Filomat 31:9 (2017), 2749–2762
https://doi.org/10.2298/FIL1709749C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Distortion and Covering Theorems of Pluriharmonic Mappings

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Abstract. The linear-invariant families of analytic functions make it possible to obtain well-known results to broader classes of functions, and are often helpful in obtaining simpler proofs along with new results. Based on this classical approach due to Pommerenke, properties (such as bounds for the derivative, covering and distortion) of a corresponding class of locally quasiconformal and planar harmonic mappings are established by Starkov. Motivated by these works, in this paper, we mainly investigate distortion and covering theorems on some classes of pluriharmonic mappings.

1. Introduction and Preliminaries

The notion of linear-invariant family (hereafter $\mathcal{LIF}$) of holomorphic functions defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ was first introduced by Pommerenke in [28] and showed a number of important properties of such families. Recall that if $\mathcal{A}$ denotes the family of all holomorphic functions f on $\mathbb{D}$ with the topology of uniform convergence of compact subsets of $\mathbb{D}$, then a subfamily $\mathcal{F}$ of $\mathcal{A}$ is called linear-invariant if it is closed under the re-normalized composition with a conformal automorphism of $\mathbb{D}$. If the modulus of the second Taylor coefficient is bounded in $\mathcal{F}$, then the order α of the $\mathcal{LIF}$ is defined to be

$$\alpha := \sup\{|f''(0)|/2 : f \in \mathcal{F}\}.$$

Many properties of a $\mathcal{LIF}$ depends on the order of the family. A universal $\mathcal{LIF}$ of order α, denoted by $\mathcal{U}_\alpha$, is the union of all $\mathcal{LIF}$'s $\mathcal{F}$ such that the order of $\mathcal{F}$ is less than or equal to α. The fact is that $\mathcal{U}_\alpha$ is empty if $\alpha < 1$ and $\mathcal{U}_1$ coincides with the family of all normalized holomorphic functions f which univalently map $\mathbb{D}$ onto convex domains, see [28]. Also, a $\mathcal{LIF}$ of order 2 is the family $\mathcal{S}$ of normalized univalent functions from $\mathcal{A}$. Moreover, it has been proved that many subfamilies of univalent mappings on $\mathbb{D}$ are linearly invariant, see for example [21] and the references therein. For the regularity growth of functions on $\mathcal{U}_\alpha$,

2010 *Mathematics Subject Classification*. Primary 31B05, 32A10, 32H02; Secondary 31C10, 30C65
Keywords. Pluriharmonic mapping, distortion theorem, linear invariant family, quasiregular mapping.
Received: 14 September 2015; Accepted: 04 September 2016
Communicated by Dragan S. Djordjević
This research of the first author was partly supported by the National Natural Science Foundation of China (No. 11401184 and No. 11571216), the Hunan Province Natural Science Foundation of China (No. 2015JJ3025), the China's fifty-ninth batch of post doctoral funding (No. 2016M590492), the Post Doctoral Foundation of Jiangsu Province (No. 1601182C), the Construct Program of the Key Discipline in Hunan Province and the Science and Technology Plan of Hunan Province (2016TP1020). The research of the second author was supported by the project RUS/RFBR/P-163 under Department of Science & Technology and Russian Foundation for Basic Research and this author is currently on leave from the Indian Institute of Technology Madras.
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we refer to [3, 30, 31]. The concept of linear invariance was generalized by many authors in many different contexts and in 1997, Pfaltzgraff [25] extended this concept for locally holomorphic functions defined on the unit ball of the complex Euclidean n-space $\mathbb{C}^n$ and many properties were further discussed in [26, 27]. In the recent years, the theory of functions of several complex variables found numerous applications in many different areas of mathematics including function spaces and quantum field theory, enriching it with far-reaching consequences. Various questions from one-dimensional to higher dimensional case remain unsolved. For our discussion, we need to deal with such problems in the higher dimensional case and thus, the article is primarily devoted to certain class of pluriharmonic mappings and their interplay with holomorphic mappings.

As with the standard practice, for $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$, we let $\overline{z} = (\overline{z}_1 \cdots \overline{z}_n)$, and $\langle z, w\rangle := \sum_{k=1}^{n} z_k\overline{w}_k$ with the associated Euclidean norm $\|z\| := \langle z, z\rangle^{1/2}$ which makes $\mathbb{C}^n$ into an n-dimensional complex Hilbert space. Throughout the discussion an element $z \in \mathbb{C}^n$ is identified as an $n\times 1$ column vector. For $a \in \mathbb{C}^n$ and $r > 0$,

$$\mathbb{B}^n(a, r) = \{z \in \mathbb{C}^n : \|z - a\| < r\}$$

denotes the (open) ball of radius r with center a. Also, we let $\mathbb{B}^n(r) := \mathbb{B}^n(0, r)$ and use $\mathbb{B}^n$ to denote the unit ball $\mathbb{B}^n(1)$, and $\mathbb{D} = \mathbb{B}^1$.

A continuous complex-valued function f defined on a domain $G \subset \mathbb{C}^n$ is said to be *pluriharmonic* if for each fixed $z \in G$ and $\theta \in \partial\mathbb{B}^n$, the function $f(z + \theta\zeta)$ is harmonic in $\{\zeta \in \mathbb{C} : \|\theta\zeta - z\| < d_G(z)\}$, where $d_G(z)$ denotes the distance from z to the boundary ∂G of G. It follows from [29, Theorem 4.4.9] that a real-valued function u defined on G is pluriharmonic if and only if it is locally the real part of a holomorphic function. If Ω is a simply connected domain in $\mathbb{C}^n$, then it is clear that a mapping $f : \Omega \to \mathbb{C}$ is pluriharmonic if and only if f has a representation $f = h + \overline{g}$, where h, g are holomorphic in Ω (cf. [34]). A *vector-valued mapping* $f = (f_1 \cdots f_N)^T$, the transpose of the $1 \times N$ row matrix $(f_1 \cdots f_N)$, defined in $\mathbb{B}^n$ is said to be pluriharmonic, if each component f_j $(1 \le j \le N)$ is a pluriharmonic mapping from $\mathbb{B}^n$ into $\mathbb{C}$, where N is a positive integer and the superscript T indicates the transpose of a matrix. We refer to [7, 9–12, 14, 17, 19, 29] for further details and recent investigations on pluriharmonic mappings.

For an $n \times n$ complex matrix A, we introduce the *operator norm*

$$\|A\| = \sup_{z\neq 0} \frac{\|Az\|}{\|z\|} = \max\{\|A\theta\| : \theta \in \partial\mathbb{B}^n\}.$$

We use $L(\mathbb{C}^n, \mathbb{C}^m)$ to denote the space of continuous *linear operators* from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the operator norm, and let I_n be the *identity operator* in $L(\mathbb{C}^n, \mathbb{C}^n)$.

We denote by $\mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ the set of all *vector-valued pluriharmonic mappings* from $\mathbb{B}^n$ into $\mathbb{C}^n$. Then every $f \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ can be written as $f = h + \overline{g}$, where h and g are holomorphic in $\mathbb{B}^n$, and this representation is unique when $g(0) = 0$. It is a simple exercise to see that the real Jacobian determinant of f can be written as

$$\det J_f = \det\begin{pmatrix} Dh & \overline{Dg} \\ Dg & \overline{Dh} \end{pmatrix}$$

and if h is locally biholomorphic (i.e. the complex Jacobian matrix $J_f(z)$ of f at each z is invertible), then the determinant of J_f has the form

$$\det J_f = |\det Dh|^2 \det\left(I_n - Dg[Dh]^{-1}\overline{Dg[Dh]^{-1}}\right). \tag{1}$$

In the case of a *planar harmonic mapping* $f = h + \overline{g}$, we find that

$$\det J_f = |h'|^2 - |g'|^2,$$

and so, f is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $|g'(z)| < |h'(z)|$ in $\mathbb{D}$; or equivalently if $h'(z) \neq 0$ and the dilatation $\omega(z) = g'(z)/h'(z)$ is analytic in $\mathbb{D}$ and has the property that $|\omega(z)| < 1$ in $\mathbb{D}$ (see [16, 22]). For $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$, the condition $\|Dg[Dh]^{-1}\| < 1$ is sufficient for $\det J_f$ to be positive and hence for f to be sense-preserving (see [17, Theorem 5]). This is indeed a natural generalization of one-variable condition.

For motivation, consider the Taylor expansion of a function $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ with $h(0) = g(0) = 0$, where

$$\begin{aligned} h(z) &= [Dh(0)]z + \frac{1}{2}[D^2h(0)](z,z) + \cdots + \frac{1}{m}[D^m h(0)](z,\ldots,z) + \cdots \\ &= A_1 z + A_2(z,z) + A_m(z,\ldots,z) + \cdots \end{aligned} \tag{2}$$

and

$$\begin{aligned} g(z) &= [Dg(0)]z + \frac{1}{2}[D^2g(0)](z,z) + \cdots + \frac{1}{m}[D^m g(0)](z,\ldots,z) + \cdots \\ &= B_1 z + B_2(z,z) + B_m(z,\ldots,z) + \cdots. \end{aligned} \tag{3}$$

As with one variable case, a $\mathcal{LIF}$ in $\mathbb{B}^n$ is a family $\mathcal{M}$ of locally biholomorphic mappings $f : \mathbb{B}^n \to \mathbb{C}^n$ such that if $f \in \mathcal{M}$ then

(i) $f(0) = 0$, $Df(0) = I_n$ and

(ii) $\Lambda_\phi(f) \in \mathcal{M}$ for all $\phi \in \mathrm{Aut}(\mathbb{B}^n)$, the holomorphic automorphism of $\mathbb{B}^n$.

Here $\Lambda_\phi(f) = [D\phi(0)]^{-1}[Df(\phi(0))]^{-1}[f(\phi(z)) - f(\phi(0))]$ denotes the *Koebe transform* of f (cf. [26, 27]) and thus, the classical definition of the order α of $\mathcal{LIF}$ introduced in the beginning is generalized as follows:

Definition 1.1. *If $\mathcal{M}$ is a $\mathcal{LIF}$, then the norm order of $\mathcal{M}$ is the quantity*

$$\|\mathrm{ord}\|_{\mathcal{M}} = \sup\left\{\frac{1}{2}\|D^2 f(0)\| : f \in \mathcal{M}\right\} = \alpha.$$

In [26, Theorem 3.1], it has been shown that $\alpha \geq 1$. As in the planar case, the universal linearly-invariant family $\mathcal{M}_\alpha$ of order α is defined as the union of all linearly invariant families of order less than or equal to α (cf. [28]).

Our main aim of this paper is to examine the higher dimensional generalizations of certain results from the classical function theory in the complex plane and in particular, we extend the corresponding results of [32] and [33] to higher dimensional case.

2. Main Results

Let $\mathcal{PH}(\alpha, k)$ denote the set of all sense-preserving mappings $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ with the normalization $h(0) = g(0) = 0$, $\|Dh(0) + \overline{Dg(0)}\| = 1$, $[Dh(0)]^{-1}h(z) \in \mathcal{M}_\alpha$, and such that for $k \in [0, 1)$,

$$\left\|Dg(z)[Dh(z)]^{-1}\right\| \leq k,$$

where h is locally biholomorphic and g is holomorphic in $\mathbb{B}^n$.

Obviously, if $n = 1$, then $\mathcal{PH}(\alpha, k)$ coincides with the set $H(\alpha, K)$ of [32] and [33]. As a generalization of [32, Theorem 1], we have the following.

Theorem 2.1. *For $\alpha < \infty$, the classes $\mathcal{PH}(\alpha, k)$ are compact with respect to the topology of almost uniform convergence in $\mathbb{B}^n$.*

The derivative of $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ in the direction of vector $\theta \in \partial\mathbb{B}^n$ at the point z will be denoted by

$$\partial_\theta f(z) = \lim_{\rho \to 0+} \frac{f(z + \rho\theta) - f(z)}{\rho} = Dh(z)\theta + \overline{Dg(z)\theta},$$

where h and g are holomorphic in $\mathbb{B}^n$. We use the standard notations:

$$\Lambda_f = \max_{\theta \in \partial\mathbb{B}^n} \left\|\partial_\theta f\right\| \quad \text{and} \quad \lambda_f = \min_{\theta \in \partial\mathbb{B}^n} \left\|\partial_\theta f\right\|.$$

With this setting, we now present a generalization of [32, Theorem 2].

Theorem 2.2. *For $\alpha < \infty$, let $f = h + \overline{g} \in \mathcal{PH}(\alpha, k)$. Then*

$$\frac{1-k}{\left\|[Dh(0)]^{-1}\right\|}\frac{(1-\|z\|)^{\alpha-1}}{(1+\|z\|)^{\alpha+1}} \leq \Lambda_f(z) \leq \left(\frac{1+k}{1-k}\right)\frac{(1+\|z\|)^{\alpha-1}}{(1-\|z\|)^{\alpha+1}} \tag{4}$$

and

$$\|f(z)\| \leq \frac{1+k}{2\alpha(1-k)}\left\{\frac{(1+\|z\|)^{\alpha}}{(1-\|z\|)^{\alpha}} - 1\right\}. \tag{5}$$

In particular, if $n = 1$, then the estimate of (4) is sharp. Moreover, if $z = re^{it}$, then the equality on the right of (4) is obtained for $f(z) = h(z) - k\overline{h(z)}$, where

$$h(z) = \frac{e^{it}}{2\alpha(1-k)}\left[\left(\frac{1+ze^{-it}}{1-ze^{-it}}\right)^{\alpha} - 1\right]$$

and the equality on the left of (4) is obtained for $f(z) = h^*(z) + k\overline{h^*(z)}$, where

$$h^*(z) = \frac{e^{it}}{2\alpha(1+k)}\left[\left(\frac{1-ze^{-it}}{1+ze^{-it}}\right)^{\alpha} - 1\right].$$

The following result is a covering theorem of $\mathcal{PH}(\alpha, k)$.

Theorem 2.3. *For $r \in (0, 1]$ and $\alpha < \infty$, if $f = h + \overline{g} \in \mathcal{PH}(\alpha, k)$, then $f(\mathbb{B}^n(r))$ contains a univalent ball $\mathbb{B}^n(R)$ with*

$$R \geq \frac{(1-k)|\det Dh(0)|}{\|Dh(0)\|^{n-1}}\int_0^r \frac{(1-x)^{(2n-1)\alpha+(n-3)/2}}{(1+x)^{(2n-1)\alpha-(n-3)/2}}\,dx.$$

In particular, if $n = 1$, then $R = (1-k)\left[1 - \left(\frac{1-r}{1+r}\right)^{\alpha}\right]/[2\alpha(1+k)]$, and the extreme function $f = h + k\overline{h}$ shows that this estimate is sharp, where

$$h(z) = \frac{\pm i}{2\alpha(1+k)}\left[\left(\frac{1 \pm iz}{1 \mp iz}\right) - 1\right].$$

We remark that Theorem 2.3 is a generalization of [32, Theorem 3].

Theorem 2.4. *For $\alpha < \infty$, if $f = h + \overline{g} \in \mathcal{PH}(\alpha, k)$, then*

$$|\det J_f(z)| \geq \frac{(1-k^2)^n}{\left(\det[Dh(0)]^{-1}\right)^2}\frac{\left(1-\|z\|\right)^{2n\alpha-n-1}}{\left(1+\|z\|\right)^{2n\alpha+n+1}}.$$

For $r \in (0, 1)$, a univalent mapping $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ with $h(0) = g(0) = 0$, $Dg(0) = 0$ and

$$\left\|Dg[Dh]^{-1}\right\| < 1$$

is called *fully starlike* if it maps every ball $\overline{\mathbb{B}^n(r)}$ onto a starlike domain with respect to the origin, where h is locally biholomorphic and g is holomorphic in $\mathbb{B}^n$ (cf. [13]). The following result is a generalization of [8, Theorem 1.3].

Theorem 2.5. *Let $r \in (0, 1)$ and $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ be fully starlike, where h is locally biholomorphic and g is holomorphic in $\mathbb{B}^n$. Then for all $z \in \overline{\mathbb{B}^n(r)}$,*

$$\|h(z)\| \leq \frac{1}{1-r}\|f(z)\|.$$

Furthermore, if $h \in \mathcal{M}_{\alpha}$, then

(a) *for* $z \in \mathbb{B}^n(r_0)$,

$$\|f(z)\| \geq r_0^2(1-r_0)\frac{\|z\|}{(r_0+\|z\|)^2},$$

where $r_0 = 4\alpha/(1+4\alpha^2)$;

(b) f *differs from zero in* $\mathbb{B}^n(r_0)\backslash\{0\}$.

We remark that

$$\frac{4\alpha}{1+4\alpha^2} = \frac{1}{\alpha} - \frac{1}{\alpha(1+4\alpha^2)} \sim \frac{1}{\alpha}$$

as $\alpha \to \infty$. Hence Theorem 2.5(b) is a generalization of [33, Theorem 1].

Definition 2.6. *A holomorphic mapping* f *of* $\mathbb{B}^n$ *into* $\mathbb{C}^n$ *is said to be normalized if* $f(0) = 0$ *and* $J_f(0) = I_n$. *A normalized holomorphic mapping* f *is said to be convex (resp. starlike) if it maps* $\mathbb{B}^n$ *univalently onto a region which is convex (resp. starlike with respect to the origin)*

If f is a convex holomorphic mapping, then for each $z \in \mathbb{B}^n$, we have

$$\frac{\|z\|}{1+\|z\|} \leq \|f(z)\| \leq \frac{\|z\|}{1-\|z\|}$$

and the estimates are sharp. See [18, Theorem 7.2.2]. Moreover, if f is a starlike holomorphic mapping, then for each $z \in \mathbb{B}^n$, then the above inequalities takes the form

$$\frac{\|z\|}{(1+\|z\|)^2} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^2}$$

and the estimates are sharp. See [1] and [18, Theorem 7.1.1].

As with the above definition, we may now introduce

Definition 2.7. *Suppose that* $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ *is univalent with* $h(0) = g(0) = 0$, $Dh(0) = I_n$, $Dg(0) = 0$ *and*

$$\left\|Dg[Dh]^{-1}\right\| < 1.$$

Then it is called convex (resp. starlike) if it maps $\mathbb{B}^n$ *onto a domain which is convex (resp. starlike with respect to the origin), where* h *is locally biholomorphic and* g *is holomorphic in* $\mathbb{B}^n$.

In view of the above results for the holomorphic case, it is natural to ask for analog theorems for the case of pluriharmonic mappings. Thus we raise the following.

Problem 2.8. *What is the sharp distortion theorem for convex (resp. starlike) pluriharmonic mappings?*

It is worth to remark that in the one dimensional case of Problem 2.8 for convex mappings, one has for $z \in \mathbb{D}$,

$$|f(z)| \leq \frac{|z|}{(1-|z|)^2}$$

and the estimate is sharp as the extreme function $f_0 = h_0 + \overline{g_0}$ demonstrates, where

$$h_0(z) = \frac{z - z^2/2}{(1-z)^2} \ \text{ and } \ g_0(z) = -\frac{z^2/2}{(1-z)^2}.$$

Again, we remark that in the one dimensional case of Problem 2.8 for starlike pluriharmonic mappings, one has for $z \in \mathbb{D}$,

$$|f(z)| \leq \frac{|z| + |z|^3/3}{(1-|z|)^3}$$

and the estimate is sharp as the extreme function

$$f_1(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \overline{\left(\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}\right)}$$

shows.

A continuous mapping $f:\ \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is called *K-quasiregular* if $f \in W^1_{n,\mathrm{loc}}(\Omega)$ and

$$\|Df(x)\|^n \leq K \det J_f(x) \text{ for almost every } x \in \Omega,$$

where $K\ (\geq 1)$ is a constant. Here $f \in W^1_{n,\mathrm{loc}}(\Omega)$ means that the distributional derivatives $\partial f_j/\partial x_k$ of the coordinates f_j of f are locally in $L^n(\Omega)$ and $J_f(x)$ denotes the Jacobian of f (cf. [35]).

Let $f = (f_1\ \cdots\ f_n)^T \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$. For $j \in \{1, \ldots, n\}$, we let $z = (z_1\ \cdots\ z_n)^T$, $z_j = x_j + iy_j$ and $f_j(z) = u_j(z) + iv_j(z)$, where u_j and v_j are real pluriharmonic functions from $\mathbb{B}^n$ into $\mathbb{R}$. We denote the real Jacobian matrix of f by

$$J_f = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial y_2} & \cdots & \frac{\partial u_1}{\partial x_n} & \frac{\partial u_1}{\partial y_n} \\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial y_2} & \cdots & \frac{\partial v_1}{\partial x_n} & \frac{\partial v_1}{\partial y_n} \\ & & & \vdots & & & \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial y_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial y_2} & \cdots & \frac{\partial u_n}{\partial x_n} & \frac{\partial u_n}{\partial y_n} \\ \frac{\partial v_n}{\partial x_1} & \frac{\partial v_n}{\partial y_1} & \frac{\partial v_n}{\partial x_2} & \frac{\partial v_n}{\partial y_2} & \cdots & \frac{\partial v_n}{\partial x_n} & \frac{\partial v_n}{\partial y_n} \end{pmatrix}.$$

Let $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$, where h and g are holomorphic in $\mathbb{B}^n$. In the following, we investigate the Bloch type Theorem and the quasiregular relationship between f and h. On the related discussions, see [2, 5, 6, 20, 24].

Theorem 2.9. *Let $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ with $\left\|Dg(z)[Dh(z)]^{-1}\right\| \leq c < 1$ for $z \in \mathbb{B}^n$, where c is a positive constant. Then*

(a) *f is a quasiregular mapping if and only if h is a quasiregular mapping;*
(b) *for $n \geq 2$, $f(\mathbb{B}^n)$ contains a univalent ball with the radius*

$$R \geq \frac{k_n \pi}{8m}\left(\frac{k_n \pi \sqrt{1-c}}{4K\sqrt{1+c}\log(1/(1-k_n))}\right)^{4n-1},$$

where $m \approx 4.2$ is the minimum of the function $(2-r^2)/(r(1-r^2))$ for $0 \leq r \leq 1$, $\det J_f(0) = 1$, h is a K-quasiregular mapping with $K \geq 1$ and $0 < k_n < 1$ is a unique root such that

$$-4n\log(1-k_n) = (4n-1)\frac{k_n}{1-k_n}. \tag{6}$$

The roots k_n in $(0, 1)$ of the equation (6) for the values of $n = 2, 3, 4, 5$ are listed in Table 1 for a ready reference.

The proofs of Theorems 2.1–2.9 will be presented in Section 3.

Value of n	Value of k_n
1	0.423166
2	0.230006
3	0.157659
4	0.119898
5	0.0967215

Table 1: Values of k_n in Equation (6) for $n = 1, 2, 3, 4, 5$

3. Proofs of the Main Theorems

Proof of Theorem 2.1

Consider a sequence $f_m = h_m + \overline{g}_m \in \mathcal{PH}(\alpha, k)$. By definition, we have the conditions $\|Dh_m(0) + \overline{Dg_m(0)}\| = 1$ and $\left\|Dg_m(z)[Dh_m(z)]^{-1}\right\| \leq k$, we see that

$$\|Dh_m(0)\| \leq 1 + \|Dg_m(0)\|$$

whereas the second condition gives

$$\|Dg_m(0)\| = \left\|Dg_m(0)[Dh_m(0)]^{-1}[Dh_m(0)]\right\| \leq k\|Dh_m(0)\|.$$

Using the last two inequalities, we easily have

$$\|Dg_m(0)\| \leq \frac{k}{1-k} \text{ and } \|Dh_m(0)\| \leq \frac{1}{1-k}. \tag{7}$$

By (7), $[Dh_m(0)]^{-1}h_m(z) \in \mathcal{M}_\alpha$ and thus by [26, Theorem 4.1], we obtain that

$$\frac{(1-\|z\|)^{\alpha-1}}{(1+\|z\|)^{\alpha+1}} \leq \left\|[Dh_m(0)]^{-1}Dh_m(z)\right\| \leq \frac{(1+\|z\|)^{\alpha-1}}{(1-\|z\|)^{\alpha+1}}, \tag{8}$$

which implies

$$\begin{aligned} \|[Dh_m(z)]\| &= \left\|Dh_m(0)[Dh_m(0)]^{-1}Dh_m(z)\right\| \\ &\leq \left\|[Dh_m(0)]^{-1}Dh_m(z)\right\| \|Dh_m(0)\| \\ &\leq \frac{1}{(1-k)}\frac{(1+\|z\|)^{\alpha-1}}{(1-\|z\|)^{\alpha+1}}. \end{aligned}$$

Moreover, by the definition of $\mathcal{PH}(\alpha, k)$, it follows that

$$\|Dg_m(z)\| \leq k\|Dh_m(z)\| \leq \frac{k}{(1-k)}\frac{(1+\|z\|)^{\alpha-1}}{(1-\|z\|)^{\alpha+1}}.$$

Hence $Dh_m(z)$ and $Dg_m(z)$ are uniformly bounded on compact subsets of $\mathbb{B}^n$, which implies $\mathcal{PH}(\alpha, k)$ are compact. □

Proof of Theorem 2.2

Let $f = h + \overline{g} \in \mathcal{PH}(\alpha, k)$ for some $\alpha < \infty$. By the definition of directional derivatives, we have

$$\begin{aligned} \left\|\partial_\theta f(z)\right\| &= \left\|Dh(z)\theta + \overline{Dg(z)[Dh(z)]^{-1}Dh(z)\theta}\right\| \\ &\geq \|Dh(z)\theta\|\left(1 - \left\|Dg(z)[Dh(z)]^{-1}\right\|\right) \\ &\geq (1-k)\|Dh(z)\theta\| \end{aligned}$$

and similarly,

$$\begin{aligned}\left\|\partial_\theta f(z)\right\| &\leq \|Dh(z)\theta\|\left(1+\left\|Dg(z)[Dh(z)]^{-1}\right\|\right)\\ &\leq (1+k)\|Dh(z)\theta\|.\end{aligned}$$

It follows that

$$(1-k)\|Dh(z)\| \leq \Lambda_f(z) = \max_{\theta\in\partial\mathbb{B}^n}\left\|\partial_\theta f(z)\right\| \leq (1+k)\|Dh(z)\|. \tag{9}$$

Again, by elementary calculations, we have

$$\|Dh(z)\| = \left\|Dh(0)[Dh(0)]^{-1}Dh(z)\right\| \leq \left\|[Dh(0)]^{-1}Dh(z)\right\| \|Dh(0)\|,$$

which gives

$$\frac{\|Dh(z)\|}{\|Dh(0)\|} \leq \left\|[Dh(0)]^{-1}Dh(z)\right\| \leq \|Dh(z)\| \left\|[Dh(0)]^{-1}\right\|. \tag{10}$$

By $[Dh(0)]^{-1}h(z) \in \mathcal{M}_\alpha$ and [26, Theorem 4.1], we deduce that

$$\frac{(1-\|z\|)^{\alpha-1}}{(1+\|z\|)^{\alpha+1}} \leq \left\|[Dh(0)]^{-1}Dh(z)\right\| \leq \frac{(1+\|z\|)^{\alpha-1}}{(1-\|z\|)^{\alpha+1}}. \tag{11}$$

By (10) and (11), we get

$$\frac{1}{\left\|[Dh(0)]^{-1}\right\|}\frac{(1-\|z\|)^{\alpha-1}}{(1+\|z\|)^{\alpha+1}} \leq \|Dh(z)\| \leq \frac{(1+\|z\|)^{\alpha-1}}{(1-\|z\|)^{\alpha+1}}\|Dh(0)\|, \tag{12}$$

which implies

$$\frac{1-k}{\left\|[Dh(0)]^{-1}\right\|}\frac{(1-\|z\|)^{\alpha-1}}{(1+\|z\|)^{\alpha+1}} \leq \Lambda_f(z) \leq \frac{(1+\|z\|)^{\alpha-1}}{(1-\|z\|)^{\alpha+1}}\|Dh(0)\|(1+k). \tag{13}$$

Applying (13) and the inequality,

$$\frac{1}{1+k} \leq \|Dh(0)\| \leq \frac{1}{1-k}, \tag{14}$$

we conclude that

$$\frac{1-k}{\left\|[Dh(0)]^{-1}\right\|}\frac{(1-\|z\|)^{\alpha-1}}{(1+\|z\|)^{\alpha+1}} \leq \Lambda_f(z) \leq \frac{1+k}{(1-k)}\frac{(1+\|z\|)^{\alpha-1}}{(1-\|z\|)^{\alpha+1}}. \tag{15}$$

Now we prove (5). Let $[0,z]$ be the segment from 0 to $z \in \mathbb{B}^n$. Then by using (15), we have

$$\begin{aligned}\|f(z)\| &= \left\|\int_{[0,z]} df(\zeta)\right\| = \left\|\int_{[0,z]} Dh(\zeta)\,d\zeta + \overline{Dg(\zeta)\,d\zeta}\right\|\\ &\leq \int_{[0,z]} \Lambda_f(\zeta)\|d\zeta\|\\ &= \frac{1+k}{1-k}\int_0^1 \frac{(1+t\|z\|)^{\alpha-1}}{(1-t\|z\|)^{\alpha+1}}\|z\|\,dt\\ &= \frac{1+k}{2\alpha(1-k)}\left\{\frac{(1+\|z\|)^{\alpha}}{(1-\|z\|)^{\alpha}}-1\right\}.\end{aligned}$$

The proof of the theorem is complete. □

Lemma 3.1. ([23, Lemma 4]) *Let A be an $n\times n$ complex* (*real*) *matrix with $\|A\| \neq 0$. Then for all unit vector $\theta \in \partial\mathbb{B}^n$, the inequality*

$$\|A\theta\| \geq \frac{|\det A|}{\|A\|^{n-1}}$$

holds.

Proof of Theorem 2.3

Let ρ be the radius of the largest univalence ball of center 0 and contained in $f(\mathbb{B}^n(r))$. Then we have $\|f(z_0)\| = \rho$ for some z_0 with $\|z_0\| = r$. Let $[0, f(z_0)]$ denote the segment from 0 to $f(z_0)$ and γ be a curve joining 0 and z_0 in $\mathbb{B}^n(r)$, which is the preimage of $[0, f(z_0)]$ for the mapping f. We use $\gamma(t)$ to denote a smooth parametrization of γ with $\gamma(0) = 0$ and $\gamma(1) = z_0$, where $t \in [0, 1]$.

Applying [26, Theorem 4.1 (4.2)] and Lemma 3.1, we get

$$\begin{aligned}
\left\|\partial_\theta f(z)\right\| &= \left\|Dh(z)\theta + \overline{Dg(z)[Dh(z)]^{-1}Dh(z)\theta}\right\| \\
&\geq \|Dh(z)\theta\|\left(1 - \left\|Dg(z)[Dh(z)]^{-1}\right\|\right) \\
&\geq (1-k)\|Dh(z)\theta\| \\
&= (1-k)\left\|Dh(0)\frac{[Dh(0)]^{-1}Dh(z)\theta}{\left\|[Dh(0)]^{-1}Dh(z)\theta\right\|}\right\|\left\|[Dh(0)]^{-1}Dh(z)\theta\right\| \\
&\geq (1-k)\frac{(1-\|z\|)^{(2n-1)\alpha+(n-3)/2}}{(1+\|z\|)^{(2n-1)\alpha-(n-3)/2}}\min_{\xi\in\mathbb{B}^n}\|Dh(0)\xi\|
\end{aligned}$$

which implies that

$$\begin{aligned}
\rho &= |f(z_0)| = \left\|\int_0^1 \frac{d}{dt} f\big(\gamma(t)\big)dt\right\| \\
&= \int_0^1 \left\|\frac{d}{dt} f\big(\gamma(t)\big)\right\| dt = \int_0^1 \left\|\partial_\theta f\big(\gamma(t)\big)\right\| |\gamma'(t)|\, dt \\
&\geq (1-k)\min_{\theta\in\mathbb{B}^n}\|Dh(\gamma(0))\theta\| \int_0^1 \frac{(1-\|\gamma(t)\|)^{(2n-1)\alpha+(n-3)/2}}{(1+\|\gamma(t)\|)^{(2n-1)\alpha-(n-3)/2}}\| d\gamma(t)\| \\
&\geq (1-k)\min_{\theta\in\mathbb{B}^n}\|Dh(0)\theta\| \int_0^r \frac{(1-\|z\|)^{(2n-1)\alpha+(n-3)/2}}{(1+\|z\|)^{(2n-1)\alpha-(n-3)/2}} d\|z\| \\
&\geq \frac{(1-k)|\det Dh(0)|}{\|Dh(0)\|^{n-1}} \int_0^r \frac{(1-\|z\|)^{(2n-1)\alpha+(n-3)/2}}{(1+\|z\|)^{(2n-1)\alpha-(n-3)/2}} d\|z\|,
\end{aligned}$$

where $\gamma'(t) = |\gamma'(t)|\theta$.

In particular, if $n = 1$, then

$$\begin{aligned}
\rho &\geq (1-k)\min_{\xi\in\mathbb{B}^n}\|Dh(0)\xi\| \int_0^r \frac{(1-\|z\|)^{(2n-1)\alpha+(n-3)/2}}{(1+\|z\|)^{(2n-1)\alpha-(n-3)/2}} d\|z\| \\
&\geq \frac{1-k}{1+k}\int_0^r \frac{(1-x)^{\alpha-1}}{(1+x)^{\alpha+1}}dx \\
&= \frac{1-k}{2\alpha(1+k)}\left[1 - \left(\frac{1-r}{1+r}\right)^\alpha\right].
\end{aligned}$$

The proof of the theorem is complete. □

Lemma 3.2. *Suppose that $A = (a_{ij})$ is an $n \times n$ matrix. Then*

$$\left(\min_{\theta\in\partial\mathbb{B}^n}\|A\theta\|\right)^n \leq |\det A| \leq \|A\|^n.$$

Proof. If $A^* = (\overline{a_{ji}})$, then the product A^*A is a positive semi-definite matrix. Let $\lambda_1, \dots, \lambda_n$ $(0 \leq \lambda_1 \leq \cdots \leq \lambda_n)$ be the n eigenvalues of the matrix A^*A. Then

$$\sqrt{\lambda_n} = \max\{\|A\theta\| : \theta \in \partial\mathbb{B}^n\} \text{ and } \sqrt{\lambda_1} = \min\{\|A\theta\| : \theta \in \partial\mathbb{B}^n\},$$

which implies that

$$\|A\|^n \geq |\det A| = \sqrt{\Pi_{k=1}^n \lambda_k} \geq \left(\sqrt{\lambda_1}\right)^n = \left(\min_{\theta\in\partial\mathbb{B}^n} \|A\theta\|\right)^n.$$

The proof of the lemma is complete. □

Proof of Theorem 2.4

In view of Lemma 3.2 and [25, Theorem 5.1], J_f given by (1) shows that

$$\begin{aligned}
|\det J_f(z)| &= |\det Dh(z)|^2 \det\left(I_n - Dg(z)[Dh(z)]^{-1}\overline{Dg(z)[Dh(z)]^{-1}}\right)\\
&\geq |\det Dh(z)|^2 \min_{\theta\in\partial\mathbb{B}^n} \left\|\left(I_n - Dg(z)[Dh(z)]^{-1}\overline{Dg(z)[Dh(z)]^{-1}}\right)\theta\right\|^n\\
&\geq |\det Dh(z)|^2 \left(1 - \left\|Dg(z)[Dh(z)]^{-1}\right\|^2\right)^n\\
&\geq |\det Dh(z)|^2(1-k^2)^n\\
&= \frac{\left|\det\left([Dh(0)]^{-1}Dh(z)\right)\right|^2 (1-k^2)^n}{(\det[Dh(0)]^{-1})^2}\\
&\geq \frac{(1-k^2)^n}{\left(\det[Dh(0)]^{-1}\right)^2} \frac{\left(1-\|z\|\right)^{2n\alpha-n-1}}{\left(1+\|z\|\right)^{2n\alpha+n+1}}.
\end{aligned}$$

The proof of the theorem is complete. □

Proof of Theorem 2.5

By the inverse mapping theorem and the assumptions of Theorem 2.5, one obtains that f^{-1} is differentiable. Let $f^{-1} = (\sigma_1 \cdots \sigma_n)^T$. Then for $j, m \in \{1, \ldots, n\}$, we use Df^{-1} and $\overline{D}f^{-1}$ to denote the two $n\times n$ matrices $\left(\partial\sigma_j/\partial z_m\right)_{n\times n}$ and $\left(\partial\sigma_j/\partial\overline{z}_m\right)_{n\times n}$, respectively.

Differentiation of the equation $f^{-1}(f(z)) = z$ yields the following relations

$$\begin{cases} Df^{-1}Dh + \overline{D}f^{-1}Dg = I_n,\\ Df^{-1}\overline{Dg} + \overline{D}f^{-1}\overline{Dh} = 0,\end{cases}$$

which give

$$\begin{cases} DhDf^{-1} = \left(I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\right)^{-1},\\ Dh\overline{D}f^{-1} = -\left(I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\right)^{-1}\overline{Dg}[\overline{Dh}]^{-1}.\end{cases} \tag{16}$$

By (16), we get

$$\begin{aligned}
\|DhDf^{-1}\| + \|Dh\overline{D}f^{-1}\| &= \left\|\left(I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\right)^{-1}\right\| + \left\|\left(I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\right)^{-1}\overline{Dg}[\overline{Dh}]^{-1}\right\|\\
&\leq \left\|\left(I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\right)^{-1}\right\|\left(1 + \|Dg[Dh]^{-1}\|\right)\\
&\leq \frac{1 + \|Dg[Dh]^{-1}\|}{1 - \left\|\overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\right\|}\\
&\leq \frac{1 + \|Dg[Dh]^{-1}\|}{1 - \|Dg[Dh]^{-1}\|^2} = \frac{1}{1 - \|Dg[Dh]^{-1}\|}.
\end{aligned} \tag{17}$$

Since $\Omega = f(\overline{\mathbb{B}^n(r)})$ is starlike (by assumption), for each point $z_0 \in \overline{\mathbb{B}^n(r)}$ and $t \in [0,1]$, we have $\varphi(t) = tf(z_0) \in \Omega$, where $f = (f_1 \cdots f_n)^T$. Let $\gamma = f^{-1} \circ \varphi$. For any fixed $\theta \in \partial\mathbb{B}^n$, let $A_\theta = Dg[Dh]^{-1}\theta$. By Schwarz's lemma, for $z \in \mathbb{B}^n(r)$, $\|A_\theta(z)\| \leq \|z\|$ if $r \in (0,1)$. The arbitrariness of $\theta \in \partial\mathbb{B}^n$ gives

$$\|Dg(z)[Dh(z)]^{-1}\| \leq \|z\| \leq r \tag{18}$$

for $z \in \mathbb{B}^n(r)$. As before, by (17) and (18), we obtain that

$$\begin{aligned}
\|h(z_0)\| &= \left\|\int_0^1 Dh(\gamma(t))\,\frac{d}{dt}\gamma(t)\,dt\right\| \\
&= \left\|\int_0^1 Dh(\gamma(t))\left[Df^{-1}(\varphi(t))D\varphi(t) + \overline{D}f^{-1}(\varphi(t))\overline{D\varphi(t)}\right]dt\right\| \\
&\leq \int_0^1 \Big(\|Dh(\gamma(t))Df^{-1}(\varphi(t))\| + \|Dh(\gamma(t))\overline{D}f^{-1}(\varphi(t))\|\Big)\|D\varphi(t)\|\,dt \\
&\leq \|f(z_0)\|\int_0^1 (1 + \|Dg(\gamma(t))[Dh(\gamma(t))]^{-1}\|)\left\|I_n - \overline{Dg(\gamma(t))}[\overline{Dh(\gamma(t))}]^{-1}Dg(\gamma(t))[Dh(\gamma(t))]^{-1}\right\|dt \\
&\leq \|f(z_0)\|\int_0^1 \frac{1 + \|Dg(\gamma(t))[Dh(\gamma(t))]^{-1}\|}{1 - \left\|\overline{Dg(\gamma(t))}[\overline{Dh(\gamma(t))}]^{-1}Dg(\gamma(t))[Dh(\gamma(t))]^{-1}\right\|}\,dt \\
&\leq \|f(z_0)\|\int_0^1 \frac{1}{1 - \left\|Dg(\gamma(t))[Dh(\gamma(t))]^{-1}\right\|}\,dt \\
&\leq \frac{1}{1-r}\|f(z_0)\|,
\end{aligned}$$

where

$$D\varphi(t) = \begin{pmatrix} f_1(z_0) & 0 & 0 & \cdots & 0 \\ 0 & f_2(z_0) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & f_{n-1}(z_0) & 0 \\ 0 & 0 & \cdots & 0 & f_n(z_0) \end{pmatrix}$$

is a diagonal matrix.

Now we prove the remaining parts of Theorem 2.5. By [26, Theorem 5.7], we know that $h(\mathbb{B}^n(r_0))$ is starlike. For $\zeta \in \mathbb{B}^n$, let $H(\zeta) = h(r_0\zeta)/r_0$. Applying [1, Theorem 2.1] to H, we know that for $\zeta \in \mathbb{B}^n$,

$$\|H(\zeta)\| \geq \frac{\|\zeta\|}{(1 + \|\zeta\|)^2},$$

which implies for $z \in \mathbb{B}^n(r_0)$,

$$\|h(z)\| \geq \frac{r_0^2\|z\|}{(r_0 + \|z\|)^2}. \tag{19}$$

Then Theorem 2.5(a) follows from (19), and Theorem 2.5(b) easily follows from Theorem 2.5(a). The proof of the theorem is complete. □

Proof of Theorem 2.9

We first prove the sufficiency of part (a). Without loss of generality, we assume that

$$\|Dh(z)\| \leq K|\det Dh(z)|^{\frac{1}{n}} \quad \text{for } z \in \mathbb{B}^n, \tag{20}$$

where $K \geq 1$ is a constant.

As in the proof of Theorem 2.4, (20) and Lemma 3.2, for $z \in \mathbb{B}^n$, we have

$$|\det J_f(z)| \quad \geq \quad |\det Dh(z)|^2(1-c^2)^n$$

so that

$$|\det Dh(z)|^{\frac{1}{n}} \leq \frac{|\det J_f(z)|^{\frac{1}{2n}}}{\sqrt{1-c^2}}.$$

Moreover,

$$\Lambda_f(z) = \max_{\theta \in \partial \mathbb{B}^{2n}_{\mathbb{R}}} \|J_f(z)\theta\| \leq \|Dh(z)\| \left(1 + \left\|Dg(z)[Dh(z)]^{-1}\right\|\right) \leq \|Dh(z)\|(1+c),$$

which by the last inequality gives that

$$\Lambda_f(z) \leq K\sqrt{\frac{1+c}{1-c}}\,|\det J_f(z)|^{\frac{1}{2n}} \tag{21}$$

and hence, f is a quasiregular mapping. Here $\mathbb{B}^{2n}_{\mathbb{R}}$ represents the unit ball of $\mathbb{R}^{2n}$. Then

$$\Lambda_f = \max_{\theta \in \partial \mathbb{B}^{2n}_{\mathbb{R}}} \|J_f\theta\| \text{ and } \lambda_f = \min_{\theta \in \partial \mathbb{B}^{2n}_{\mathbb{R}}} \|J_f\theta\|.$$

Next we prove the necessity of part (a). We assume that for $z \in \mathbb{B}^n$,

$$\Lambda_f(z) \leq K_1|\det J_f(z)|^{\frac{1}{2n}}, \tag{22}$$

where $K_1 \geq 1$ is a constant.

As in the proof of Theorem 2.4, for $z \in \mathbb{B}^n$, by calculations and Lemma 3.2, we get

$$\begin{aligned}
|\det J_f(z)| &= |\det Dh(z)|^2 \left|\det\left(I_n - Dg(z)[Dh(z)]^{-1}\overline{Dg(z)[Dh(z)]^{-1}}\right)\right| \\
&\leq |\det Dh(z)|^2 \left\|I_n - Dg(z)[Dh(z)]^{-1}\overline{Dg(z)[Dh(z)]^{-1}}\right\|^n \\
&\leq |\det Dh(z)|^2 \left(1+c^2\right)^n
\end{aligned}$$

so that

$$|\det Dh(z)|^{\frac{1}{n}} \geq \frac{|\det J_f(z)|^{\frac{1}{2n}}}{\sqrt{1+c^2}}.$$

Furthermore,

$$\Lambda_f(z) = \max_{\theta \in \partial \mathbb{B}^{2n}_{\mathbb{R}}} \|J_f(z)\theta\| \geq \|Dh(z)\| \left(1 - \left\|Dg(z)[Dh(z)]^{-1}\right\|\right) \geq \|Dh(z)\|(1-c),$$

which, by (22), implies that

$$\|Dh(z)\|(1-c) \leq \Lambda_f(z) \leq K_1|\det J_f(z)|^{\frac{1}{2n}} \leq K_1\sqrt{1+c^2}\,|\det Dh(z)|^{\frac{1}{n}}.$$

Hence

$$\|Dh(z)\| \leq \frac{K_1\sqrt{1+c^2}}{1-c}|\det Dh(z)|^{\frac{1}{n}},$$

which shows that h is a quasiregular mapping.

Now we prove part (b). By (21), we know that f is a pluriharmonic K_2-quasiregular mapping, where $K_2 = K\sqrt{\frac{1+c}{1-c}}$. Applying [4, Theorem 6], we know that $f(\mathbb{B}^n)$ contains a univalent ball with the radius R with

$$R \geq \frac{k_n\pi}{8m}\left(\frac{k_n\pi}{4K_2\log(1/(1-k_n))}\right)^{4n-1},$$

where $m \approx 4.2$ is the minimum of the function $(2-r^2)/(r(1-r^2))$ for $0 \leq r \leq 1$ and $0 < k_n < 1$ is a unique root such that

$$4n \log \frac{1}{1-k_n} = (4n-1)\frac{k_n}{1-k_n}.$$

The proof of the theorem is complete. □

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