] for $(\alpha-\mu-\psi-H-\mathcal{F})-$contractive, $\alpha-$admissible and $\mu-$subadmissible mappings using compatible mappings.

2010 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25
Keywords. Coupled Coincidence Point, $\alpha-\varphi$-Contractive Mapping, Compatible Mappings
Received: 23 November 2015; Accepted: 03 July 2016
Communicated by Dragan S. Djordjević
Research supported by MNTRRS-174009.
Email addresses: analsisamirmath2@gmail.com (Arslan Hojat Ansari), diana.dolicanin@pr.ac.rs (Diana Dolićanin-Djekić), tatjanad@tf.uns.ac.rs (Tatjana Došenović), Corresponding author: radens@beotel.rs (Stojan Radenović*)

Definition 1.1. *[9] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. Then, a map F is said to have mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,*

$$x_1, x_2 \in X,\ x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1, y_2 \in X,\ y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

Definition 1.2. *[9] An element $(x, y) \in X \times X$ is said to be coupled fixed point of the mapping $F : X^2 \to X$ if*

$$F(x, y) = x \text{ and } F(y, x) = y.$$

Theorem 1.3. *[9] Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)],$$

for all $x \succeq u$ and $y \preceq v$. If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0), y_0 \succeq F(y_0, x_0),$$

then there exist $x, y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$.

Theorem 1.4. *[9] Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property:*

(i) if a non-decreasing sequence $(x_n) \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$,
(ii) if a non-increasing sequence $(y_n) \to y$, then $y \preceq y_n$ for all $n \in \mathbb{N}$.

Let $F : X^2 \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)]$$

for all $x \succeq u$ and $y \preceq v$. If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0), y_0 \succeq F(y_0, x_0),$$

then there exist $x, y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$.

Let Ψ be the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the n-th iterate of ψ and satisfying:
(i) $\psi^{-1}(\{0\}) = \{0\}$,
(ii) $\psi(t) < t$ for all $t > 0$,
(iii) $\lim_{r \to t^+} \psi(r) < t$ for all $t > 0$.

Lemma 1.5. *[27] If $\psi : [0, +\infty] \to [0, +\infty]$ is nondecreasing and right continuous then $\psi^n(t) \to 0$ as $n \to +\infty$ for all $t \geq 0$ if and only if $\psi(t) < t$ for all $t > 0$.*

Definition 1.6. *[24] Let (X, d) be a partially ordered set, $F : X^2 \to X$ and $g : X \to X$ be two mappings. Then, a map F is said to have mixed $g-$monotone property if $F(x, y)$ is monotone $g-$ non-decreasing in x and is monotone $g-$non-increasing in y, that is, for any $x, y \in X$,*

$$x_1, x_2 \in X,\ gx_1 \preceq gx_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1, y_2 \in X,\ gy_1 \preceq gy_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

Definition 1.7. *[24] An element $(x, y) \in X^2$ is said to be coupled coincidence point of mapping $F : X^2 \to X$ and $g : X \to X$ if*

$$F(x, y) = gx, F(y, x) = gy.$$

Definition 1.8. *[10] The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be compatible if*

$$\lim_{n\to+\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0,$$

and

$$\lim_{n\to+\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that

$$\begin{aligned} \lim_{n\to+\infty} F(x_n, y_n) &= \lim_{n\to+\infty} gx_n, \\ \lim_{n\to+\infty} F(y_n, x_n) &= \lim_{n\to+\infty} gy_n. \end{aligned}$$

Definition 1.9. *[27] Let (X, d) be a partially ordered metric space and $F : X^2 \to X$ be a mapping. Then, a map F is said to be $(\alpha, \psi)-$ contractive if there exist two functions $\alpha : X^2 \times X^2 \to [0, +\infty)$ and $\psi \in \Psi$ such that*

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq \psi\Big(\frac{d(x, u) + d(y, v)}{2}\Big),$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$.

Definition 1.10. *[27] Let $F : X^2 \to X$ and $\alpha : X^2 \times X^2 \to [0, +\infty)$ be two mappings. Then, F is said to be (α)-admissible if*

$$\alpha((x, y), (u, v)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1,$$

for all $x, y, u, v \in X$.

Definition 1.11. *[33] Let $F : X^2 \to X$, $g : X \to X$ and $\alpha : X^2 \times X^2 \to [0, +\infty)$ be mappings. Then F and g are said to be (α)-admissible if*

$$\alpha((gx, gy), (gu, gv)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1,$$

for all $x, y, u, v \in X$.

2. Main Results

In this section we will first introduce new functions, and than using the concepts of a pair upclass, $\alpha-$admissible and $\mu-$subadmissible mappings we will prove a coupled coincidence point results on metric space endowed with partial order.

Definition 2.1. *Let $F : X^2 \to X$ and $\mu : X^2 \times X^2 \to [0, +\infty)$ be two mappings. Then F is said to be $(\mu)-$subadmissible if*

$$\mu((x, y), (u, v)) \leq 1 \Rightarrow \mu((F(x, y), F(y, x)), (F(u, v), F(v, u))) \leq 1,$$

for all $x, y, u, v \in X$.

Definition 2.2. *Let $F : X^2 \to X$, $g : X \to X$ and $\alpha : X^2 \times X^2 \to [0, +\infty)$ be mappings. Then F and g are said to be $(\mu)-$subadmissible if*

$$\mu((gx, gy), (gu, gv)) \leq 1 \Rightarrow \mu((F(x, y), F(y, x)), (F(u, v), F(v, u))) \leq 1,$$

for all $x, y, u, v \in X$.

Definition 2.3. *A function $H : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ is a function of subclass of type I if it is continuous and $x \geq 1 \Rightarrow H(1, y) \leq H(x, y)$, for all $x \in \mathbb{R}$, $y \in [0, +\infty)$.*

Example 2.4. *We have the following functions of subclass of type I, for all $x \in \mathbb{R}$, $y \in [0, +\infty)$:*

- $H(x, y) = (y + l)^x, l > 1,$
- $H(x, y) = (x + l)^y, l > 1,$
- $H(x, y) = xy^n,$
- $H(x, y) = xy,$
- $H(x, y) = y,$
- $H(x, y) = (\frac{x+1}{2})y,$
- $H(x, y) = \frac{2x+1}{3}y,$
- $H(x, y) = (\frac{\sum_{i=0}^{n} x^{n-i}}{n+1})y,$
- $H(x, y) = (\frac{\sum_{i=0}^{n} x^{n-i}}{n+1} + l)^y, l > 1.$

Definition 2.5. *Let $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a mapping. We say that the pair $(\mathcal{F}, H)$ is a upclass of type I if $\mathcal{F}$ is continuous, H is a function of subclass of type I and satisfies:*
(1) $0 \leq x \leq 1 \Longrightarrow \mathcal{F}(x, y) \leq \mathcal{F}(1, y)$
(2) $H(1, y_1) \leq \mathcal{F}(x, y_2) \Longrightarrow y_1 \leq xy_2,$
for all $x, y, y_1, y_2 \in \mathbb{R}^+$.

Example 2.6. *The following the functions of a upclass of type I, for all $x \in \mathbb{R}$, $y, t \in [0, +\infty)$, $s \in [0, 1]$:*

- $H(x, y) = (y + l)^x, l > 1, \mathcal{F}(s, t) = st + l,$
- $H(x, y) = (x + l)^y, l > 1, \mathcal{F}(s, t) = (1 + l)^{st},$
- $H(x, y) = xy^n, \mathcal{F}(s, t) = s^n t^n,$

- $H(x,y) = xy, \mathcal{F}(s,t) = st,$
- $H(x,y) = y, \mathcal{F}(s,t) = st,$
- $H(x,y) = \frac{2x+1}{3}y, \mathcal{F}(s,t) = st,$
- $H(x,y) = (\frac{x+1}{2})y, \mathcal{F}(s,t) = st,$
- $H(x,y) = (\frac{\sum_{i=0}^{n} x^{n-i}}{n+1})y, \mathcal{F}(s,t) = st,$
- $H(x,y) = (\frac{\sum_{i=0}^{n} x^{n-i}}{n+1} + l)^y, l > 1, \mathcal{F}(s,t) = (1+l)^{st}.$

Definition 2.7. *Let $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a mapping. We say that the pair $(\mathcal{F}, H)$ is a special upclass of type I if $\mathcal{F}$ is continuous, H is a function of subclass of type I and satisfies:*
(1) $0 \le s \le 1 \Rightarrow \mathcal{F}(s,t) \le \mathcal{F}(1,t)$;
(2) $H(1,y) \le \mathcal{F}(1,t) \Longrightarrow y \le t$,
for all $y, t \in \mathbb{R}^+$.

Example 2.8. *We have following functions of special upclass of type I, for all $x \in \mathbb{R}$, $y,t,s \in \mathbb{R}^+$:*

- $H(x,y) = (y^k + l)^{x^n}, l > 1, \mathcal{F}(s,t) = s^m t^k + l,$
- $H(x,y) = (x^n + l)^{y^k}, l > 1, \mathcal{F}(s,t) = (1+l)^{s^m t^k},$
- $H(x,y) = x^n y^k, \mathcal{F}(s,t) = s^p t^k,$
- $H(x,y) = xy, \mathcal{F}(s,t) = st,$
- $H(x,y) = y, \mathcal{F}(s,t) = st,$
- $H(x,y) = \frac{2x+1}{3}y, \mathcal{F}(s,t) = st,$
- $H(x,y) = (\frac{x+1}{2})y, \mathcal{F}(s,t) = st,$
- $H(x,y) = (\frac{\sum_{i=0}^{n} x^{n-i}}{n+1})^m y^k, \mathcal{F}(s,t) = s^p t^k,$
- $H(x,y) = (\frac{\sum_{i=0}^{n} x^{n-i}}{n+1} + l)^{y^k}, l > 1, \mathcal{F}(s,t) = (1+l)^{s^p t^k}.$

Remark 2.9. *Each pair $(\mathcal{F}, H)$ of upclass of type I is pair $(\mathcal{F}, H)$ of special upclass of type I but converse is not true.*

Theorem 2.10. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X^2 \to X$ and $g : X \to X$ be a mappings, and let F have a mixed $g-$monotone property on X. Suppose that there exists $\psi \in \Psi$, pair $(\mathcal{F}, H)$ is a upclass of type I and α, $\mu : X^2 \times X^2 \to [0, +\infty)$ such that for $x, y, u, v \in X$, the following holds:*

$$\begin{aligned} & H(\alpha((gx,gy),(gu,gv)), \frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2}) \\ \le \ & \mathcal{F}(\mu((gx,gy),(gu,gv)), \psi\Big(\frac{d(gx,gu)+d(gy,gv)}{2}\Big)), \end{aligned} \tag{1}$$

for all $gx \succeq gu$ and $gy \preceq gv$. Suppose also that

(i) F and g are (α)-admissible and (μ)-subadmissible,

(ii) there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$ with

$$\alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1, \alpha((gy_0, gx_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1,$$

$$\mu((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \leq 1, \mu((gy_0, gx_0), (F(y_0, x_0), F(x_0, y_0))) \leq 1,$$

(iii) $F(X^2) \subseteq g(X)$, g is continuous, and F and g are compatible in X,

(iv) F is continuous.

Then F and g have a coupled coincidence point, that is there exist $x, y \in X$ such that

$$F(x, y) = gx, F(y, x) = gy.$$

Proof. Let $x_0, y_0 \in X$ be such that

$$\alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1, \quad \alpha((gy_0, gx_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1,$$

and

$$\mu((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \leq 1, \quad \mu((gy_0, gx_0), (F(y_0, x_0), F(x_0, y_0))) \leq 1,$$

and using (iii) there exist $x_1, y_1 \in X$, such that $gx_0 \preceq F(x_0, y_0) = gx_1$ and $gy_0 \succeq F(y_0, x_0) = gy_1$. Also, there exist $x_2, y_2 \in X$ such that $F(x_1, y_1) = gx_2$ and $F(y_1, x_1) = gy_2$. Continuing this process, we can construct two sequences (x_n) and (y_n) in X as follows

$$gx_{n+1} = F(x_n, y_n), \; gy_{n+1} = F(y_n, x_n),$$

for all $n \geq 0$. We shall show that

$$gx_n \preceq gx_{n+1}, \; gy_n \succeq gy_{n+1}, \tag{2}$$

for all $n \geq 0$. We will use the mathematical induction. Let $n = 0$. Since $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$ and as $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, we have $gx_0 \preceq gx_1$ and $gy_0 \succeq gy_1$. Thus (2) hold for $n = 0$. Now suppose that (2) hold for some fixed $n \geq 0$. Then, since $gx_n \preceq gx_{n+1}$ and $gy_n \succeq gy_{n+1}$ and by mixed monotone property of F, we have

$$gx_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_n, y_{n+1}) \succeq F(x_n, y_n) = gx_{n+1}$$

and

$$gy_{n+2} = F(y_{n+1}, x_{n+1}) \preceq F(y_n, x_{n+1}) \preceq F(y_n, x_n) = gy_{n+1}.$$

From above, we conclude that

$$gx_{n+1} \preceq gx_{n+2} \; , \; gy_{n+1} \succeq gy_{n+2}.$$

Thus by the mathematical induction we conclude that (2) hold for all $n \geq 0$. If for some n, we have $(gx_{n+1}, gy_{n+1}) = (gx_n, gy_n)$, then $F(x_n, y_n) = gx_n$ and $F(y_n, x_n) = gy_n$, that is $x_n = x$ and $y_n = y$ are coupled coincidence point of F and g. Now, we assumed that $(gx_{n+1}, gy_{n+1}) \neq (gx_n, gy_n)$ for all $n \geq 0$. Since F is (α)-admissible and (μ)-subadmissible we have

$$\alpha((gx_0, gy_0), (gx_1, gy_1)) = \alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1 \Rightarrow$$

$$\alpha((F(x_0, y_0), F(y_0, x_0)), (F(x_1, y_1), F(y_1, x_1))) = \alpha((gx_1, gy_1), (gx_2, gy_2)) \geq 1.$$

Thus, by mathematical induction, we have

$$\alpha((gx_n, gy_n), (gx_{n+1}, gy_{n+1})) \geq 1 \tag{3}$$

and similarly

$$\alpha((gy_n, gx_n), (gy_{n+1}, gx_{n+1})) \geq 1. \tag{4}$$

Also,

$$\mu((gx_0, gy_0), (gx_1, gy_1)) = \mu((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \leq 1 \Rightarrow$$

$$\mu((F(x_0, y_0), F(y_0, x_0)), (F(x_1, y_1), F(y_1, x_1))) = \mu((gx_1, gy_1), (gx_2, gy_2)) \leq 1.$$

So, by mathematical induction, we have,

$$\mu((gx_n, gy_n), (gx_{n+1}, gy_{n+1})) \leq 1 \tag{5}$$

and similarly

$$\mu((gy_n, gx_n), (gy_{n+1}, gx_{n+1})) \leq 1, \tag{6}$$

for all $n \in \mathbb{N}$. Using (1), (3) and (5), we obtain

$$\begin{aligned}
&H(1, \frac{d(gx_n, gx_{n+1}) + d(gy_{n+1}, gy_n)}{2}) \\
= \;& H(1, \frac{d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) + d(F(y_{n-1}, x_{n-1}), F(y_n, x_n))}{2}) \\
\leq \;& H(\alpha((gx_{n-1}, gy_{n-1}), (gx_n, gy_n)), \frac{d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) + d(F(y_{n-1}, x_{n-1}), F(y_n, x_n))}{2}) \\
\leq \;& \mathcal{F}(\mu((gx_{n-1}, gy_{n-1}), (gx_n, gy_n)), \psi\Big(\frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}{2}\Big)) \\
\leq \;& \mathcal{F}(1, \psi\Big(\frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}{2}\Big)).
\end{aligned}$$

So, using condition (1) from Definition 2.5 we conclude that

$$\frac{d(gx_n, gx_{n+1}) + d(gy_{n+1}, gy_n)}{2} \leq \psi\Big(\frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}{2}\Big).$$

Repeating the above process, we get

$$\frac{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})}{2} \leq \psi^n\Big(\frac{d(gx_0, gx_1) + d(gy_0, gy_1)}{2}\Big)$$

for all $n \in \mathbb{N}$. For $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq n(\epsilon)} \psi^n\Big(\frac{d(gx_0, gx_1) + d(gy_0, gy_1)}{2}\Big) < \varepsilon/2.$$

Let $n, m \in \mathbb{N}$ be such that $m > n > n(\varepsilon)$. Then, by using the triangle inequality, we have

$$\begin{aligned}
\frac{d(gx_n, gx_m) + d(gy_n, gy_m)}{2} \;&\leq\; \sum_{k=n}^{m-1} \frac{d(gx_k, gx_{k+1}) + d(gy_k, gy_{k+1})}{2} \\
&\leq\; \sum_{k=n}^{m-1} \psi^k\Big(\frac{d(gx_0, gx_1) + d(gy_0, gy_1)}{2}\Big) \\
&\leq\; \sum_{n \geq n(\epsilon)} \psi^n\Big(\frac{d(gx_0, gx_1) + d(gy_0, gy_1)}{2}\Big) < \varepsilon/2.
\end{aligned}$$

This implies that $d(gx_n, gx_m) + d(gy_n, gy_m) < \varepsilon$. So,

$$d(gx_n, gx_m) < \varepsilon$$

and

$$d(gy_n, gy_m) < \epsilon.$$

Therefore, (gx_n) and (gy_n) are Cauchy sequences in (X, d). Since (X, d) is complete metric space we have that the sequences (gx_n) and (gy_n) are convergent in (X, d). Then, there exist $x,\ y \in X$ such that

$$\begin{aligned} \lim_{n\to+\infty} F(x_n, y_n) &= \lim_{n\to+\infty} gx_n = x, \\ \lim_{n\to+\infty} F(y_n, x_n) &= \lim_{n\to+\infty} gy_n = y. \end{aligned}$$

Since F and g are compatible mappings, we have:

$$\lim_{n\to+\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0, \tag{7}$$

and

$$\lim_{n\to+\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0. \tag{8}$$

Now, we use the assumption that F and g are continuous mappings. Also we have that $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$. Now, taking limit $n \to +\infty$ in (7) and (8) we get

$$d(gx, F(x, y)) = 0.$$

Similarly, we have $d(gy, F(y, x)) = 0$. So, $F(x, y) = gx$ and $F(y, x) = gy$. Thus we proved that F and g have a coupled coincidence point. □

If we choice $H(x, y) = xy$ and $\mathcal{F}(s, t) = st$, we obtain a corrigendum of Theorem 3.2 in [33] with $\mu(x, y) = 1$ for all $x,\ y \in X$.

Theorem 2.11. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^2 \to X$ and $g : X \to X$ be two mappings such that F have the mixed $g-$monotone property on X. Suppose that there exists $\psi \in \Psi$ and $\alpha : X^2 \times X^2 \to [0, +\infty)$ such that for $x, y, u, v \in X$, the following holds:*

$$\alpha((gx, gy), (gu, gv))\frac{d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))}{2}) \leq \psi\Big(\frac{d(gx, gu) + d(gy, gv)}{2}\Big)$$

for all $gx \succeq gu$ and $gy \preceq gv$. Suppose also that

(i) F and g are $(\alpha)-$admissible,

(ii) there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$ with

$$\alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1,\ \ \alpha((gy_0, gx_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1,$$

(iii) $F(X^2) \subseteq g(X)$, g is continuous and F and g are compatible in X,

(iv) F is continuous.

Then F and g have a coupled coincidence point, that is, there exist $x,\ y \in X$ such that

$$F(x, y) = gx,\ \ F(y, x) = gy.$$

In the next theorem, we omit the continuity hypothesis of F.

Theorem 2.12. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^2 \to X$ and $g : X \to X$ be two mappings such that F has a mixed g−monotone property. Assume that there exists $\psi \in \Psi$, pair $(\mathcal{F}, H)$ is a upclass of type I and a mapping $\alpha : X^2 \times X^2 \to [0, +\infty)$ such that*

$$\begin{aligned} &H(\alpha((gx, gy), (gu, gv)), \frac{d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))}{2}) \\ &\leq \mathcal{F}(\mu((gx, gy), (gu, gv)), \psi\Big(\frac{d(gx, gu) + d(gy, gv)}{2}\Big)), \end{aligned} \tag{9}$$

for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$. Suppose that

(a) conditions (i),(ii) and (iii) of Theorem 2.10 hold,
(b) if (x_n) and (y_n) are sequences in X such that

$$\alpha((ggx_n, ggy_n), (ggx_{n+1}, ggy_{n+1})) \geq 1, \ \alpha((ggy_n, ggx_n), (ggy_{n+1}, ggx_{n+1})) \geq 1,$$

$$\mu((ggx_n, ggy_n), (ggx_{n+1}, ggy_{n+1})) \leq 1, \ \mu((ggy_n, ggx_n), (ggy_{n+1}, ggx_{n+1})) \leq 1,$$

for all $n \in \mathbb{N}$, and $\lim_{n\to+\infty} gx_n = x \in X$ and $\lim_{n\to+\infty} gy_n = y \in X$ then

$$\alpha((ggx_n, ggy_n), (gx, gy)) \geq 1, \ \alpha((ggy_n, ggx_n), (gy, gx)) \geq 1,$$

and

$$\mu((ggx_n, ggy_n), (gx, gy)) \leq 1, \ \mu((ggy_n, ggx_n), (gy, gx)) \leq 1.$$

Then there exist $x, y \in X$ such that $F(x, y) = gx$ and $F(y, x) = gy$, that is, F and g have a coupled coincidence in X.

Proof. Proceeding the same lines as in the proof of Theorem 2.10 we know that (gx_n) and (gy_n) are Cauchy sequences in the complete metric space (X, d). Then, there exist $x, y \in X$ such that

$$\lim_{n\to+\infty} gx_n = x, \ \lim_{n\to+\infty} gy_n = y.$$

On the other hand, from (3) and hypothesis (b), we obtain

$$\alpha((gx_n, gy_n), (gx, gy)) \geq 1, \tag{10}$$

and similarly

$$\alpha((gy_n, gx_n), (gy, gx)) \geq 1. \tag{11}$$

Also,

$$\mu((gx_n, gy_n), (gx, gy)) \leq 1, \tag{12}$$

and similarly

$$\mu((gy_n, gx_n), (gy, gx)) \leq 1, \tag{13}$$

for all $n \in \mathbb{N}$. Also, as in Theorem 2.10 we have

$$\lim_{n\to+\infty} F(x_n, y_n) = x \ \text{ and } \ F(y_n, x_n) = y.$$

Using (11), (13), property of $\psi(t) < t$ for all $t > 0$ and (9) we get

$$H(1, \frac{d(F(gx_n, gy_n), F(x,y)) + d(F(gy_n, gx_n), F(y,x))}{2}) \tag{14}$$

$$\begin{aligned} &\leq H(\alpha((ggx_n, ggy_n),(gx,gy)), \frac{d(F(gx_n, gy_n), F(x,y)) + d(F(gy_n, gx_n), F(y,x))}{2}) \\ &\leq \mathcal{F}(\mu((ggx_n, ggy_n),(gx,gy)), \psi(\frac{d(ggx_n, gx) + d(ggy_n + gy)}{2})) \\ &\leq \mathcal{F}(1, \psi\Big(\frac{d(ggx_n, gx) + d(ggy_n + gy)}{2}\Big)). \end{aligned}$$

Therefore, from condition (1) of Definition 2.5 we have

$$\frac{d(F(gx_n, gy_n), F(x,y)) + d(F(gy_n, gx_n), F(y,x))}{2} \leq \psi\Big(\frac{d(ggx_n, gx) + d(ggy_n + gy)}{2}\Big) \tag{15}$$

$$\leq \frac{d(ggx_n, gx) + d(ggy_n + gy)}{2}.$$

From triangle inequality, and from condition that F and g are compatible we have

$$d(gx, F(x,y)) \leq d(gx, ggx_{n+1}) + d(ggx_{n+1}, F(gx_n, gy_n)) + d(F(gx_n, gy_n), F(x,y)) \tag{16}$$

$$= d(gx, ggx_{n+1}) + d(gF(x_n, y_n), F(gx_n, gy_n)) + d(F(gx_n, gy_n), F(x,y)),$$

and

$$d(gy, F(y,x)) \leq d(gy, ggy_{n+1}) + d(ggy_{n+1}, F(gy_n, gx_n)) + d(F(gy_n, gx_n), F(y,x)) \tag{17}$$

$$= d(gy, ggy_{n+1}) + d(gF(y_n, x_n), F(gy_n, gx_n)) + d(F(gy_n, gx_n), F(y,x)).$$

Adding (16) and (17), and using (15) we get

$$d(gx, F(x,y)) + d(gy, F(y,x)) \tag{18}$$

$$\begin{aligned} &\leq d(gx, ggx_{n+1}) + d(gF(x_n, y_n), F(gx_n, gy_n)) + d(gy, ggy_{n+1}) + d(gF(y_n, x_n), F(gy_n, gx_n)) \\ &+ d(ggx_n, gx) + d(ggy_n + gy) \end{aligned}$$

Letting $n \to +\infty$ in (18), and using using properties that g is continuous, and F and g are compatible we have

$$d(F(x,y), gx) + d(gy, F(y,x)) = 0. \tag{19}$$

Hence, $F(x,y) = gx$ and $F(y,x) = gy$. Thus, F and g have a coupled coincidence in X. □

Remark 2.13. *We noted, that papers ([27], [33]) have imprecisions. Namely, the proof of Theorem 3.3. from [33] is not completely correct. This irregularity is able to correct if the condition (2) in the Theorem 3.3 replace by other condition in accordance with above proven theorem in this paper. Also, in the paper [28] although the authors fix mistakes, both of the examples have inaccuracies, because in both cases selected functions F does not satisfy the feature of mixed-monotone property. For more details also see [23].*

If we choice $H(x,y) = xy$ and $\mathcal{F}(s,t) = st$, we obtain a corrigendum of Theorem 3.5 in [33] with $\mu(x,y) = 1$ for all $x, y \in X$.

Theorem 2.14. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^2 \to X$ and $g : X \to X$ be two mappings such that F has a mixed $g-$monotone property. Assume that there exists $\psi \in \Psi$ and a mapping $\alpha : X^2 \times X^2 \to [0, +\infty)$ such that*

$$\alpha((gx, gy), (gu, gv))\frac{d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))}{2}) \quad \leq \quad \psi\Big(\frac{d(gx, gu) + d(gy, gv)}{2}\Big)$$

for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$. Suppose that

(a) conditions (i), (ii) and (iii) of Theorem 2.10 hold,
(b) if (x_n) and (y_n) are sequences in X such that

$$\alpha((gx_n, gy_n),\ (gx_{n+1}, gy_{n+1})) \geq 1,\ \alpha((gy_n, gx_n), (gy_{n+1}, gx_{n+1})) \geq 1,$$

for all $n \in \mathbb{N}$, $\lim_{n\to+\infty} gx_n = x \in X$ and $\lim_{n\to+\infty} gy_n = y \in X$ then

$$\alpha((gx_n, gy_n), (gx, gy)) \geq 1,\ \alpha((gx_n, gy_n), (gx, gy)) \geq 1.$$

Then there exist $x,\ y \in X$ such that $F(x, y) = gx$ and $F(y, x) = gy$, that is, F and g have a coupled coincidence in X.

In the following theorem, we shall prove the uniqueness of the coupled fixed point. If $(X, \preceq)$ is a partially ordered set then we endow the product X^2 with following partial order relation:

$$(x, y) \sqsubseteq (u, v) \Leftrightarrow x \preceq u,\ \ y \succeq v,$$

for all $(x, y), (u, v) \in X^2$.

Theorem 2.15. *In adding to the hypothesis of Theorem 2.10, suppose that for every $(x, y), (s, t)$ in X^2, there exists (u, v) in X^2 such that*

$$\alpha((gx, gy), (gu, gv)) \geq 1,\ \alpha((gs, gt), (gu, gv)) \geq 1, \tag{20}$$

$$\mu((gx, gy), (gu, gv)) \leq 1,\ \mu((gs, gt), (gu, gv)) \leq 1, \tag{21}$$

and (u, v) is comparable to (x, y) and (s, t). Then F and g have a unique coupled coincidence point.

Proof. From Theorem 2.10, the set of coupled coincidence point is nonempty. Suppose that (x, y) and (s, t) are different coupled coincidence points of the mappings $F : X^2 \to X$ and $g : X \to X$ that is, $gx = F(x, y)$, $gy = F(y, x)$, $gs = F(s, t)$ and $gt = F(t, s)$. By assumption there exists (u, v) in X^2 such that (u, v) is comparable to (x, y) and (s, t). Put $u = u_0$ and $v = v_0$ and choose $u_1, v_1 \in X$ such that $gu_1 = F(u_0, v_0)$ and $gv_1 = F(v_0, u_0)$. Thus, we can define two sequences (gu_n) and (gv_n) as

$$gu_{n+1} = F(u_n, v_n),\ gv_{n+1} = F(v_n, u_n).$$

Since (u, v) is comparable to (x, y) then it is easy to show that $gx \preceq gu_1$ and $gy \succeq gv_1$. Thus, $gx \preceq gu_n$ and $gy \succeq gv_n$ for all $n \geq 1$. Since, for every $(x, y), (s, t) \in X^2$ there exists $(u, v) \in X \times X$ such that (20) and (21) are satisfied. Since F and g are (α)-admissible, from (20) and (21), we have

$$\alpha((gx, gy), (gu, gv)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1$$

$$\mu((gx, gy), (gu, gv)) \leq 1 \Rightarrow \mu((F(x, y), F(y, x)), (F(u, v), F(v, u))) \leq 1$$

Thus, for $u = u_0$ and $v = v_0$, we get

$$\alpha((gx, gy), (gu_0, gv_0)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u_0, v_0), F(v_0, u_0))) \geq 1,$$

and

$$\mu((gx,gy),(gu_0,gv_0)) \leq 1 \Rightarrow \mu((F(x,y),F(y,x)),(F(u_0,v_0),F(v_0,u_0))) \leq 1.$$

Thus, we have

$$\alpha((gx,gy),(gu_0,gv_0)) \geq 1 \Rightarrow \alpha((gx,gy),(gu_1,gv_1)) \geq 1,$$

and

$$\mu((gx,gy),(gu_0,gv_0)) \leq 1 \Rightarrow \mu((gx,gy),(gu_1,gv_1)) \leq 1.$$

Therefore, by mathematical induction, we obtain

$$\alpha((gx,gy),(gu_n,gv_n)) \geq 1, \tag{22}$$

and

$$\mu((gx,gy),(gu_n,gv_n)) \geq 1, \tag{23}$$

for all $n \in \mathbb{N}$, and similarly, $\alpha((gy,gx),(gv_n,gu_n)) \geq 1$. From (20), (21), (22) and (23) we get

$$\begin{aligned} H(1, \frac{d(gx,gu_{n+1}) + d(gy,gv_{n+1})}{2}) &= H(1, \frac{d(F(x,y),F(u_n,v_n)) + d(F(y,x),F(v_n,u_n))}{2}) \\ &\leq H(\alpha((gx,gy),(gu_n,gv_n)), \frac{d(F(x,y),F(u_n,v_n)) + d(F(y,x),F(v_n,u_n))}{2}) \\ &\leq \mathcal{F}(\mu((gx,gy),(gu_n,gv_n)), \psi\Big(\frac{d(gx,gu_n) + d(gy,gv_n)}{2}\Big)) \\ &\leq \mathcal{F}(1, \psi\Big(\frac{d(gx,gu_n) + d(gy,gv_n)}{2}\Big)). \end{aligned}$$

So, using condition (2) of Definition (2.5) we conclude that

$$\frac{d(gx,gu_{n+1}) + d(gy,gv_{n+1})}{2} \leq \psi\Big(\frac{d(gx,gu_n) + d(gy,gv_n)}{2}\Big). \tag{24}$$

Thus, from (24) we have

$$\frac{d(gx,gu_{n+1}) + d(gy,gv_{n+1})}{2} \leq \psi^n\Big(\frac{d(gx,gu_1) + d(gy,gv_1)}{2}\Big), \tag{25}$$

for each $n \geq 1$. Letting $n \to +\infty$ in (25) and using Lemma 1.5, we get

$$\lim_{n\to+\infty} [d(gx,gu_{n+1}) + d(gy,gv_{n+1})] = 0.$$

This implies that

$$\lim_{n\to+\infty} d(gx,gu_{n+1}) = \lim_{n\to+\infty} d(gy,gv_{n+1}) = 0. \tag{26}$$

Similarly, one can show that

$$\lim_{n\to+\infty} d(gs,gu_{n+1}) = \lim_{n\to+\infty} d(gt,gv_{n+1}) = 0. \tag{27}$$

From (26) and (27), we conclude that $x = gx = gs = s$ and $y = gy = gt = t$. Hence, F and g have a unique coupled coincidence point. □

Corollary 2.16. *Let $(X,\preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F: X^2 \to X$ and $g: X \to X$ be two mappings such that F has a mixed $g-$monotone property on X. Suppose that there exists $\psi \in \Psi$ and $\alpha,\ \mu : X^2 \times X^2 \to [0,+\infty)$ such that for $x,y,u,v \in X$, the following holds:*

$$(\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2} + l)^{\alpha((gx,gy),(gu,gv))} \quad \leq \quad \mu((gx,gy),(gu,gv))\psi\Big(\frac{d(gx,gu)+d(gy,gv)}{2}\Big)+l$$

for all $gx \succeq gu$ and $gy \preceq gv$. Suppose also that

(i) F and g are $(\alpha)-$admissible and $(\mu)-$subadmissible,

(ii) there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0,y_0)$ and $gy_0 \succeq F(y_0,x_0)$ with

$$\alpha((gx_0,gy_0),(F(x_0,y_0),F(y_0,x_0))) \geq 1,\ \alpha((gy_0,gx_0),(F(y_0,x_0),F(x_0,y_0))) \geq 1$$

and

$$\mu((gx_0,gy_0),(F(x_0,y_0),F(y_0,x_0))) \leq 1,\ \mu((gy_0,gx_0),(F(y_0,x_0),F(x_0,y_0))) \leq 1,$$

(iii) $F(X^2) \subseteq g(X)$, g is continuous and F and g are compatible in X.

(iv) F is continuous, or if (x_n) and (y_n) are sequences in X such that

$$\alpha((gx_n,gy_n),(gx_{n+1},gy_{n+1})) \geq 1,\ \alpha((gy_n,gx_n),(gy_{n+1},gx_{n+1})) \geq 1,$$

and

$$\mu((gx_n,gy_n),(gx_{n+1},gy_{n+1})) \leq 1,\ \mu((gy_n,gx_n),(gy_{n+1},gx_{n+1})) \leq 1,$$

for all $n \in \mathbb{N}$, and $\lim_{n\to+\infty} gx_n = x \in X$ and $\lim_{n\to+\infty} gy_n = y \in X$ then

$$\alpha((gx_n,gy_n),(gx,gy)) \geq 1,\ \alpha((gx_n,gy_n),(gx,gy)) \geq 1$$

and

$$\mu((gx_n,gy_n),(gx,gy)) \leq 1,\ \mu((gx_n,gy_n),(gx,gy)) \leq 1.$$

Then F and g have a coupled coincidence point, that is, there exist $x,\ y \in X$ such that

$$F(x,y) = gx,\ F(y,x) = gy.$$

Corollary 2.17. *Let $(X,\preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F: X^2 \to X$ and $g: X \to X$ be two mappings such that F have a mixed $g-$monotone property on X. Suppose that there exists $\psi \in \Psi$ and $\alpha,\mu : X^2 \times X^2 \to [0,+\infty)$ such that for $x,y,u,v \in X$, the following holds:*

$$(\alpha((gx,gy),(gu,gv)) + l)^{\frac{d(F(x,y),F(u,v))+d(F(y,x),F(v,u))}{2}} \quad \leq \quad (1+l)^{\mu((gx,gy),(gu,gv))\psi\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right)}$$

for all $gx \succeq gu$ and $gy \preceq gv$. Suppose also that

(i) F and g are (α)-admissible and (μ)-subadmissible,

(ii) there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0,y_0)$ and $gy_0 \succeq F(y_0,x_0)$ with

$$\alpha((gx_0,gy_0),(F(x_0,y_0),F(y_0,x_0))) \geq 1,\ \alpha((gy_0,gx_0),(F(y_0,x_0),F(x_0,y_0))) \geq 1,$$

and

$$\mu((gx_0,gy_0),(F(x_0,y_0),F(y_0,x_0))) \leq 1,\ \mu((gy_0,gx_0),(F(y_0,x_0),F(x_0,y_0))) \leq 1,$$

(iii) $F(X^2) \subseteq g(X)$, g is continuous and F and g are compatible in X,

(iv) F is continuous, or if (x_n) and (y_n) are sequences in X such that

$$\alpha((gx_n, gy_n), (gx_{n+1}, gy_{n+1})) \geq 1, \ \alpha((gy_n, gx_n), (gy_{n+1}, gx_{n+1})) \geq 1,$$

and

$$\mu((gx_n, gy_n), (gx_{n+1}, gy_{n+1})) \leq 1, \ \mu((gy_n, gx_n), (gy_{n+1}, gx_{n+1})) \leq 1,$$

for all $n \in \mathbb{N}$, and $\lim_{n\to+\infty} gx_n = x \in X$, $\lim_{n\to+\infty} gy_n = y \in X$ then

$$\alpha((gx_n, gy_n), (gx, gy)) \geq 1, \ \alpha((gx_n, gy_n), (gx, gy)) \geq 1$$

and

$$\mu((gx_n, gy_n), (gx, gy)) \leq 1, \ \mu((gx_n, gy_n), (gx, gy)) \leq 1.$$

Then F and g have a coupled coincidence point, that is, there exist $x, y \in X$ such that

$$F(x, y) = gx, \ F(y, x) = gy.$$

Example 2.18. *Let $X = [0, 1]$ and $d : X^2 \to \mathbb{R}$ be a standard metric. Define a mapping $F : X^2 \to X$ and $g : X \to X$ by $F(x, y) = \frac{x-y}{8}$ and $g(x) = x$ for all $x, y \in X$. Let $H(x, y) = xy$ and $\mathcal{F}(s, t) = st$. Consider a mapping $\alpha : X^2 \times X^2 \to [0, +\infty)$ and $\mu : X^2 \times X^2 \to [0, +\infty)$ be such that*

$$\alpha((x, v), (y, u)) = \begin{cases} 1, & \text{if } x \geq y, u \geq v, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mu((x, v), (y, u)) = \begin{cases} 1, & \text{if } x \geq y, u \geq v, \\ 10, & \text{otherwise.} \end{cases}$$

Now we have,

$$d(F(x, v), F(y, u)) = \left| \frac{x - v}{8} - \frac{y - u}{8} \right| \leq \frac{1}{8}(|x - y| + |v - u|) = \frac{1}{8}(d(x, y) + d(v, u)).$$

Analogous,

$$d(F(v, x), F(y, u)) \leq \frac{1}{8}(d(x, y) + d(v, u)).$$

So, it follows that

$$\begin{aligned} &\alpha((x, v), (y, u)) \frac{d(F(x, v), F(y, u)) + d(F(v, x), F(u, y))}{2} \\ &\leq \frac{1}{4}(d(x, y) + d(v, u)) \\ &\leq \frac{1}{2}\mu((x, v), (y, u)) \frac{(d(x, y) + d(v, u))}{2}. \end{aligned}$$

Thus all hypothesis of Theorem 2.10 for $\psi(t) = \frac{t}{2}$ are fulfilled. Then, there exists a coupled coincidence point $(0, 0)$ of F and g.

Example 2.19. *Let $X = [0,1]$ Then $(X, \preceq)$ is a partially ordered set with the natural ordering of real numbers. Let $d(x,y) = \left|x-y\right|$ for $x, y \in [0,1]$. Then (X,d) is a complete metric space. Let $F : X^2 \to X$ and $g : X \to X$ be defined as $g(x) = x^3$, and $F(x,y) = \frac{x^3-y^3}{4} + \frac{3}{4}$, for all $x \in X$. Let $H(x,y) = xy$ and $\mathcal{F}(s,t) = st$.*

Let (x_n) and (y_n) be sequences in X such that

$$\begin{aligned} \lim_{n\to+\infty} F(x_n, y_n) &= \lim_{n\to+\infty} gx_n = a, \\ \lim_{n\to+\infty} F(y_n, x_n) &= \lim_{n\to+\infty} gy_n = b. \end{aligned}$$

Thus it follows that $a = \frac{3}{4}$, and $b = \frac{3}{4}$.

Therefore,

$$\lim_{n\to+\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0$$

$$\lim_{n\to+\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0$$

Hence, the mappings F and g are compatible in X. Consider a mapping α, $\mu : X^2 \times X^2 \to [0, +\infty)$ be such that

$$\alpha((gx, gv), (gy, gu)) = \begin{cases} 1, & \text{if } x \geq y, u \geq v, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mu((gx, gv), (gy, gu)) = \begin{cases} 1, & \text{if } x \geq y, u \geq v, \\ 10, & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned} \alpha((gx, gv), (gy, gu))\frac{d(F(x,v), F(y,u)) + d(F(v,x), F(u,y))}{2} &= \frac{\left|\frac{x^3-v^3}{4} - \frac{y^3-u^3}{4}\right| + \left|\frac{v^3-x^3}{4} - \frac{u^3-y^3}{4}\right|}{2} \\ &\leq \frac{1}{4}(\left|x^3-y^3\right| + \left|u^3-v^3\right|) \\ &= \frac{1}{4}(d(gx,gy) + d(gu,gv)) \\ &\leq \frac{1}{2}\mu((gx,gv),(gy,gu))\frac{(d(gx,gy)+d(gu,gv))}{2}. \end{aligned}$$

Thus $\psi(t) = \frac{t}{2}$. Also we can see that $F(X^2) \subseteq g(X)$ and F satisfies mixed $g-$monotone property. So, all conditions of Theorem 2.10 are satisfied, and $(0,0)$ is a coupled coincidence point for F and g.

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors read and approved the final manuscript.

Acknowledgements

The authors thank the editor and the referees for their valuable comments and suggestions. The third author supported by MNTRRS-174009.

References

[1] Abbas, M., Khan, A. M., Radenović, S., Common coupled fixed point theorems in cone metric spaces for w-compatible mappings, Appl. Math. Comput. 217 (2010) 195–202.

[2] Abdeljawad, T., Coupled Fixed Point Theorems for Partially Contractive Mappings, Fixed Point Theory Appl., 2012:148 (2012).

[3] Alsulami, H. H., Roldan, A., Karapinar, E., Radenović, S., Some inevitable remarks on "Tripled fixed point theorems for mixed monotone Kannan type contractive mappings", Journal of Applied Mathematics, (2014) Article ID 392301, 7 pages.

[4] Amini-Harandi, A., Coupled and Tripled Fixed Point Theory in Partially Ordered Metric Spaces with Application to Initial Value Problem, Math. Comput. Model., 57 (2013) 2343–2348.

[5] Ansari, A. H., Note on "α-admissible mappings and related fixed point theorems", The 2nd Regional Conference on Mathematics And Applications,PNU, September 2014, (2014) 373–376.

[6] Ansari, A. H., Shukla, S., Some fixed point theorems for ordered $F-(F;h)$-contraction and subcontractions in $0-f$-orbitally complete partial metric spaces, J. Adv. Math. Stud. 9(1) (2016) 37–53.

[7] Aydi, H., Samet, B. and Vetro, C., Coupled Fixed Point Results in Cone Metric Spaces for $\tilde{W}$-Compatible Mappings, Fixed Point Theory Appl., 2011:27 (2011).

[8] Banach, S., Sur les op*é*rations dans les ensembles abstraits et leur application aux *é*quations int*é*grales, Fundam Math., 3 (1922) 133–181.

[9] Bhaskar, T.G., Lakshmikantham, V., Fixed Point Theorems in Partially Ordered Metric Spaces and Applications, Nonlinear Anal., 65(7) (2006) 1379–1393.

[10] Choudhury, B.S., Kundu, A., A Coupled Coincidence Point Result in Partially Ordered Metric Spaces for Compatible Mappings, Nonlinear Anal. TMA., 73(8) (2010) 2524–2531.

[11] Choudhury, B.S., Das, K., Das, P., Coupled Coincidence Point Results for Compatible Mappings in Partially Ordered Fuzzy Metric Spaces, Fuzzy Set. Syst., 222 (2013) 84–97.

[12] Ćojbašić-Rajić, V., Radenović, S., A note on tripled fixed point of w-compatible mappings in tvs-cone metric spaces, Thai Journal of Mathematics, 12(3) (2014) 717–728.

[13] Ding, H. S., Li, L., Radenović, S., Coupled coincidence points for mixed g-monotone mappings in partially ordered metric spaces, Fixed Point Theory Appl., 2012:96 (2012).

[14] Fadail, Z. M., Rad, G. S., Ozturk, V., Radenović, S., Some remarks on coupled, tripled and n-tupled fixed points theorems in ordered abstract metric spaces, Far East Journal of Mathematical Sciences, 97(7) (2015) 809–839.

[15] Guo, D., Lakshmikantham, V., Coupled Fixed Points of Nonlinear Operators with Applications, Nonlinear Anal. TMA., 11(5) (1987) 623–632.

[16] Huang, H., Radenović, S., Vujaković, J., On some recent coincidence and coupled coincidence point results in partially ordered b-metric spaces, Fixed Point Theory Appl., 2015:63 (2015).

[17] Jungck, G., Compatible Mappings and Common Fixed Points, Internat. J. Math. Math. Sci., 9(4) (1986) 771-779.

[18] Kadelburg, Z., Radenović, S., Coupled fixed point results under tvs-cone metric spaces and w-cone-distance, Advances in Fixed Point Theory, 2(1) (2012) 29–46.

[19] Kadelburg, Z., Kumam, P., Radenović, S., Sintunavarat, W., Common coupled fixed point theorems for Geraghty's type contraction mappings without mixed monotone property, Fixed Point Theory Appl., 2015:27 (2015).

[20] Kadelburg, Z., Radenović, S., *Fixed point and tripled fixed point theorems under Pata-type conditions in ordered metric spaces,* Int. J. Anal. Appl., **6**(1), 113–122, 2014.

[21] Kadelburg, Z., Nashine, H. K., Radenović, S., Some new coupled fixed point results in 0-complete ordered partial metric spaces, J. Adv. Math. Studies, 6(1) (2013) 159–172.

[22] Karapinar, E., Samet, B., Generalized $\alpha-\psi-$Contractive Type Mappings and Related Fixed Point Theorems with Applications, Abstr. Appl. Anal., Article ID: 793486 (2012).

[23] Karapinar, E., Agarwal, R.P., A note on "Coupled fixed point theorems for $\alpha-\psi-$contractive-type mappings in partially ordered metric spaces, Fixed Point Theory Appl., 2013:216 (2013).

[24] Lakshmikantham, V., Ćirić, Lj., Coupled Fixed Point Theorems for Nonlinear Contractions in Partially Ordered Metric Spaces, Nonlinear Anal., 70 (2009) 4341-4349.

[25] Luong, N.V., Thuan, N.X., Coupled Fixed Points in Partially Ordered Metric Spaces and Application, Nonlinear Anal. TMA., 74(3) (2011) 983–992.

[26] Marín, J., Romaguera, S., Tirado, P., Q-Functions on Quasimetric Spaces and Fixed Points for Multivalued Maps, Fixed Point Theory Appl., Article ID 603861, doi:10.1155/2011/603861, 2011.

[27] Mursaleen, M., Mohiuddine, S.A., Agarwal, R.P., Coupled Fixed Point Theorems for $\alpha-\psi$-Contractive Type Mappings in Partially Ordered Metric Spaces, Fixed Point Theory Appl., 2012:228 (2012).

[28] Mursaleen, M., Mohiuddine, S.A., Agarwal, R.P., Corrigendum to "Coupled fixed point theorems for $\alpha-\psi-$contractive type mappings in partially ordered metric spaces", Fixed Point Theory Appl. 2013:127 (2013).

[29] Nashine, H.K., Samet, B., Vetro, C., Coupled Coincidence Points for Compatible Mappings Satisfying Mixed Monotone Property, J. Nonlinear Sci. Appl., 5 (2012) 104–114.

[30] Nashine, H.K., Kadelburg, Z., Radenović, S., Coupled Common Fixed Point Theorems for W^*-Compatible Mappings in Ordered Cone Metric Spaces, Appl. Math. Comput., 218(9) (2012) 5422–5432.

[31] Nieto, J.J., Rodríguez-López, R., Contractive Mapping Theorems in Partially Ordered Sets and Applications to Ordinary Differential Equations. Order, 22 (2005) 223-239.

[32] Nieto, J.J., Rodriguez-López, R., Existence and Uniqueness of Fixed Point in Partially Ordered Sets and Applications to Ordinary Differential Equations, Acta Math. Sin., 23(12) (2007) 2205–2212.

[33] Preeti, Kumar S., Coupled Fixed Point for (α, ψ)–Contractive in Partially Ordered Metric Spaces Using Compatible Mappings, Applied Mathematics, 6 (2015) 1380–1388.
[34] Ran, A.C.M., Reurings, M.C.B., A Fixed Point Theorem in Partially Ordered Sets and Some Applications to Matrix Equations, Proc. Am. Math. Soc., 132 (2004) 1435-1443.
[35] Radenović, S., Coupled fixed point theorems for monotone mappings in partially ordered metric spaces, Kragujevac Journal of Mathematics, 38(2) (2014) 249–257.
[36] Radenović, S., Some coupled coincidence points results of monotone mappings in partially ordered metric spaces, Int. J. Anal. Appl., 5(2) (2014) 174–184.
[37] Radenović, S., *Bhaskar-Lakshmikantham type-results for monotone mappings in partially ordered metric spaces*, Int. J. Nonlinear Anal. Appl., **5**(2), 37–49, 2014.
[38] Radenović, S., *Remarks on some coupled fixed point results in partial metric spaces*, Nonlinear Funct. Anal. and Appl. **18**(1), 39–50, 2013.
[39] Radenović, S., *Remarks on some recent coupled coincidence point results in symmetric G-metric spaces*, Journal of Operators, **2013**, Article ID 290525, 8 pages.
[40] Radenović, S., *Remarks on some coupled coincidence point results in partially ordered metric spaces*, Arab. J. Math. Sci., **20**(1), 29–39, 2014.
[41] Radenović, S., A note on tripled coincidence and tripled common fixed point theorems in partially ordered metric spaces, Appl. Math. Comput., 236 (2014) 367–372.
[42] Salimi, P., Latif, A., Hussain,N., Modified $\alpha - \psi$-contractive mappings with applications, Fixed Point Theory Appl., 2013:151 (2013).
[43] Samet, B., Vetro, C., Coupled Fixed Point, F-Invariant Set and Fixed Point of N-Order, Ann. Funct. Anal., 1(2) (2010) 46–56.
[44] Samet, B., Vetro, C., Coupled Fixed Point Theorems for Multi-Valued Nonlinear Contraction Mappings in Partially Ordered Metric Spaces, Nonlinear Anal. TMA., 74(12) (2011) 4260–4268.
[45] Samet, B., Vetro, C., Vetro, P., Fixed Point Theorems for $\alpha - \psi$-Contractive Type Mappings, Nonlinear Anal. TMA., 75(4) (2012) 2154–2165.
[46] Samreen, M., Kiran, Q., Kamran, T., Fixed point theorems for φ-contractions, J. Inequal. Appl., 2014:266 (2014).
[47] Sedghi, S., Kumam, P., Došenović, T., Shobe N., Two extensions of coupled coincidence point results for nonlinear contractive mappings, J. Egyptian Math. Soc., 23(2) (2015) 343–351.
[48] Shatanawi, W., Samet, B., Abbas, M., Coupled Fixed Point Theorems for Mixed Monotone Mappings in Ordered Partial Metric Spaces, Math. Comput. Model., 55(3) (2012) 680–687.
[49] Sintunavarat, W., Cho, Y.J., Kumam, P., Coupled Fixed Point Theorems for Nonlinear Contractions without Mixed Monotone Property, Fixed Point Theory Appl., 2012:170 (2012).
[50] Turinici, M., Abstract Comparison Principles and Multivariable Gronwall-Bellman Inequalities, J. Math. Anal. Appl., 117(1) (1986) 100–127.