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# Compatible Adjacency Relations for Digital Products

**Sang-Eon Han**[a]

[a]*Department of Mathematics Education, Institute of Pure and Applied Mathematics*
*Chonbuk National University, Jeonju-City Jeonbuk, 54896, Republic of Korea*

**Abstract.** The present paper studies compatible adjacency relations for digital products such as a $C$-compatible adjacency (or the $L_C$-property in [21]), an $S$-compatible adjacency in [27] (or the $L_S$-property in [21]), which are used to study product properties of digital images. Furthermore, to study an automorphism group of a Cartesian product of two digital coverings which do not satisfy a radius 2 local isomorphism, which remains open, the paper uses some properties of an ultra regular covering in [24]. By using this approach, we can substantially classify digital products.

## 1. Introduction

Motivated by the strong adjacency of a ordinary graph product in [1], the paper [11] firstly introduced the notion of a digital product with a normal adjacency from the viewpoint of digital topology. The normal adjacency of a digital product contributed to the study of topological properties of a Cartesian product of two digital images [6, 20, 23, 26]. In relation to the study of this topic, several approaches have been used as follows: in case a Cartesian product (or digital product) has a normal adjacency in [11] (or an $S$-compatible adjacency in [27], or the $L_S$-property [23]) or the $L_C$-property in [21], many works including [6, 20, 23, 26] dealt with digital topological properties of digital products by using a digital fundamental group [4], digital coverings [10, 11, 25], an automorphism group of a digital covering [18] and a digital $k$-surface structure [2, 3, 8, 14, 31]. Computing Hyper-crossed complex pairings in digital images was studied in [35]. However, these approaches could not be enough to deal with the issue because the other cases remain open.

Indeed, a digital image $(X, k)$ is naturally considered to be a set $X \subset \mathbf{Z}^n, n \in \mathbf{N}$ with one of the $k$-adjacency relations of $\mathbf{Z}^n$ [37], where $\mathbf{Z}^n$ (resp. $\mathbf{N}$) is the set of points in the Euclidean $n$D space with integer coordinates (resp. the set of natural numbers). It is well known that a Cartesian product of graphs [30] has substantially contributed to the study of a Cartesian product of graphs. Motivated by this approach, for digital products having the $L_C$-property in [21], their digital topological properties were studied under some hypothesis. To be specific, consider a Cartesian product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$, where $SC_{k_i}^{n_i,l_i}$ is a simple closed $k_i$-curve with $l_i$ elements in $\mathbf{Z}^{n_i}, i \in \{1, 2\}$ and further, is not $k_i$-contractible. Then, if the Cartesian product $(SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}, k)$ has the property $L_S$ or $L_C$, its digital $k$-fundamental group $\pi^k(SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}, (c_0, d_0))$ is

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*Email address:* sehan@jbnu.ac.kr (Sang-Eon Han)

proved to be isomorphic to $\pi^{k_1}(SC_{k_1}^{n_1,l_1}, c_0) \times \pi^{k_2}(SC_{k_2}^{n_2,l_2}, d_0)$ [21] in terms of various properties from digital homotopy and digital covering theory. Although the paper [7] studied some properties for digital products, the tools are kinds of the properties of $L_C$ and $L_S$ in [21, 23]. Besides, more generalized cases with the $L_{HS}$- or the $L_{HC}$-property were also studied in [23]. Unlike these properties, we now need to study the other cases which remains open. For instance, given two digital images $(X_i, k_i), i \in \{1, 2\}$, let us consider the Cartesian product $X_1 \times X_2 \subset \mathbf{Z}^{n_1+n_2}$ with a $k$-adjacency. In details, consider the following cases:
(1) it has neither the $L_S$-property (or a normal $k$-adjacency) nor the $L_{HS}$-property; and
(2) it does not have the $L_{HC}$-property.

To address these topics, the present paper, motivated by the Cartesian product adjacency in [30], uses the notion of a $C$-compatible adjacency of a digital product which is different from both a normal $k$-adjacency and the $L_S$-property. Besides, the paper investigates some properties of a $C$-compatible adjacency for studying the product property of digital images.

The present paper is organized as follows: Section 2 provides basic notions on digital topology in a graph-theoretical approach. Section 3 recalls various properties of a $C$-compatible $k$-adjacency relation of a digital product which play important roles in studying digital topological properties of digital products. Besides, we prove that none of an $S$-compatible adjacency and a $C$-compatible $k$-adjacency on $X_1 \times X_2$ implies the other. Section 4 studies the product property of two digital coverings. Section 5 studies an automorphism group of a Cartesian product of digital coverings by using a $C$-compatible adjacency. Finally, Section 6 concludes the paper with a summary.

## 2. Preliminaries

A (binary) digital image $(X, k)$ can be regarded as a subset $X \subset \mathbf{Z}^n$ with one of the $k$-adjacency relations of $\mathbf{Z}^n$ (see (2.2)) below. For $a, b \in \mathbf{Z}$ with $a \precneqq b$, the set $[a, b]_{\mathbf{Z}} = \{n \in \mathbf{Z} \,|\, a \leq n \leq b\}$ is called a digital interval [4]. In this paper we shall use the symbol " :=" in order to introduce new notions without mentioning the fact. Further, let us recall the process of establishing $k$-adjacency relations of $\mathbf{Z}^n$ which is a generalization of $k$-adjacency relations of $\mathbf{Z}^2$ and $\mathbf{Z}^3$ in [37], as follows: let $p := (p_i)_{i \in [1,n]_{\mathbf{Z}}}$ be a point of $\mathbf{Z}^n$ and $m$ an integer in $[1, n]_{\mathbf{Z}}$.

For a natural number $m$, $1 \leq m \leq n$, two distinct points

$$p = (p_1, p_2, \cdots, p_n) \text{ and } q = (q_1, q_2, \cdots, q_n) \in \mathbf{Z}^n,$$

are $k(m, n)$-($k$-, for brevity)adjacent [10, 11] if

$$\text{at most } m \text{ of their coordinates differs by } \pm 1, \text{ and all others coincide.} \tag{2.1}$$

The number of such points is [20] (for more details, see [22])

$$k := k(m, n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \text{ where } C_i^n = \frac{n!}{(n-i)!\, i!}. \tag{2.2}$$

For consistency with the nomenclature "$k$-adjacent", $k \in \{4, 8\}$, well developed in the context of 2D integer grids, we will say that two points $p, q \in \mathbf{Z}^n$ are *$k$-adjacent* if they satisfy the condition (2.1), where $k := k(m, n)$ is suggested in (2.2) [11] (see also [20, 22]).

For instance [20, 22],

$$(m, n, k) \in \left\{ \begin{array}{l} (1,4,8), (2,4,32), (3,4,64), (4,4,80); \\ (1,5,10), (2,5,50), (3,5,130), (4,5,210), (5,5,242). \end{array} \right\} \tag{2.3}$$

Owing to the phrase "*at most m*" in (2.1), it is obvious that the points $p = (p_1, p_2, ..., p_n)$ and $q = (q_1, q_2, ..., q_n) \in \mathbf{Z}^n$ may differ in as many as $m$ coordinates. Thus, in general, we obtain the following: if two distinct points $x, y \in \mathbf{Z}^n$ are $k(m,n)$-adjacent, then they are obviously $k(m',n)$-adjacent [22], where $m \leq m'$. This observation will be often used in Section 5.

By using the $k$-adjacency relations of $\mathbf{Z}^n$ in (2.2), we can study digital topological properties of a set $X \subset \mathbf{Z}^n$ with a $k$-adjacency, $n \in \mathbf{N}$. This has been often used to represent digital continuity, a digital isomorphism, a digital homotopy, a digital $k$-surface structure, etc. Owing to the digital $k$-connectivity paradox in [33], we remind the reader that $k \neq \bar{k}$ except the case $(\mathbf{Z}, 2, 2, X)$. However, in this paper we are not concerned with $\bar{k}$-adjacency between two points in $\mathbf{Z}^n \setminus X$. We say that a set $X \subset \mathbf{Z}^n$ is $k$-connected if it is not a union of two disjoint non-empty subsets of $X$ that are not $k$-adjacent to each other [33]. For $(X, k)$, a point $x \in X$ is called *isolated* if it is not $k$-adjacent with any point in $X$ [33]. For a $k$-adjacency relation of $\mathbf{Z}^n$, a simple $k$-path with $l+1$ elements in $\mathbf{Z}^n$ is assumed to be an injective sequence $(x_i)_{i \in [0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that $x_i$ and $x_j$ are $k$-adjacent if and only if $|\, i - j\,| = 1$ [33]. If $x_0 = x$ and $x_l = y$, then we say that the length of the simple $k$-path is $l$. A simple closed $k$-curve with $l$ elements in $\mathbf{Z}^n, n \geq 2$, denoted by $SC_k^{n,l}$ [11] (see also [14]), is the simple $k$-path $(x_i)_{i \in [0,l-1]_{\mathbf{Z}}}$, where $x_i$ and $x_j$ are $k$-adjacent if and only if $|i - j| = 1 (mod\, l)$ [33]. Besides, for $\mathbf{Z}^n$ we remind the following [33]:

$$\left\{ \begin{array}{l} N_k(x) := \{x' \,|\, x \text{ is } \ k\text{-adjacent to } x' \text{ in } \mathbf{Z}^n\} \text{ and} \\ N_k^*(x) := N_k(x) \cup \{x\}. \end{array} \right\} \tag{2.4}$$

As a generalization of $N_k^*(x)$ in $\mathbf{Z}^n$, for a multi-dimensional digital image $(X, k)$ and a point $x \in X \subset \mathbf{Z}^n$, the notion of a (digital) $k$-neighborhood of a point $x$ with radius $\varepsilon \in \mathbf{N}$ was established [9] (see also [11]), as follows.

$$N_k(x_0, \varepsilon) := \{x \in X|\ l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}, \tag{2.5}$$

where $l_k(x_0, x)$ is the length of a shortest simple $k$-path from $x_0$ to $x$ in $X$. For instance, for $x \in (X, k)$ we observe [19]

$$N_k^*(x) \cap X = N_k(x, 1). \tag{2.6}$$

If a point $x$ in a digital image $(X, k)$ is isolated, then for any $\varepsilon \in \mathbf{N}$ we observe that $N_k(x, \varepsilon)$ is a singleton $\{x\}$. The $k$-neighborhood of (2.6) will be often used to establish a compatible adjacency for a Cartesian product of two digital images (see Section 4) and digital continuity.

The original version of digital continuity was firstly developed in [37]: let $(X, k_0)$ and $(Y, k_1)$ be digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Let $f : (X, k_0) \to (Y, k_1)$ be a function. We say that $f$ is $(k_0, k_1)$-continuous if the image under $f$ of every $k_0$-connected subset of $X$ is $k_1$-connected (see Theorem 2.4 of [38]).

By using the property (2.6), we can represent the notion of digital continuity, as follows:

**Definition 2.1.** ([11]) (see also [19]) Let $(X, k_0)$ and $(Y, k_1)$ be digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function $f : X \to Y$ is $(k_0, k_1)$-continuous if for every $x \in X$, $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.

We have often used the following notion of a $(k_0, k_1)$-*isomorphism* instead of a $(k_0, k_1)$-homeomorphism used in [4]: for two digital images $(X, k_0)$ in $\mathbf{Z}^{n_0}$ and $(Y, k_1)$ in $\mathbf{Z}^{n_1}$, a map $h : X \to Y$ is called a $(k_0, k_1)$-*isomorphism* [12, 36] if $h$ is a $(k_0, k_1)$-continuous bijection and further, $h^{-1} : Y \to X$ is $(k_1, k_0)$-continuous [12] (see also [19]), and we use the notation $X \approx_{(k_0,k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then we call it a $k_0$-*isomorphism*.

## 3. *C*-Compatible Adjacency Relation for a Digital Product

In graph theory two compatible adjacencies of a Cartesian product such as the *normal adjacency* in [1] and the *Cartesian product adjacency* in [30] play important roles in studying graphs. Motivated by the normal adjacency in [1], the paper [11] developed a normal adjacency for a digital product to study a digital fundamental group of a digital product, as follows:

**Definition 3.1.** ([11]) Given two digital images $(X, k_1)$ in $\mathbf{Z}^{n_1}$, $(Y, k_2)$ in $\mathbf{Z}^{n_2}$, consider the digital product $X \times Y \subset \mathbf{Z}^{n_1+n_2}$. Then we say that two points $(x, y) \in X \times Y$, $(x', y') \in X \times Y$ are normally $k$-adjacent to each other if and only if

(1) $x$ is $k_1$-adjacent to $x'$ and $y = y'$; or
(2) $y$ is $k_2$-adjacent to $y'$ and $x = x'$; or
(3) $x$ is $k_1$-adjacent to $x'$ and $y$ is $k_2$-adjacent to $y'$.

As an equivalent version of the normal adjacency of Definition 3.1, we have the following presentation named by an "$S$-compatible adjacency" for a digital product, which can be substantially used in studying a normal adjacency of a digital product in terms of a matric presentation.

**Remark 3.2.** ([27]) Given two digital images $(X, k_1)$ in $\mathbf{Z}^{n_1}$ and $(Y, k_2)$ in $\mathbf{Z}^{n_2}$, consider a Cartesian product $X \times Y \subset \mathbf{Z}^{n_1+n_2}$. We say that a $k$-adjacency of $X \times Y$ is strongly compatible (for brevity, $S$-compatible) with the $k_i$-adjacency, $i \in \{1, 2\}$ if every point $(x, y)$ in $X \times Y$ satisfies the following property: for two distinct points $(x, y)$ and $(x', y')$ in $X \times Y$

$$(x', y') \in N_k((x, y), 1) \Leftrightarrow x' \in N_{k_1}(x, 1), y' \in N_{k_2}(y, 1).$$

The following simple closed 4- and 8-curves in $\mathbf{Z}^2$ [9] and a simple closed 18- and 26-curves in $\mathbf{Z}^3$ [9, 16] will be often used later in the paper, especially in Examples 3.3, 3.5, and 4.8 (see also Figure 1).

$$\left\{\begin{array}{c} SC_4^{2,8} \approx_4 ((0,0),(1,0),(2,0),(2,1),(2,2),(1,2),(0,2),(0,1)), \\ SC_8^{2,6} := MSC_8 \approx_8 ((0,0),(1,1),(1,2),(0,3),(-1,2),(-1,1)), \\ SC_8^{2,4} \approx_8 ((0,0),(1,1),(2,0),(1,-1)), \\ SC_8^{2,8} := ((0,0),(1,1),(2,2),(1,3),(0,4),(-1,3),(-2,2),(-1,1)), \\ MSC_{18} := ((0,0,0),(1,-1,0),(1,-1,1),(2,0,1),(1,1,1),(1,1,0)), \\ SC_{18}^{3,6} := ((0,0,0),(1,0,1),(1,1,2),(0,2,2),(-1,1,2),(-1,0,1)), \\ SC_{26}^{3,4} := ((0,0,0),(1,1,1),(0,2,2),(-1,1,1)), \\ SC_{26}^{3,6} := ((0,0,0),(1,1,1),(1,1,2),(0,2,3),(-1,1,2),(-1,1,1)). \end{array}\right\} \tag{3.1}$$

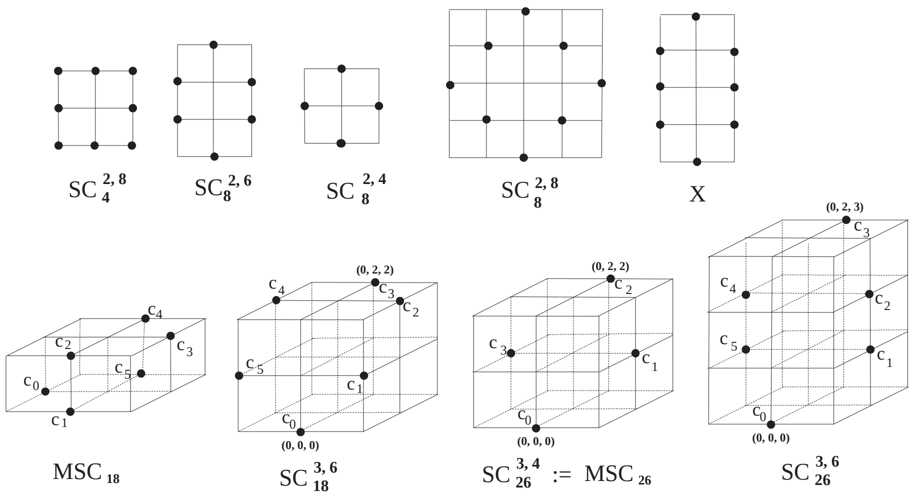


Figure 1: Several kinds of simple closed $k$-curves [9, 17, 22, 23].

To represent a Cartesian product of two digital images as a matrix, we use the notation

$$SC_{k_1}^{n_1,l_1} := (a_i)_{i \in [1,l_1]_{\mathbf{Z}}} \text{ and } SC_{k_2}^{n_2,l_2} := (b_j)_{j \in [1,l_2]_{\mathbf{Z}}} \text{ (see (3.1))}.$$

Then take the Cartesian product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2} \subset \mathbf{Z}^{n_1+n_2}$ which can be represented as the following matrix:

$$SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2} := (c_{i\,j})_{(i,j)\in[1,l_1]_{\mathbf{Z}}\times[1,l_2]_{\mathbf{Z}}},\ \text{for brevity},\ (c_{i\,j}) \tag{3.2}$$

where $c_{i\,j} := (a_i, b_j)$.

In view of Definition 3.1, not every digital product has an $S$-compatible $k$-adjacency, as follows:

**Example 3.3.** ([26, 27]) (1) $([a,b]_{\mathbf{Z}} \times [c,d]_{\mathbf{Z}}, 8)$,
(2) $(SC_8^{2,6} \times [a,b]_{\mathbf{Z}}, 26)$,
(3) $(SC_8^{2,6} \times SC_8^{2,6}, 80)$, $(SC_8^{2,8} \times SC_8^{2,8}, 80)$ and $(SC_8^{2,6} \times SC_8^{2,8}, 80)$,
(4) $(SC_{18}^{3,6} \times SC_8^{2,4}, 242)$, $(SC_8^{2,6} \times SC_{18}^{3,6}, 242)$
(5) None of $MSC_{18} \times MSC_{18}$, $SC_4^{2,8} \times SC_8^{2,6}$, and $SC_4^{2,8} \times SC_4^{2,8}$ has an $S$-compatible $k$-adjacency [7, 21, 23].

Even though a normal adjacency of a digital product were established for studying digital products, there are many digital products which do not have normal adjacencies. Thus we need to propose another new adjacency for a digital product from the viewpoint of digital topology. Motivated by the Cartesian product adjacency in [30], we now develop its digital version for strongly compatible product adjacency. The following is another presentation of the $L_C$-property of digital products in [21].

**Definition 3.4.** For two digital images $(X, k_1)$ in $\mathbf{Z}^{n_1}$ and $(Y, k_2)$ in $\mathbf{Z}^{n_2}$, consider the Cartesian product $X \times Y \subset \mathbf{Z}^{n_1+n_2}$. We say that a $k$-adjacency of $X \times Y$ is strongly Cartesian compatible (for brevity, $C$-compatible) with the given $k_i$-adjacency, $i \in \{1,2\}$ if every point $(x, y)$ in $X \times Y$ satisfies the following property:

$$N_k((x,y),1) = (N_{k_1}(x,1) \times \{y\}) \cup (\{x\} \times N_{k_2}(y,1)).$$

In view of Example 3.3, not every digital product has a $C$-compatible $k$-adjacency, as follows:

**Example 3.5.** ([21]) (1) $([a,b]_{\mathbf{Z}} \times [c,d]_{\mathbf{Z}}, 4)$,
(2) $(SC_4^{2,8} \times [a,b]_{\mathbf{Z}}, 6)$,
(3) $(SC_8^{2,6} \times SC_{26}^{3,4}, 130)$, $(SC_8^{2,4} \times SC_8^{2,6}, 32)$ and $(SC_8^{2,8} \times SC_8^{2,8}, 32)$, and
(4) None of $SC_4^{2,8} \times SC_8^{2,6}$, $SC_8^{2,6} \times SC_8^{2,6}$, and $SC_8^{2,6} \times MSC_{18}$ has a $C$-compatible $k$-adjacency.

By Definition 3.2 and Example 3.5, we obviously obtain the following:

**Remark 3.6.** Consider two digital images $(X_i, k_i)$ in $\mathbf{Z}^{n_i}$, $i \in \{1,2\}$, where $k_i := k(m_i, n_i)$ via (2.1). Assume $k_i = 2n_i$, *i.e.* $m_i = 1, i \in \{1,2\}$. Then the digital product $X_1 \times X_2 \subset \mathbf{Z}^{n_1+n_2}$ has a $C$-compatible $k(1, n_1 + n_2)$-adjacency [21]. For instance, see the case $(SC_4^{2,8} \times SC_4^{2,8}, 8)$ in Example 3.5

**Remark 3.7.** [Merits of a $C$-compatible adjacency of a digital product] Given two digital images $(X_i, k_i), i \in \{1,2\}$, consider a digital product $(X_1 \times X_2, k)$. If the given $k$-adjacency is a $C$-compatible adjacency, then each of the projection maps $p_i : (X_1 \times X_2, k) \to (X_i, k_i), i \in \{1,2\}$ is always a $(k, k_i)$-continuous map (see Corollary 3.11).

According to Remarks 3.6 and 3.7, since the existence of a $C$-compatible adjacency for a digital product depends on the situation, we now propose the following:

**Theorem 3.8.** *Consider* $SC_{k_i}^{n_i,l_i}, i \in \{1,2\}, k_i := k(m_i, n_i)$ *from* (2.2). *Assume* $k_i \neq 2n_i, i \in \{1,2\}$ *and* $m_1 \leq m_2$. *Then we obtain the following cases supporting a C-compatible adjacency for* $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$.

*(Case 1) Consider the case* $m_1 = m_2$ *and* $m_1 \neq n_1$, *i.e.* $k_1 \neq 3^{n_1} - 1$. *For each element* $y_j \in SC_{k_2}^{n_2,l_2} := (y_j)_{j\in[0,l_2-1]_{\mathbf{Z}}}$ *assume the number of different coordinates of every pair of the consecutive points* $y_j$ *and* $y_{j+1(mod\, l_2)}$ *in* $SC_{k_2}^{n_2,l_2}$ *is constant as the number* $m_2$ *instead of "at most* $m_2$*". Then the product* $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ *has a C-compatible* $k(m, n_1+n_2)$*-adjacency, where* $m = m_1$.

*(Case 2) In case $m_1 = n_1$, i.e. $k_1 = 3^{n_1} - 1$, assume that for each element $y_j \in SC_{k_2}^{n_2,l_2} := (y_j)_{j\in[0,l_2-1]_Z}$, the number of different coordinates of every pair of the consecutive points $y_j$ and $y_{j+1(mod\, l_2)}$ is constant as the number $m_2$ instead of "at most $m_2$".*
*Then the product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ has a C-compatible $k(m_2, n_1 + n_2)$-adjacency.*

*(Case 3) In case $m_i = n_i, i \in \{1, 2\}$, i.e. $k_i = 3^{n_i} - 1$ (or $m_i \notin [0, n_i - 1]_Z$), then we can consider two cases:*
*(Case 3-1) Assume that for each element $y_j \in SC_{k_2}^{n_2,l_2} := (y_j)_{j\in[0,l_2-1]_Z}$, the number of different coordinates of every pair of the consecutive points $y_j$ and $y_{j+1(mod\, l_2)}$ is constant as the number $m_2$ instead of "at most $m_2$".*
*Then the product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ has a C-compatible $k(m_2, n_1 + n_2)$-adjacency.*
*(Case 3-2) We assume that for each element $x_i \in SC_{k_1}^{n_1,l_1} := (x_i)_{i\in[0,l_1-1]_Z}$, the number of different coordinates of every consecutive points $x_i$ and $x_{i+1(mod\, l_1)}$ in $SC_{k_1}^{n_1,l_1}$ is constant as the number $n_1$ instead of "at most $n_1$" and further, for each element $y_j \in SC_{k_2}^{n_2,l_2} := (y_j)_{j\in[0,l_2-1]_Z}$, the number of different coordinates of every pair of the consecutive points $y_j$ and $y_{j+1(mod\, l_2)}$ is constant as the number $m_2$ instead of "at most $m_2$".*
*Then the product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ has a C-compatible $k(t, n_1 + n_2)$-adjacency, where $m_2 \leq t \leq m_1 + m_2 - 1$.*

Before proving this theorem, we need to explain the hypotheses relating to the four cases of this theorem such as "the number of different coordinates of every pair of the consecutive points $y_j$ and $y_{j+1(mod\, l_2)}$ in $SC_{k_2}^{n_2,l_2}$ is constant as the number $m_2$" and another relating to the $k_1$-adjacency of $SC_{k_1}^{n_1,l_1}$. In relation to the hypothesis of Case 1, for instance, consider the digital image $MSC_{18} := (c_i)_{i\in[0,5]_Z}$ in Figure 1. Then we observe that it does not have the property that for each element $c_i$ the number of different coordinates of the points $c_i$ and $c_{i+1(mod\, 6)}$ in $MSC_{18}$ is constant as the number 2. To be specific, consider the consecutive two points $c_1 = (1, 0, 0)$ and $c_2 = (1, 0, 1)$ of $MSC_{18}$ in (3.1). Then we observe that the number of different coordinates of these points in $MSC_{18}$ is 1 instead of the number 2. However, consider another space such as $SC_8^{2,8}$ in Figure 1. Then we observe that for each element $x_i \in SC_8^{2,8} := (x_i)_{i\in[0,7]_Z}$ the number of different coordinates of the points $x_i$ and $x_{i+1(mod\, 8)} \in SC_8^{2,8}$ is constant as the number 2. This property is substantially required to make this theorem valid.

*Proof.* (Case 1): Consider the case $m_1 = m_2$, $m_1 \neq n_1$ and given $SC_{k_2}^{n_2,l_2}$ satisfies the hypothesis. Let us now prove that for each point $(x, y) \in SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$, the following property is valid.

$$N_k((x, y), 1) = (N_{k_1}(x, 1) \times \{y\}) \cup (\{x\} \times N_{k_2}(y, 1)),$$

where $k := k(m, n_1 + n_2)$.

Consider the point $(x, y) := (x_i, y_j) := p \in SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$. Then we obviously observe that all of the points $(x_{i-1}, y_j), (x_{i+1}, y_j), (x_i, y_{j-1}), (x_i, y_{j+1})$ and $(x_i, y_j)$ belong to $N_k(p, 1)$. However, since the following points $(x_{i+2}, y_j), (x_{i-2}, y_j), (x_i, y_{j-2}), (x_i, y_{j+2})$ cannot belong to $N_k(p, 1)$, let us only examine if each of the points

$$(x_{i-1}, y_{j-1}), (x_{i-1}, y_{j+1}), (x_{i+1}, y_{j-1}), (x_{i+1}, y_{j+1}) \tag{3.3}$$

belongs to the set $N_k(p, 1)$. First of all, consider the point $(x_{i-1}, y_{j-1})$. Then we see that the number of different coordinates of the following points $(x_{i-1}, y_{j-1})$ and $(x_i, y_j)$ is at least 2, which implies that the point $(x_{i-1}, y_{j-1}) \notin N_k(p, 1)$.

Similarly, we can prove the other points in (3.3) cannot belong to the set $N_k(p, 1)$ either by using the method similar to the proof of $(x_{i-1}, y_{j-1}) \notin N_k(p, 1)$.

(Case 2) For each element $y_j \in SC_{k_2}^{n_2,l_2} := (y_j)_{j\in[0,l_2-1]_Z}$ assume that the number of different coordinates of every pair of the consecutive points $y_j$ and $y_{j+1(mod\, l_2)}$ is constant as the number $m_2$ instead of "at most $m_2$". Then for each point $(x, y) \in SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$, by using the method similar to the proof of Case 1, we obtain that

$$N_k((x, y), 1) = (N_{k_1}(x, 1) \times \{y\}) \cup (\{x\} \times N_{k_2}(y, 1)),$$

where $k := k(m_2, n_1 + n_2)$, which implies that the product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ has a $C$-compatible $k(m_2, n_1 + n_2)$-adjacency.

(Case 3): Before proving this case, we need to explain the hypothesis in Case 4. Let us consider the digital product $SC_8^{2,6} \times SC_{26}^{3,6} \subset \mathbf{Z}^5$. Then we observe that none of $SC_8^{2,6}$ and $SC_{26}^{3,6}$ in Figure 1 satisfies the hypothesis, which implies that the digital product $SC_{18}^{3,6} \times SC_{26}^{3,6}$ could not have any $C$-compatible $k$-adjacency.
(Case 3-1): By using the method similar to the proof of Cases 2, the proof is completed.
(Case 3-2): By using the method similar to the proof of Cases 2 and 3, we obtain the result. □

**Example 3.9.** Consider the digital images $SC_{18}^{3,6}, SC_8^{2,8}, SC_{26}^{3,4}, MSC_{18}$ in Figure 1. Then we obtain the following cases related to the existence of a $C$-compatible adjacency:
(1) $(MSC_{18} \times SC_8^{2,8}, k(2,5) := 50)$, and $(SC_8^{2,6} \times SC_8^{2,8}, 32 := k(2,4))$,
(2) $(SC_8^{2,8} \times SC_{26}^{3,4}, 130 := k(3,5))$,
(3) $(SC_{18}^{3,6} \times SC_{26}^{3,4}, k), k \in \{232 := k(3,6), 472 := k(4,6)\}$,
(4) $(SC_8^{2,4} \times SC_{26}^{3,4}, k), k \in \{130 := k(3,5), 210 := k(4,5)\}$, and
(5) None of $SC_8^{2,6} \times MSC_{18}$, $MSC_{18} \times MSC_{18}$ and $MSC_{18} \times SC_{26}^{3,4}$ has a $C$-compatible adjacency.

Since the concepts of $S$-and $C$-compatible adjacency for a digital product play important roles in studying digital products, let us now compare "$S$-compatible" with "$C$-compatible".

**Theorem 3.10.** *Given two digital images $(X_i, k_i)$ in $\mathbf{Z}^{n_i}, i \in \{1,2\}$, none of an $S$-compatible adjacency and a $C$-compatible $k$-adjacency on $X_1 \times X_2$ implies the other.*

*Proof.* As examples related to the assertion, consider the following two products $SC_4^{2,8} \times SC_4^{2,8}$ and $SC_8^{2,6} \times SC_8^{2,6}$ in Example 3.5. While $SC_4^{2,8} \times SC_4^{2,8}$ has a $C$-compatible 8-adjacency, it has no $S$-compatible $k$-adjacency, $k \in \{8, 32, 64, 80\}$. Besides, while $SC_8^{2,6} \times SC_8^{2,6}$ has an $S$-compatible $80 := k(4,4)$-adjacency, it has no $C$-compatible $k$-adjacency, $k \in \{8, 32, 64, 80\}$. □

By Definitions 2.1 and 3.1, we obtain the following:

**Corollary 3.11.** *Assume a $C$-compatible $k$-adjacency on $X_1 \times X_2$. Then the natural projection map $P_i : X_1 \times X_2 \to X_i$ is a $(k, k_i)$-continuous map, $i \in \{1,2\}$.*

*Proof.* If we take a $C$-compatible $k$-adjacency on $X_1 \times X_2$, then each of the natural projection maps is clearly $(k, k_i)$-continuous, $i \in \{1,2\}$. □

## 4. Digital Product Properties of Digital Coverings

Even though the paper [6] studied product properties of two digital coverings, we need to make it advanced in the present section. In this section, we show that a $C$-compatible $k$-adjacency relation of a digital product can be substantially used to study Cartesian product properties of digital coverings. Hereafter, each digital image $(X, k)$ is assumed to be $k$-connected. Let us recall the notion of a digital covering space in [24] which is an advanced and simplified version of the earlier versions in [6, 10, 11, 18] (see Remark 4.2).

**Definition 4.1.** ([10, 11, 14, 24]) Let $(E, k_0)$ and $(B, k_1)$ be digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Let $p : E \to B$ be a surjection. Suppose that for every $b \in B$ there exists $\varepsilon \in \mathbf{N}$ such that
(DC 1) for some index set $M$, $p^{-1}(N_{k_1}(b, \varepsilon)) = \cup_{i \in M} N_{k_0}(e_i, \varepsilon)$ with $e_i \in p^{-1}(b)$;
(DC 2) if $i, j \in M$ and $i \neq j$, then $N_{k_0}(e_i, \varepsilon) \cap N_{k_0}(e_j, \varepsilon) = \emptyset$; and
(DC 3) the restriction map $p|_{N_{k_0}(e_i,\varepsilon)} : N_{k_0}(e_i, \varepsilon) \to N_{k_1}(b, \varepsilon)$ is a $(k_0, k_1)$-isomorphism for all $i \in M$.
Then the map $p$ is called a $(k_0, k_1)$-covering map, $(E, p, B)$ is said to be a $(k_0, k_1)$-covering, and $(E, k_0)$ is called a $(k_0, k_1)$-covering space over $(B, k_1)$.

In Definition 4.1, $N_{k_1}(b, \varepsilon)$ is called an *elementary $k_1$-neighborhood* of $b$ with radius $\varepsilon$ [11]. The collection $\{N_{k_0}(e_i, \varepsilon) \,|\, i \in M\}$ is a partition of $p^{-1}(N_{k_1}(b, \varepsilon))$ into slices. Furthermore, we may take $\varepsilon = 1$ which is a special case of Definition 4.1 [14]. For pointed digital images $((E, e_0), k_0)$ and $((B, b_0), k_1)$ if $p : (E, e_0) \to (B, b_0)$ is a $(k_0, k_1)$-covering map such that $p(e_0) = b_0$, then the map $p$ is a pointed $(k_0, k_1)$-covering map [11]. Besides, consider the map $p : (\mathbf{Z}, 0) \to (SC_k^{n,l}, c_0)$ with $p(i) = c_{i(mod\, l)}$, where $SC_k^{n,l} := (c_i)_{i\in[0,l-1]_\mathbf{Z}}$. Then $(\mathbf{Z}, p, SC_k^{n,l})$ is a pointed $(2, k)$-covering [11] because we may take $\varepsilon = 1$ in Definition 4.1.

**Remark 4.2.** ([24]) We observe that the digital covering map of Definition 4.1 is more simplified than the versions of [6, 11, 12, 14, 19]. Namely, owing to the property (DC 3), the "$(k_0, k_1)$-continuous surjection" of the earlier versions in [6, 9, 12, 17] is replaced by a "surjection".

**Definition 4.3.** ([11]) We say that a $(k_0, k_1)$-covering map $p : E \to B$ is an *n-fold $(k_0, k_1)$-covering map* if the cardinality of the index set $M$ is $n$.

For instance, consider the map

$$p : SC_{k_0}^{n_0,lm} := ((d_i)_{i\in[0,lm-1]}, d_0) \to SC_{k_1}^{n_1,l} := ((c_j)_{j\in[0,l-1]}, c_0)$$

with $p(d_i) = c_{i(mod\, l)}, m \in \mathbf{N}$. Then $(SC_{k_0}^{n_0,ml}, p, SC_{k_1}^{n_1,l})$ is an $m$-fold $(k_0, k_1)$-covering. This property will be often used in Section 5.

**Definition 4.4.** ([10]) For $n \in \mathbf{N}$, a $(k_0, k_1)$-covering $(E, p, B)$ is a radius $n$ local isomorphism if the restriction map $p|_{N_{k_0}(e_i,n)} : N_{k_0}(e_i, n) \to N_{k_1}(b, n)$ is a $(k_0, k_1)$-isomorphism for all $i \in M$.

**Definition 4.5.** ([19]) A $(k_0, k_1)$-covering $(E, p, B)$ is called a *radius $n$-$(k_0, k_1)$-covering* if $\varepsilon \geq n$.

For instance, while $(\mathbf{Z}, p, SC_8^{2,6})$ is a radius 2-$(2, 8)$-covering, $(\mathbf{Z}, p, SC_8^{2,4})$ cannot be a radius 2-$(2, 8)$-covering. The case of $\varepsilon = 2$ in Definition 4.5 has been substantially used to establish the digital homotopy lifting theorem (see Lemma 5.6 of the present paper).

By Definitions 4.4 and 4.5, we can say that a $(k_0, k_1)$-covering satisfying a radius $n$ local isomorphism is equivalent to a *radius $n$-$(k_0, k_1)$-covering*. Namely, in this case we may take $\varepsilon = n$ in Definition 4.1. In addition, we clearly observe that a $(k_0, k_1)$-covering is a radius 1-$(k_0, k_1)$-covering.

Let us now study some Cartesian product properties of digital coverings.

**Remark 4.6.** ([23]) Consider four digital images $(E_1, k_0)$ in $\mathbf{Z}^{n_0}$, $(B_1, k_1)$ in $\mathbf{Z}^{n_1}$, $(E_2, k_2)$ in $\mathbf{Z}^{n_2}$ and $(B_2, k_3)$ in $\mathbf{Z}^{n_3}$. Furthermore, the digital products $(E_1 \times E_2, k_4), (B_1 \times B_2, k_5)$ are assumed with some $k_4$- and $k_5$-adjacencies of $\mathbf{Z}^{n_0+n_2}$ and $\mathbf{Z}^{n_1+n_3}$, respectively. Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be $(k_0, k_1)$- and $(k_2, k_3)$-covering maps, respectively. Precisely, we observe that the digital product map $p := p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$ need not be a $(k_4, k_5)$-covering map.

To make Remark 4.6 self-contained, we can consider the following $(8, 4)$-covering

$$(X := (c_i)_{i\in[0,7]_\mathbf{Z}}, p, SC_4^{2,8} := (d_j)_{j\in[0,7]_\mathbf{Z}}) \text{ given by } p(c_i) = d_i,$$

where $(X, 8)$ is the digital image in Figure 1. Using the presentation of (3.2), let us consider the product map

$$p \times p : X \times X := (c_{ij}) \to SC_4^{2,8} \times SC_4^{2,8} := (d_{ij}) \tag{4.1}$$

given by $p \times p(c_i, c_j) = (d_i, d_j)$. Consider two Cartesian products $(X \times X, k)$, $(SC_4^{2,8} \times SC_4^{2,8}, k')$ with any $k$- and $k'$-adjacencies of $\mathbf{Z}^4$ in (2.2), respectively. Then we can observe that none of $k$- and $k'$-adjacencies of $\mathbf{Z}^4$ in (2.2) makes the product map $p \times p$ a $(k, k')$-covering map.

With some hypothesis of a Cartesian product map we obtain the following:

**Theorem 4.7.** *Let $(E_1, p_1, B_1)$ be a $(k_0, k_1)$-covering and let $(E_2, p_2, B_2)$ be a $(k_2, k_3)$-covering, where $(E_1, k_0)$, $(B_1, k_1)$, $(E_2, k_2)$ and $(B_2, k_3)$ are considered in $\mathbf{Z}^{n_0}$, $\mathbf{Z}^{n_1}$, $\mathbf{Z}^{n_2}$ and $\mathbf{Z}^{n_3}$, respectively. Then the Cartesian product map $p := p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$ is a $(k, k')$-covering map given by $p(e_1, e_2) = (p_1(e_1), p_2(e_2))$, where $k$ and $k'$ are C-compatible adjacencies of $E_1 \times E_2 \subset \mathbf{Z}^{n_0+n_2}$ and $B_1 \times B_2 \subset \mathbf{Z}^{n_1+n_3}$, respectively.*

*Proof.* In Definition 4.1 as already referred, we may take $\varepsilon = 1$. With the hypothesis, let us now examine if the Cartesian product map $p := p_1 \times p_2$ is a $(k, k')$-covering map.

First, we observe that the map $p := p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$ is clearly a surjection because both $p_1$ and $p_2$ are surjections.

Second, owing to the existence of a $C$-compatible adjacency of $B_1 \times B_2$, for any $(b_1, b_2) \in B_1 \times B_2$, we obtain

$$N_{k'}((b_1, b_2), 1) = (N_{k_1}(b_1, 1) \times \{b_2\}) \cup (\{b_1\} \times N_{k_3}(b_2, 1)) \subset B_1 \times B_2.$$

Due to the given digital coverings $(E_1, p_1, B_1)$ and $(E_2, p_2, B_2)$, for some index sets $M_1$ and $M_2$, we obtain

$$\left\{\begin{array}{l} p_1^{-1}(N_{k_1}(b_1, 1)) = \cup_{i\in M_1} N_{k_0}(e_i, 1) \text{ and} \\ p_2^{-1}(N_{k_3}(b_2, 1)) = \cup_{j\in M_2} N_{k_2}(e_j, 1), \end{array}\right\} \tag{4.2}$$

where $N_{k_1}(b_1, 1)$ and $N_{k_3}(b_2, 1)$ are, respectively, elementary $k_1$- and $k_3$-neighborhoods of $b_1$ and $b_2$, and $e_i \in p_1^{-1}(b_1)$, $e_j \in p_2^{-1}(b_2)$.

Then for the index set $M_1 \times M_2$ and $(e_i, e_j) \in p^{-1}((b_1, b_2))$, by (4.2) and the hypothesis of an existence of $C$-compatible adjacencies of $E_1 \times E_2$ and $B_1 \times B_2$, we obtain

$$\left\{\begin{array}{l} p^{-1}(N_{k'}((b_1, b_2), 1)) = p^{-1}((N_{k_1}(b_1, 1) \times \{b_2\}) \cup (\{b_1\} \times N_{k_3}(b_2, 1))) \\ \quad = (p_1^{-1}(N_{k_1}(b_1, 1)) \times \cup_{j\in M_2}\{e_j\}) \cup (\cup_{i\in M_1}\{e_i\} \times (p_2^{-1}(N_{k_3}(b_2, 1)))) \\ \quad = (\cup_{i\in M_1} N_{k_0}(e_i, 1) \times \cup_{j\in M_2}\{e_j\}) \cup (\cup_{i\in M_1}\{e_i\} \times \cup_{j\in M_2} N_{k_2}(e_j, 1)) \\ \quad = \cup_{(i,j)\in M_1\times M_2}((N_{k_0}(e_i, 1) \times \{e_j\}) \cup (\{e_i\} \times N_{k_2}(e_j, 1))), \\ \quad = \cup_{(i,j)\in M_1\times M_2} N_k((e_i, e_j), 1). \end{array}\right\} \tag{4.3}$$

Third, if $(i_1, j_1)$ and $(i_2, j_2) \in M_1 \times M_2$ and $(i_1, j_1) \neq (i_2, j_2)$, then we prove that $N_k((e_{i_1}, e_{j_1}), 1) \cap N_k((e_{i_2}, e_{j_2}), 1) = \emptyset$.

To be specific, due to both the $(k_0, k_1)$-covering $(E_1, p_1, B_1)$ and the $(k_2, k_3)$-covering $(E_2, p_2, B_2)$, for $(i_1, j_1) \neq (i_2, j_2)$ we suffice to consider two cases as follows.

In case $i_1 \neq i_2$, we obviously obtain $N_{k_0}(e_{i_1}, 1) \cap N_{k_0}(e_{i_2}, 1) = \emptyset$, where $N_{k_0}(e_{i_1}, 1), N_{k_0}(e_{i_2}, 1) \in p_1^{-1}(N_{k_1}(b_1, 1))$ and $e_{i_1}, e_{i_2} \in p_1^{-1}(b_1)$.

In case $j_1 \neq j_2$, we also have $N_{k_2}(e_{j_1}, 1) \cap N_{k_2}(e_{j_2}, 1) = \emptyset$, where $N_{k_2}(e_{j_1}, 1), N_{k_2}(e_{j_2}, 1) \in p_2^{-1}(N_{k_3}(b_2, 1))$, and $e_{j_1}, e_{j_2} \in p_2^{-1}(b_2)$.

Consequently, if $(i_1, j_1) \neq (i_2, j_2)$, then we get

$$N_k((e_{i_1}, e_{j_1}), 1) \cap N_k((e_{i_2}, e_{j_2}), 1) = \emptyset, \tag{4.4}$$

because

$$\left\{\begin{array}{l} \quad N_k((e_{i_1}, e_{j_1}), 1) = (N_{k_0}(e_{i_1}, 1) \times \{e_{j_1}\}) \cup (\{e_{i_1}\} \times N_{k_2}(e_{j_1}, 1)), \\ \quad N_k((e_{i_2}, e_{j_2}), 1) = (N_{k_0}(e_{i_2}, 1) \times \{e_{j_2}\}) \cup (\{e_{i_2}\} \times N_{k_2}(e_{j_2}, 1)), \\ \text{where } (e_{i_1}, e_{j_1}), (e_{i_2}, e_{j_2}) \in p^{-1}(b_1, b_2). \end{array}\right\}$$

Fourth, due to the given digital coverings $(E_1, p_1, B_1)$ and $(E_2, p_2, B_2)$, for any $N_k((e_i, e_j), 1) \in p^{-1}(N_{k'}((b_1, b_2), 1))$, by (4.3), we obtain the restriction map

$$p\,|_{N_k((e_i,e_j),1)} : N_k((e_i, e_j), 1) \to N_{k'}((b_1, b_2), 1) \tag{4.5}$$

which is a $(k, k')$-isomorphism for all $(i, j) \in M_1 \times M_2$ because $p_1\,|_{N_{k_0}(e_i,1)}$ and $p_2\,|_{N_{k_2}(e_j,1)}$ are $(k_0, k_1)$- and $(k_2, k_3)$-isomorphisms, respectively. Therefore, by (4.3), (4.4) and (4.5), the surjection $p : E_1 \times E_2 \to B_1 \times B_2$ is proved to be a $(k, k')$-covering map. □

**Example 4.8.** (1) Consider the following two maps $p_1$ and $p_2$ which are $(4, 8)$- and $(2, 8)$-covering maps, respectively:

$p_1 : SC_4^{2,8} := (c_i)_{i\in[0,7]_Z} \to SC_8^{2,4} := (d_j)_{j\in[0,3]_Z}$ with $p_1(c_i) = d_{i(mod\,4)}$ and

$p_2 : \mathbf{Z} \to SC_8^{2,6} := (c_i)_{i\in[0,5]_Z}$ with $p_2(t) = c_{t(mod\,6)}$.

By Example 3.5, we observe that $SC_4^{2,8} \times \mathbf{Z}$ and $SC_8^{2,4} \times SC_8^{2,6}$ have $C$-compatible 6- and 32-adjacencies, respectively. Thus, by Theorem 4.7, the Cartesian product map $p_1 \times p_2 : SC_4^{2,8} \times \mathbf{Z} \to SC_8^{2,4} \times SC_8^{2,6}$ is a $(6, 32)$-covering map.

(2) Consider the following two maps $q_1$ and $q_2$ which are $(2, 8)$-covering maps, respectively:

$q_1 : \mathbf{Z} \to SC_8^{2,4} := (c_i)_{i\in[0,3]_Z}$ with $q_1(i) = c_{i(mod\,4)}$ and

$q_2 : \mathbf{Z} \to SC_8^{2,6} := (d_j)_{i\in[0,5]_Z}$ with $q_2(t) = d_{t(mod\,6)}$.

By Example 3.5, we observe that $\mathbf{Z}\times\mathbf{Z}$ and $SC_8^{2,4}\times SC_8^{2,6}$ have $C$-compatible 4- and 32-adjacencies, respectively. Thus, by Theorem 4.7, the Cartesian product map $q_1 \times q_2 : \mathbf{Z} \times \mathbf{Z} \to SC_8^{2,4} \times SC_8^{2,6}$ is a $(4, 32)$-covering map.

(3) Consider the two digital products in (4.1). Since the following Cartesian products $(SC_8^{2,8} \times SC_8^{2,8}, 32)$, $(SC_4^{2,8} \times SC_4^{2,8}, 8)$ have only 32- and 8-compatible adjacencies, respectively, we obtain the product map which is a $(32, 8)$-covering map.

## 5. Automorphism Group of a Cartesian Product of Digital Coverings

Since the study of an automorphism group of a digital covering space plays an important role in classifying digital images [18], we need to study an automorphism group of a Cartesian product of two digital coverings in terms of a $C$-compatible $k$-adjacency, we can consider the following two cases: for a Cartesian product of two digital coverings

(Case 1) it satisfies a radius 2 local isomorphism.

(Case 2) it does not satisfy a radius 2 local isomorphism.

Before studying these cases, let us now recall the notion of a digital covering homomorphism in [18, 19]. While the notion of a $(k_1, k_2)$-covering homomorphism from $(E_1, p_1, B)$ into $(E_2, p_2, B)$ is motivated by that of a covering homomorphism in [34, 39], it can be substantially used to study a digital homomorphism between two digital coverings $(E_1, p_1, B)$ and $(E_2, p_2, B)$ because the present digital covering is very different from a covering from algebraic topology. In this section we show that a $C$-compatible $k$-adjacency of a digital product can be substantially used to study an automorphism group of a Cartesian product of digital coverings.

**Definition 5.1.** ([18, 19]) For three digital images $(B, k)$, $(E_1, k_1)$, and $(E_2, k_2)$, let $(E_1, p_1, B)$ and $(E_2, p_2, B)$ be $(k_1, k)$- and $(k_2, k)$-coverings, respectively. Then, we say that a $(k_1, k_2)$-continuous map $\phi : E_1 \to E_2$ such that $p_2 \circ \phi = p_1$ is a $(k_1, k_2)$-covering homomorphism from $(E_1, p_1, B)$ into $(E_2, p_2, B)$.

Let us now study an automorphism group of a Cartesian product of two digital coverings of the above Case 1. First of all, in case a digital covering satisfies a radius 2 local isomorphism (see the above Case 1), since the study of an automorphism group of a digital covering is very related to the calculation of a digital fundamental group, let us now recall it. Motivated by the pointed digital homotopy in [4, 32], the following notion of *$k$-homotopy relative to a subset $A \subset X$* has been substantially used to study a $k$-homotopic thinning [14, 16, 18] and a strong $k$-deformation retract of a digital image $(X, k)$ in $\mathbf{Z}^n$ [14, 16, 18, 20].

**Definition 5.2.** ([16, 20]) Let $((X, A), k_0)$ and $(Y, k_1)$ be a digital image pair and a digital image, respectively. Let $f, g : X \to Y$ be $(k_0, k_1)$-continuous functions. Suppose there exist $m \in \mathbf{N}$ and a function $F : X\times[0, m]_{\mathbf{Z}} \to Y$ such that

(1) for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$;

(2) for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \to Y$ given by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbf{Z}}$ is $(2, k_1)$-continuous;

(3) for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ given by $F_t(x) = F(x, t)$ for all $x \in X$ is $(k_0, k_1)$-continuous.
Then we say that $F$ is a $(k_0, k_1)$-homotopy between $f$ and $g$ [4].
(4) Furthermore, $F_t(x) = f(x) = g(x)$ for all $x \in A$ and for all $t \in [0, m]_{\mathbf{Z}}$.
Then we call $F$ a $(k_0, k_1)$-homotopy relative to $A$ between $f$ and $g$, and we say $f$ and $g$ are $(k_0, k_1)$-homotopic relative to $A$ in $Y$, $f \simeq_{(k_0,k_1)relA} g$ in symbols.

In Definition 5.2, if $A = \{x_0\} \subset X$, then we say that $F$ is a pointed $(k_0, k_1)$-homotopy at $\{x_0\}$ [4]. When $f$ and $g$ are pointed $(k_0, k_1)$-homotopic in $Y$, we denote by $f \simeq_{(k_0,k_1)} g$ the homotopic relation. In addition, if $k_0 = k_1$ and $n_0 = n_1$, then we say that $f$ and $g$ are pointed $k_0$-homotopic in $Y$ and use the notation $f \simeq_{k_0} g$ and $f \in [g]$ which means the $k_0$-homotopy class of $g$. If, for some $x_0 \in X$, $1_X$ is $k$-homotopic to the constant map with the set $\{x_0\}$ relative to $\{x_0\}$, then we say that $(X, x_0)$ is pointed $k$-contractible [4]. Indeed, the notion of $k$-contractibility is slightly different from the contractibility in Euclidean topology [18].

In classical topology any circles in the $n$D real space are homeomorphic to each other, $n \geq 2$. However, in digital topology simple closed $k$-curves in $\mathbf{Z}^n$ need not be $k$-isomorphic to each other because a $k$-isomorphism between them depends on the cardinality of $SC_k^{n,l}, n \geq 2$. Thus, motivated by both 8-contractibility of $SC_8^{2,4}$ [4] and non-8-contractibility of $SC_8^{2,6}$ [9], the paper [11] proved that $\pi^k(SC_k^{n,l})$ is an infinite cyclic group such as $(l\mathbf{Z}, +)$, where $SC_k^{n,l}$ is not $k$-contractible. More precisely, for $SC_k^{n,l}$ not $k$-contractible, by using the digital homotopy lifting theorem [10], the digital unique lifting theorem [11], and some properties of the $(2, k)$-covering map $(\mathbf{Z}, p, SC_k^{n,l})$ we obtain [11]

$$\pi^k(SC_k^{n,l}, x_0) \simeq (\mathbf{Z}, +) \simeq (l\mathbf{Z}, +), \tag{5.1}$$

where " $\simeq$" means a group isomorphism. In addition, $\pi^k(SC_k^{n,4})$ is trivial [4] if $k = 3^n - 1, n \in \mathbf{N} \setminus \{1\}$ because it is $k$-contractible.

The following notion of '*simply k-connected*' in [11] has been often used in digital topology for calculating digital fundamental groups of some digital images, classifying digital images, studying an automorphism group of a digital covering and so forth [5, 6, 11, 14, 17].

**Definition 5.3.** ([11]) A pointed $k$-connected digital image $(X, x_0)$ is called *simply k-connected* if $\pi^k(X, x_0)$ is a trivial group.

Since our definition of $k$-contractility requires a digital image $(X, k)$ to shrink $k$-continuously to a point over a finite time interval, we cannot say that $\mathbf{Z}$ is 2-contractible. However, we can establish the simply 2-connectedness of $\mathbf{Z}$ [11]. Even though the paper [6] used the property of simply $k$-connectedness of $\mathbf{Z}^n$, the present paper speaks out that the property was already proven in [13] (see [28]) as follows:

**Lemma 5.4.** ([13, 28]) *For each $n \in \mathbf{N}$, $(\mathbf{Z}^n, 0_n)$ is simply $k(m, n)$-connected, where $0_n$ is the origin in $\mathbf{Z}^n$ and $m \in [1, n]_{\mathbf{Z}}$.*

Since both *unique lifting theorem* and *digital homotopy lifting theorem* have often used to study digital images from the viewpoint of digital homotopy theory, let us recall them as follows.

**Lemma 5.5.** (Unique lifting theorem) ([11]) *For pointed digital images $((E, e_0), k_0)$ in $\mathbf{Z}^{n_0}$ and $((B, b_0), k_1)$ in $\mathbf{Z}^{n_1}$, let $p : (E, e_0) \to (B, b_0)$ be a pointed $(k_0, k_1)$-covering map. Any $k_1$-path $f : [0, m]_{\mathbf{Z}} \to B$ beginning at $b_0$ has a unique digital lifting to a $k_0$-path $\tilde{f}$ in $E$ beginning at $e_0$.*

**Lemma 5.6.** (Digital homotopy lifting theorem) [10] *Let $((E, e_0), k_0)$ and $((B, b_0), k_1)$ be pointed digital images. Let $p : (E, e_0) \to (B, b_0)$ be a radius 2-$(k_0, k_1)$-covering map. For $k_0$-paths $g_0, g_1$ in $(E, e_0)$ that start at $e_0$, if there is a $k_1$-homotopy in $B$ from $p \circ g_0$ to $p \circ g_1$ that holds the endpoints fixed, then $g_0$ and $g_1$ have the same terminal point, and there is a $k_0$-homotopy in $E$ from $g_0$ to $g_1$ that holds the endpoints fixed.*

By the use of the digital homotopy lifting theorem, we obtain the following.

**Lemma 5.7.** ([5, 19]) *Let* $((E, e_0), k_0)$, $((B, b_0), k_1)$ *be pointed digital images in* $\mathbf{Z}^{n_0}$ *and* $\mathbf{Z}^{n_1}$*, respectively. Let* $p : (E, e_0) \to (B, b_0)$ *be a pointed radius* 2-$(k_0, k_1)$*-covering map, then the map* $p_* : \pi^{k_0}(E, e_0) \to \pi^{k_1}(B, p(e_0))$ *is a monomorphism.*

**Definition 5.8.** ([14]) For digital images $((E, e_0), k_0)$ and $((B, b_0), k_1)$, let $p : (E, e_0) \to (B, b_0)$ be a pointed $(k_0, k_1)$-covering map. If $p_*\pi^{k_0}(E, e_0)$ is a normal subgroup of $\pi^{k_1}(B, b_0)$, then $((E, e_0), p, (B, b_0))$ is called a *regular* $(k_0, k_1)$*-covering*.

Let us now recall generalized digital lifting theorem in [19] as follows.

**Theorem 5.9.** ([6, 19]) *Let* $((E_1, e_1), p_1, (B, b_0))$ *and* $((E_2, e_2), p_2, (B, b_0))$ *be a pointed* $(k_1, k)$*- and a pointed radius* 2-$(k_2, k)$*-covering, respectively. Then, there is a* $(k_1, k_2)$*-covering homomorphism* $\phi : (E_1, k_1) \to (E_2, k_2)$ *such that* $\phi(e_1) = e_2$ *and* $p_2 \circ \phi = p_1$ *if and only if* $(p_1)_*\pi^{k_1}(E_1, e_1) \subset (p_2)_*\pi^{k_2}(E_2, e_2)$.

Let $((E, e_0), k_0)$ and $((B, b_0), k_1)$ be two digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Consider a $(k_0, k_1)$-covering map $p : ((E, e_0), k_0) \to ((B, b_0), k_1)$. A self $k_0$-isomorphism of the $(k_0, k_1)$-covering map $p$, denoted by $h : (E, k_0) \to (E, k_0)$, such that $p \circ h = p$ is called an *automorphism or a covering transformation* [19].

**Definition 5.10.** ([19]) A $(k_1, k_2)$-covering homomorphism $\phi : (E_1, k_1) \to (E_2, k_2)$ over a base space $(B, k)$ is called a $(k_1, k_2)$-covering isomorphism if there is a $(k_2, k_1)$-covering homomorphism $\psi : (E_2, k_2) \to (E_1, k_1)$ such that both compositions $\psi \circ \phi$ and $\phi \circ \psi$ are identity maps. Two digital coverings $((E_1, k_1), p_1, (B, k))$ and $((E_2, k_2), p_2, (B, k))$ are called $(k_1, k_2)$-isomorphic if there is a $(k_1, k_2)$-covering isomorphism $\phi : (E_1, k_1) \to (E_2, k_2)$ over the given space $(B, k)$.

The set of all covering transformations of a pointed digital $(k_0, k_1)$-covering $p$ forms a group under the operation of composition, called an automorphism group (or deck transformation group) of $((E, e_0), k_0)$ over $((B, b_0), k_1)$ [19], and denoted by $Aut((E, e_0)|\,(B, b_0))$, for brevity, $Aut(E|\,B)$ if there is no danger of ambiguity [19].

Using Massey's program of an automorphism group [34], we obtain a connection between $Aut(E\,|\,B)$ and the action of $\pi^{k_1}(B, b_0)$ on $p^{-1}(b_0)$ which represents digital topological versions of Proposition 7.1, Theorem 7.2 and Corollary 7.3 in [34], as follows: Let $((E, e_0), p, (B, b_0)))$ be a radius 2-$(k_0, k_1)$-covering. For any automorphism $\phi \in Aut(E\,|\,B)$, any point $\tilde{e} \in p^{-1}(b_0)$ and any $\alpha \in \pi^{k_1}(B, b_0)$, we obtain that [13, 17]

$$\phi(\tilde{e} \cdot \alpha) = (\phi\tilde{e}) \cdot \alpha, \tag{5.2}$$

where the operation "$\cdot$" in (5.2) is easily induced from [17] as follows: Take $\alpha = [f] \in \pi^{k_1}(B, b_0)$, where $f : [0, m_f]_{\mathbf{Z}} \to (B, b_0)$ represents $\alpha$. Consider the digital lifting $\tilde{f}$ of $f$ [10, 11] and $\phi\tilde{f} : [0, m_f]_{\mathbf{Z}} \to E$ such that $\phi\tilde{f}(0) = \phi(\tilde{e})$, $\phi\tilde{f}(m_f) = \phi(\tilde{e} \cdot \alpha)$ and $p\phi\tilde{f} = p\tilde{f} = f$. Thus $\phi\tilde{f}$ is a digital lifting of $f$. Therefore, we obtain $\phi(\tilde{e}) \cdot \alpha = \phi\tilde{f}(m_f) = \phi(\tilde{e} \cdot \alpha)$.

In other words, each element $\phi \in Aut(E\,|\,B)$ induces an automorphism of the set $p^{-1}(b_0)$ which is considered as a right $\pi^{k_1}(B, b_0)$-space [17] (see also [24]). Further, we can state the following:

**Theorem 5.11.** ([17]) *Let* $((E, e_0), p, (B, b_0))$ *be a radius* 2-$(k_0, k_1)$*-covering. Then,* $Aut(E\,|\,B)$ *is isomorphic to the group of automorphisms of the set* $p^{-1}(b_0)$*, which is considered as a right* $\pi^{k_1}(B, b_0)$*-space.*

In relation to the study of an automorphism group of a digital covering space, this kind of approach used in Theorem 5.11 has some limitations because Theorem 5.11 are only valid under the hypothesis that given a digital covering is a radius 2-$(k_0, k_1)$-covering. However, if a $(k_0, k_1)$-covering does not satisfy a radius 2 local isomorphism, then we have an obstacle to the study of the digital homotopic properties of a digital covering as well as its automorphism group (see [19]) because we cannot use digital homotopic tools such as the digital homotopy lifting theorem in [10] (see [24] for more details).

Further, in the study of a transitive action $\pi^{k_1}(B, b_0)$ on the set $p^{-1}(b_0)$, we defined the following:

**Definition 5.12.** ([24]) For a $(k_0, k_1)$-covering $((E, e_0), p, (B, b_0))$ we say that $Aut(E|B)$ acts transitively on $p^{-1}(b_0)$ if for any two distinct points $e_0$ and $e_1$ in $p^{-1}(b_0)$ there is $\phi \in Aut(E|B)$ such that $\phi(e_0) = e_1$.

**Lemma 5.13.** ([14, 24]) *If a radius 2-$(k_0, k_1)$-covering map $p : (E, e_0) \to (B, b_0)$ is regular, then $Aut(E|B)$ acts transitively on $p^{-1}(b_0)$.*

**Theorem 5.14.** ([13, 14]) *(see also [6, 29]) For two digital images $(E, k_0)$ in $\mathbf{Z}^{n_0}$ and $(B, k_1)$ in $\mathbf{Z}^{n_1}$, let $p : (E, e_0) \to (B, b_0)$ be a radius 2-$(k_0, k_1)$-covering map and let $(B, b_0)$ be $k_1$-connected. Then for any $b_0, b_1 \in B$, the sets $p^{-1}(b_0)$ and $p^{-1}(b_1)$ have the same cardinality.*

Even though the following property was used in [6], the original version was shown in [13] (see also [29] for more details).

**Theorem 5.15.** ([14]) *Let $((E, e_0), k_0)$ and $((B, b_0), k_1)$ be pointed digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Let $p : (E, e_0) \to (B, b_0)$ be a pointed radius 2-$(k_0, k_1)$-covering map and $(B, b_0)$ be $k_1$-connected. Then, there is a surjection $\Phi : \pi^{k_1}(B, b_0) \to p^{-1}(b_0)$. If $(E, e_0)$ is simply $k_0$-connected, then $\Phi$ is a bijection.*

For a digital product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ with a $C$-compatible $k$-adjacency, since the calculation of its digital $k$-fundamental group is very important in digital homotopy theory, we need to study the following:

**Theorem 5.16.** ([21]) *Assume that $SC_{k_i}^{n_i,l_i}$ is not $k_i$-contractible, $i \in \{1, 2\}$. If there is a $C$-compatible $k$-adjacency of $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$, then $\pi^k(SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}, (c_0, d_0))$ is isomorphic to $\pi^{k_1}(SC_{k_1}^{n_1,l_1}, c_0) \times \pi^{k_2}(SC_{k_2}^{n_2,l_2}, d_0)$.*

By Theorem 5.16, we obtain the following:

**Theorem 5.17.** *Let $(SC_{k_1}^{n_1,l_1}, p_1, SC_{k_2}^{n_2,l_2})$ be a radius 2-$(k_1, k_2)$-covering and let $(SC_{k_3}^{n_3,l_3}, p_2, SC_{k_4}^{n_4,l_4})$ be a radius 2-$(k_3, k_4)$-coverings, where $SC_{k_1}^{n_1,l_1} := (a_i)_{i\in[0,l_1-1]_\mathbf{Z}}$, $SC_{k_2}^{n_2,l_2} := (b_i)_{i\in[0,l_2-1]_\mathbf{Z}}$, $SC_{k_3}^{n_3,l_3} := (c_i)_{i\in[0,l_3-1]_\mathbf{Z}}$, and $SC_{k_4}^{n_4,l_4} := (d_i)_{i\in[0,l_4-1]_\mathbf{Z}}$. Assume that $SC_{k_1}^{n_1,l_1} \times SC_{k_3}^{n_3,l_3}$ and $SC_{k_2}^{n_2,l_2} \times SC_{k_4}^{n_4,l_4}$ have $C$-compatible $k$- and $k'$-adjacencies, respectively. Then we obtain that $Aut(SC_{k_1}^{n_1,l_1} \times SC_{k_3}^{n_3,l_3} |\, SC_{k_2}^{n_2,l_2} \times SC_{k_4}^{n_4,l_4})$ is isomorphic to $(\mathbf{Z}_m \times \mathbf{Z}_n, +)$, where $l_1 := ml_2$ and $l_3 := nl_4$.*

*Proof.* Using (5.1), Theorem 5.16 and the method similar to the proof of Theorem 4.7, we prove this theorem. First, with the hypothesis we need to prove that the natural product map $p := p_1 \times p_2 : SC_{k_1}^{n_1,l_1} \times SC_{k_3}^{n_3,l_3} \to SC_{k_2}^{n_2,l_2} \times SC_{k_4}^{n_4,l_4}$ is a radius 2-$(k, k')$-covering map. By Theorem 4.7, since the product map $p$ is a $(k, k')$-covering map, we suffice to prove that the map $p$ is a radius 2 local $(k, k')$-isomorphism. Namely, we need to prove that

$$p|_{N_k((a_i,c_j),2)} : N_k((a_i, c_j), 2) \to N_{k'}((b_i, d_j), 2) \tag{5.4}$$

is a $(k, k')$-isomorphism, where $(a_i, c_j) \in p^{-1}(b_i, d_j)$.

By the hypothesis, since $p_1$ and $p_2$ are, respectively, radius 2 local $(k_1, k_2)$- and radius 2 local $(k_3, k_4)$-isomorphisms, we obtain
$p_1|_{N_{k_1}(a_i,2)} : N_{k_1}(a_i, 2) \to N_{k_2}(b_j, 2)$ as a $(k_1, k_2)$-isomorphism and
$p_2|_{N_{k_3}(c_i,2)} : N_{k_3}(c_i, 2) \to N_{k_4}(d_j, 2)$ as a $(k_3, k_4)$-isomorphism.

Since $SC_{k_1}^{n_1,l_1} \times SC_{k_3}^{n_3,l_3}$ and $SC_{k_2}^{n_2,l_2} \times SC_{k_4}^{n_4,l_4}$ have, respectively, $C$-compatible $k$- and $k'$-adjacencies, we obtain

$$\left.\begin{cases} N_k((a_i, c_i), 2) = N_k((a_i, c_i), 1) \cup N_k((a_{i\pm1(mod\, l_1)}, c_i), 1) \cup N_k((a_i, c_{i\pm1(mod\, l_3)}), 1) \subset SC_{k_1}^{n_1,l_1} \times SC_{k_3}^{n_3,l_3} \\ \text{and} \\ N_{k'}((b_j, d_j), 2) = N_{k'}((b_j, d_j), 1) \cup N_{k'}((b_{j\pm1(mod\, l_2)}, d_j), 1) \cup N_{k'}((b_j, d_{j\pm1(mod\, l_4)}), 1) \subset SC_{k_2}^{n_2,l_2} \times SC_{k_4}^{n_4,l_4} \end{cases}\right\} \tag{5.5}$$

so that $N_k((a_i, c_i), 2)$ is $(k, k')$-isomorphic to $N_{k'}((b_j, d_j), 2)$. Thus the product map $p$ is proved to be a radius 2 local $(k, k')$-isomorphism.

Indeed, in view of the given two digital coverings $(SC_{k_1}^{n_1,l_1}, p_1, SC_{k_2}^{n_2,l_2})$ and $((SC_{k_3}^{n_3,l_3}, p_2, SC_{k_4}^{n_4,l_4}))$, we may assume that

$$\begin{cases} p_1 : SC_{k_1}^{n_1,l_1} \to SC_{k_2}^{n_2,l_2} \text{ is an } m\text{-fold } (k_1,k_2)\text{-covering map; and} \\ p_2 : SC_{k_3}^{n_3,l_3} \to SC_{k_4}^{n_4,l_4} \text{ is an } n\text{-fold } (k_3,k_4)\text{-covering map,} \\ \text{where } l_1 := ml_2, l_3 := nl_4. \end{cases}$$

Due to a $C$-compatible $k$-adjacency of $SC_{k_1}^{n_1,l_1} \times SC_{k_3}^{n_3,l_3}$ and a $C$-compatible $k'$-adjacency of $SC_{k_2}^{n_2,l_2} \times SC_{k_4}^{n_4,l_4}$, by (5.4) and (5.5) we observe that $p := p_1 \times p_2 : SC_{k_1}^{n_1,l_1} \times SC_{k_3}^{n_3,l_3} \to SC_{k_2}^{n_2,l_2} \times SC_{k_4}^{n_4,l_4}$ is a radius 2-$(k,k')$-covering map.

Furthermore, by (5.1) and Lemmas 5.4 and 5.7 and Theorem 5.16, the map $p$ is a regular $(k,k')$-covering map and further, we obtain that $Aut(SC_{k_1}^{n_1,l_1} \times SC_{k_3}^{n_3,l_3} | SC_{k_2}^{n_2,l_2} \times SC_{k_4}^{n_4,l_4})$ is isomorphic to the quotient group

$$\begin{cases} \quad N[p_*\pi^k(SC_{k_1}^{n_1,l_1} \times SC_{k_3}^{n_3,l_3})]/p_*\pi^k(SC_{k_1}^{n_1,l_1} \times SC_{k_3}^{n_3,l_3}) \\ = \pi^{k'}(SC_{k_2}^{n_2,l_2} \times SC_{k_4}^{n_4,l_4})/p_*\pi^k(SC_{k_1}^{n_1,l_1} \times SC_{k_3}^{n_3,l_3}) \\ \simeq l_2\mathbf{Z} \times l_4\mathbf{Z}/l_1\mathbf{Z} \times l_3\mathbf{Z} \simeq \mathbf{Z}_m \times \mathbf{Z}_n, \end{cases}$$

where $l_2/l_1 := m$ and $l_4/l_3 := n$.

Concretely, we obtain that $Aut(SC_{k_1}^{n_1,l_1} \times SC_{k_3}^{n_3,l_3} | SC_{k_2}^{n_2,l_2} \times SC_{k_4}^{n_4,l_4})$ is isomorphic to $(\mathbf{Z}_m \times \mathbf{Z}_n, +)$. □

**Example 5.18.** In this paper we assume that $SC_8^{2,12}$ is the set $N_4((x,y),3) - N_4((x,y),2)$. Further, consider $SC_8^{2,6}$ and $SC_8^{2,8}$ in Figure 1. Let us consider an $(8,8)$-covering $(SC_8^{2,12} := (a_i)_{i\in[0,11]_\mathbf{Z}}, p_1, SC_8^{2,6} := (b_j)_{j\in[0,5]_\mathbf{Z}})$ and an $(8,8)$-covering $(SC_8^{2,16} := (c_i)_{i\in[0,15]_\mathbf{Z}}, p_2, SC_8^{2,8} := (d_j)_{j\in[0,7]_\mathbf{Z}})$, where

$$\begin{cases} p_1 : SC_8^{2,12} \to SC_8^{2,6} \text{ with } p_1(a_i) = b_{i(mod\,6)}; \\ p_2 : SC_8^{2,16} \to SC_8^{2,8} \text{ with } p_2(c_i) = d_{i(mod\,8)}. \end{cases}$$

By Theorem 3.8, each of the digital products $SC_8^{2,12} \times SC_8^{2,16}$ and $SC_8^{2,6} \times SC_8^{2,8}$ has a $C$-compatible 32-adjacency. Furthermore, we obtain a regular radius 2-32-covering map $p_1 \times p_2 : SC_8^{2,12} \times SC_8^{2,16} \to SC_8^{2,6} \times SC_8^{2,8}$ with $p_1 \times p_2(a_i, c_j) = (b_{i(mod\,6)}, d_{j(mod\,8)})$. Thus, by Theorem 5.16, we obtain that $Aut(SC_8^{2,12} \times SC_8^{2,12} | SC_{18}^{3,6} \times SC_8^{2,6})$ is isomorphic to $(\mathbf{Z}_2 \times \mathbf{Z}_2, +)$.

**Theorem 5.19.** *In Theorem 5.17, let us replace $SC_{k_3}^{n_3,l_3}$ by $\mathbf{Z}$ and put $k_1 = 2n_1$. Let $(\mathbf{Z}, p_2, SC_{k_4}^{n_4,l_4})$ be a radius 2-$(2,k_4)$-covering. Then $Aut(SC_{k_1}^{n_1,l_1} \times \mathbf{Z} | SC_{k_2}^{n_2,l_2} \times SC_{k_4}^{n_4,l_4})$ is isomorphic to $(\mathbf{Z}_m \times l_4\mathbf{Z}, +)$, where $l_1 := ml_2$.*

*Proof.* With the hypothesis, we can assume that

$$\begin{cases} p_1 : SC_{k_1}^{n_1,l_1} \to SC_{k_2}^{n_2,l_2} \text{ is an } m\text{-fold } (k_0,k_1)\text{-covering map; and} \\ p_2 : \mathbf{Z} \to SC_{k_4}^{n_4,l_4} \text{ is an infinite fold } (2,k_4)\text{-covering map,} \\ \text{where } l_1 := ml_2. \end{cases}$$

Since the product $SC_{2n_1}^{n_1,l_1} \times \mathbf{Z}$ has a $C$-compatible $2(n_1+1)$-adjacency and $SC_{k_2}^{n_2,l_2} \times SC_{k_4}^{n_4,l_4}$ has a $C$-compatible $k'$-adjacency, by using the method similar to the proof of Theorem 5.17, we complete the assertion. □

**Example 5.20.** Let us consider the maps in Example 5.18 replacing $SC_4^{2,16}$ by $\mathbf{Z}$ (see the map $p_2$). Let us consider a $(4,8)$-covering $(SC_4^{2,12} := (a_i)_{i\in[0,11]_\mathbf{Z}}, p_1, SC_8^{2,6} := (b_j)_{j\in[0,5]_\mathbf{Z}})$ and a $(2,8)$-covering $(\mathbf{Z}, p_2, SC_8^{2,8} := (d_j)_{j\in[0,7]_\mathbf{Z}})$, where

$$\begin{cases} p_1 : SC_4^{2,12} \to SC_8^{2,6} \text{ with } p_1(a_i) = b_{i(mod\,6)}; \\ p_2 : \mathbf{Z} \to SC_8^{2,8} \text{ with } p_2(i) = d_{i(mod\,8)}. \end{cases}$$

Then the digital products $SC_4^{2,12} \times \mathbf{Z}$ and $SC_8^{2,6} \times SC_8^{2,8}$ are considered with $C$-compatible 6- and 32-adjacencies, respectively. Furthermore, by Theorem 5.19, we obtain a regular radius 2-(6, 32)-covering map $p_1 \times p_2$ : $SC_8^{2,12} \times \mathbf{Z} \to SC_8^{2,6} \times SC_8^{2,8}$ with $p_1 \times p_2(a_i, j) = (b_{i(mod\,6)}, d_{j(mod\,8)})$. Thus, by Theorem 5.17, we obtain that $Aut(SC_4^{2,12} \times \mathbf{Z} |\, SC_{18}^{3,6} \times SC_8^{2,6})$ is isomorphic to $(\mathbf{Z}_2 \times 6\mathbf{Z}, +)$.

**Corollary 5.21.** *In Theorem 5.17, let us replace both $SC_{k_1}^{n_1,l_1}$ and $SC_{k_3}^{n_3,l_3}$ by $\mathbf{Z}$. Let $(\mathbf{Z}, p_2, SC_{k_2}^{n_2,l_2})$ and $(\mathbf{Z}, p_2, SC_{k_4}^{n_4,l_4})$ be radius 2-$(2, k_2)$- and radius 2-$(2, k_4)$-coverings, respectively. Then we obtain that $Aut(\mathbf{Z} \times \mathbf{Z} |\, SC_{k_2}^{n_2,l_2} \times SC_{k_4}^{n_4,l_4})$ is isomorphic to $(l_2\mathbf{Z} \times l_4\mathbf{Z}, +)$).*

In general, for a $(k_0, k_1)$-covering $((E, e_0), p, (B, b_0))$ $Aut(E \,|\, B)$ need not act transitively on $p^{-1}(b_0)$ (see Example 2 in [24]). In order to deal with this problem, we need to establish the following notion which is different from the notion of a regular $(k_0, k_1)$-covering.

**Definition 5.22.** ([24]) A $(k_0, k_1)$-covering $((E, e_0), p, (B, b_0))$ is called an ultra regular (for brevity, UR-) $(k_0, k_1)$-covering if $Aut(E \,|\, B)$ acts transitively on $p^{-1}(b_0)$.

Let us now recall the following property of a UR-$(k_0, k_1)$-covering which characterizes a UR-$(k_0, k_1)$-covering.

**Theorem 5.23.** ([24]) *The following are equivalent.*
*(1) A $(k', k)$-covering $((E, e_0), p, (B, b_0))$ is ultra regular.*
*(2) For a $(k', k)$-covering $((E, e_0), p, (B, b_0))$ we assume a closed $k$-curve $\alpha : [0, m]_{\mathbf{Z}} \to (B, k)$ with $\alpha(0) = b_0 \in B$. Either each of all the liftings of $\alpha$ on $(E, k')$ is a $k'$-closed curve or none of them is a $k'$-closed curve.*

Due to Theorem 5.23 hereafter, regardless of the requirement of a radius 2 local isomorphism of a $(k', k)$-covering, we now have a very convenient method of determining if a digital covering is UR-$(k', k)$-regular and further, we can study $Aut(E \,|\, B)$ without using the digital homotopic tools of a digital covering $(E, p, B)$. Since the following theorem holds in case $SC_{k_i}^{n_i,l_i}$ is $k_i$-contractible, its utility can be expanded.

**Theorem 5.24.** *With the hypothesis of Theorem 5.23, $Aut(\mathbf{Z}\times\mathbf{Z} \,|\, SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2})$ is isomorphic to the group $l_1\mathbf{Z}\times l_2\mathbf{Z}$, where $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ has a $C$-compatible $k$-adjacency and the $(2, k_i)$-covering map $(\mathbf{Z}, p_i, SC_{k_i}^{n_i,l_i})$ need not satisfy a 2 local $(2, k_i)$-isomorphism, $i \in \{1, 2\}$.*

*Proof.* By Theorem 5.23, the Cartesian product map $p_1 \times p_2 : \mathbf{Z} \times \mathbf{Z} \to SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2} := (c_i)_{i\in[0,1_1-1]_Z} \times (d_j)_{j\in[0,1_2-1]_Z}$ is an ultra regular $(4, k)$- covering map. For convenience, for the point $(c_0, d_0) \in SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$, consider the set $(p_1 \times p_2)^{-1}((c_0, d_0))$. Then the set $\{(p_1 \times p_2)^{-1}((c_0, d_0))\}$ is equal to the set $l_1\mathbf{Z}\times l_2\mathbf{Z}$. Thus we can see that Aut$(\mathbf{Z} \times \mathbf{Z} \,|\, SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2})$ is isomorphic to the group $l_1\mathbf{Z} \times l_2\mathbf{Z}$. To be specific, since the restriction map of $p_1 \times p_2$ on $[0, l_1 - 1]_{\mathbf{Z}} \times [0, l_2 - 1]_{\mathbf{Z}}$, *i.e.* $p_1 \times p_2|_{[0,l_1-1]_{\mathbf{Z}}\times[0,l_2-1]_{\mathbf{Z}}} : [0, l_1 - 1]_{\mathbf{Z}} \times [0, l_2 - 1]_{\mathbf{Z}} \to SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$, is a $(4, k)$-isomorphism because $\mathbf{Z} \times \mathbf{Z}$ has a $C$-compatible 4-adjacency, we can consider transformations of $[0, l_1 - 1]_{\mathbf{Z}} \times [0, l_2 - 1]_{\mathbf{Z}}$ in $\mathbf{Z} \times \mathbf{Z}$ associated with the ultra regular $(4, k)$-covering map $p_1 \times p_2$ in terms of a vertical or a parallel movement such as

$$[0, l_1 - 1]_{\mathbf{Z}} \times [0, l_2 - 1]_{\mathbf{Z}} \to [l_1, 2l_1 - 1]_{\mathbf{Z}} \times [0, l_2 - 1]_{\mathbf{Z}} \text{ or } [0, l_1 - 1]_{\mathbf{Z}} \times [l_2, 2l_2 - 1]_{\mathbf{Z}},$$

and so forth, we can conclude that Aut$(\mathbf{Z} \times \mathbf{Z} \,|\, SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2})$ is isomorphic to the infinite group $(l_1\mathbf{Z} \times l_2\mathbf{Z}, +)$. □

Let us move into the case that the given $(k_1, k_2)$-covering does not satisfy a local 2-isomorphism.

**Corollary 5.25.** *Let $(SC_{k_1}^{n_1,l_1}, p_1, SC_{k_2}^{n_2,l_2})$ be a $(k_1, k_2)$-covering and let $(SC_{k_3}^{n_3,l_3}, p_2, SC_{k_4}^{n_4,l_4})$ be a $(k_3, k_4)$-covering, where each of the spaces $SC_{k_i}^{n_i,l_i}$, $i \in \{1, 2, 3, 4\}$ may be $k_i$-contractible. Assume that $SC_{k_1}^{n_1,l_1} \times SC_{k_3}^{n_3,l_3}$ and $SC_{k_2}^{n_2,l_2} \times SC_{k_4}^{n_4,l_4}$ have $C$-compatible $k$- and $k'-$ adjacencies, respectively. Then we obtain that*
*$Aut(SC_{k_1}^{n_1,l_1} \times SC_{k_3}^{n_3,l_3} |\, SC_{k_2}^{n_2,l_2} \times SC_{k_4}^{n_4,l_4})$ is isomorphic to $(\mathbf{Z}_m \times \mathbf{Z}_n, +)$, where $l_1 := ml_2$ and $l_3 := nl_4$.*

**Example 5.26.** For $(x, y) \in \mathbf{Z}^2$, in this paper we assume that $SC_8^{2,12}$ is the set $N_4((x, y), 3) - N_4((x, y), 2)$ in $\mathbf{Z}^2$. Let us consider the $(8, 18)$-covering $(SC_8^{2,12} := (a_i)_{i\in[0,11]_\mathbf{Z}}, p_1, SC_{18}^{3,6} := (b_j)_{j\in[0,5]_\mathbf{Z}})$(see Figure 1) and the $(8, 8)$-covering $(SC_8^{2,12} := (c_i)_{i\in[0,12]_\mathbf{Z}}, p_2, SC_8^{2,6} := (d_j)_{j\in[0,5]_\mathbf{Z}})$(see Figure 1), where $p_1 : SC_8^{2,12} \to SC_{18}^{3,6}$ given by $p_1(a_i) = b_{i(mod\,6)}$ and $p_2 : SC_8^{2,12} \to SC_8^{2,6}$ with $p_2(c_i) = d_{i(mod\,6)}$. Then digital products $SC_8^{2,12} \times SC_8^{2,12}$ and $SC_{18}^{3,6} \times SC_8^{2,6}$ have $C$-compatible 32- and 50-adjacencies, respectively. Furthermore, we obtain an ultra regular $(32, 50)$-covering map, $p_1 \times p_2 : SC_8^{2,12} \times SC_8^{2,12} \to SC_{18}^{3,6} \times SC_8^{2,6}$ with $p_1 \times p_2(a_i, a_j) = (b_{i(mod\,6)}, d_{j(mod\,6)})$. Thus, by Corollary 5.25, we obtain that $Aut(SC_8^{2,12} \times SC_8^{2,12} | SC_{18}^{3,6} \times SC_8^{2,6})$ is isomorphic to $(\mathbf{Z}_2 \times \mathbf{Z}_2, +)$.

## 6. Concluding Remarks

Motivated by the normal product adjacency in [1], we have used a $C$-compatible adjacency of a digital product to study both the multiplicative property of a digital fundamental group and an automorphism group of a digital covering which need not satisfy a radius local 2-isomorphism. Owing to Theorems 3.8, 3.10, 4.7, 5.17 and 5.24 and Corollary 5.25, the paper address the unsolved problem related to the product property of digital topological properties of digital products (see Remark 3.7). Thus it turns out that a $C$-compatible $k$-adjacency of a digital product can be a suitable property of which a digital product of two digital coverings makes a digital covering and leads to the multiplicative property of the digital $k$-fundamental group.

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