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On Blowing-Up Solutions for Multi–Time Nonlinear Hyperbolic Equations and Systems

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Abstract. For a two–dimensional time nonlinear hyperbolic equation with a power nonlinearity, a threshold exponent depending on the space dimension is presented. Furthermore, the analysis is extended not only to a system of two equations but also to a two–time fractional nonlinear equation with different time order derivatives.

1. Introduction

In this paper we are concerned with the nonexistence of global weak solutions for multi–time hyperbolic equations of the type

$$\left\{\begin{array}{rcll} Lu & := & u_{tt} + u_{ss} - \Delta u = |u|^p, & (t;x) \in \mathcal{D}, \\ u(t,0;x) & = & u_{10}(t;x), \quad u_t(t,0;x) = u_{20}(t;x), & (t;x) \in \Omega, \\ u(0,s;x) & = & u_{01}(s;x), \quad u_s(0,s;x) = u_{02}(s;x), & (s;x) \in \Omega, \end{array}\right. \tag{1}$$

where $u := u(t,s;x)$, $u : \mathcal{D} \to \mathbb{R}$, $\mathcal{D} = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N, \Omega = \mathbb{R}^+ \times \mathbb{R}^N$, $N \in \mathbb{N}$, $N \geq 1; p \in \mathbb{R}$, $p > 1$; u_z, u_{zz} stand for the first and second partial derivatives in the variable z; and Δ stands for the N–dimensional Laplacian with respect to x. The nonlinearity $|u|^p$ is a prototype of nonlinearities that has been considered by John [9].

We will prove that no nontrivial global week solution of (1) exists under certain conditions depending on p and N.

The result will be extended to a 2×2 system of two–time hyperbolic nonlinear equations of the form

$$\left\{\begin{array}{rcll} Lu & = & |v|^p, & \text{in} \quad \mathcal{D}, \\ Lv & = & |u|^q, & \text{in} \quad \mathcal{D}, \end{array}\right. \tag{2}$$

with initial conditions

$$\left\{\begin{array}{rcll} u(0,s;x) & = & u_{01}(s;x), \quad u_t(0,s;x) = u_{02}(s;x), & (s;x) \in \Omega, \\ u(t,0;x) & = & u_{10}(t;x), \quad u_s(t,0;x) = u_{20}(t;x), & (t;x) \in \Omega, \\ v(0,s;x) & = & v_{01}(s;x), \quad v_t(0,s;x) = v_{02}(s;x), & (s;x) \in \Omega, \\ v(t,0;x) & = & v_{10}(t;x), \quad v_s(t,0;x) = v_{20}(t;x), & (t;x) \in \Omega, \end{array}\right. \tag{3}$$

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where $u := u(t,s;x)$, $v := v(t,s;x)$, $u, v : \mathcal{D} \to \mathbb{R}$; and $p, q > 1$. The blowing–up conditions will depend on p, q, and $\mathbb{N}$.

We will also consider equations involving fractional time derivatives in both time variables, with the nonlinear term $|u|^p$ as in (1),

$$D^{1+\alpha_1}_{0|t}(u - u_{01}(s;x) - tu_{02}(s;x)) + D^{1+\alpha_2}_{0|s}(u - u_{10}(t;x) - su_{20}(t;x)) - \Delta u = |u|^p, \tag{4}$$

for $(t,s;x) \in \mathcal{D}$, subject to the initial conditions

$$\begin{cases} u(t,0;x) = u_{10}(t;x), & u_s(t,0;x) = u_{20}(t;x), \quad (t;x) \in \Omega, \\ u(0,s;x) = u_{01}(s;x), & u_t(0,s;x) = u_{02}(s;x), \quad (s;x) \in \Omega, \end{cases} \tag{5}$$

where $u := u(t,s;x)$, $u : \mathcal{D} \to \mathbb{R}$; $p \in \mathbb{R}$, $p > 1$, and $D^{1+\delta}_{0|t}$, $0 < 1+\delta < 2$, stands for the fractional time derivative of order $1+\delta$ in the variable t in the sense of Riemann–Liouville.

Finally we consider the system of fractional equations

$$\begin{cases} F_{\alpha_1,\alpha_2} u = |v|^p, & \text{in } \mathcal{D}, \\ F_{\beta_1,\beta_2} v = |u|^q, & \text{in } \mathcal{D}, \end{cases} \tag{6}$$

where for $0 < \mu, \varrho < 1$,

$$F_{\mu,\varrho} u := D^{1+\mu}_{0|t}(u - u_{01}(s;x) - tu_{02}(s;x)) + D^{1+\varrho}_{0|s}(u - u_{10}(t;x) - su_{20}(t;x)),$$

subject to the initial conditions

$$\begin{cases} u(0,s;x) = u_{01}(s;x), & u_t(0,s;x) = u_{02}(s;x), \quad (s;x) \in \Omega, \\ u(t,0;x) = u_{10}(t;x), & u_s(t,0;x) = u_{20}(t;x), \quad (t;x) \in \Omega, \\ v(0,s;x) = v_{01}(s;x), & v_t(0,s;x) = v_{02}(s;x), \quad (s;x) \in \Omega, \\ v(t,0;x) = v_{10}(t;x), & v_s(t,0;x) = v_{20}(t;x), \quad (t;x) \in \Omega, \end{cases} \tag{7}$$

where $u := u(t,s;x)$, $v := v(t,s;x)$, $u, v : \mathcal{D} \to \mathbb{R}$, and $p, q > 1$.

Recent investigations on multi–time differential equations shed light on their applications to different fields of sciences such as mechanics, physics, biomathematics, and cosmology, see for example the works of Barashenkov [3, 4], Báez, Segal, and Zhou [1], Hillion [10, 11], Newton [18], Rendall [19], Uglum [22], Matei and Udrişte [16], Craig and Weinstein [6], Tucker [21], and the recent paper of Foster and Müller [8]; evolution equations with fractional time derivative or fractional space derivative have been discussed in [5, 13–15]. Let us mention that in [6], the authors pointed out the role played by the nonlinearities imposed in [8] for the existence of a unique solution of the homogeneous ultra–hyperbolic wave equation. Concerning blowing–up solutions for one time nonlinear hyperbolic equations with power nonlinearities, a lot has been said. For a review of the literature on the equation

$$\partial_t^2 u - \Delta u = |u|^p, \quad p > 1,$$

and a final result concerning a conjecture that lasted for twenty years, see the important paper [23].

The paper is organized as follows. Section 2 is devoted to study blowing–up solutions of a scalar nonlinear hyperbolic equation with two time variables, while the extension of this result to a 2×2 system of such equations is discussed in Section 3. In Section 4 we consider a nonlinear equation and a system of nonlinear equations involving fractional time derivatives with two time variables.

Throughout the paper we use the following notations: for any $p > 1$ we denote by p' the conjugate exponent of p, that is, $p + p' = pp'$. The symbol C denotes a positive constant which may vary from line to line.

We will use the notation:

$P = (t,s;x)$, $\mathrm{d}P := \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x$, $\mathrm{d}P_0 = \mathrm{d}t\,\mathrm{d}x$ or $\mathrm{d}P_0 = \mathrm{d}s\,\mathrm{d}x$;

$\tilde{P} = (\tau, \sigma; y)$, $\mathrm{d}\tilde{P} := \mathrm{d}\tau\,\mathrm{d}\sigma\,\mathrm{d}y$, $\mathrm{d}\tilde{P}_0 = \mathrm{d}\tau\,\mathrm{d}y$ or $\mathrm{d}\tilde{P}_0 = \mathrm{d}\tau\,\mathrm{d}y$.

Σ : the space of non–negative regular functions $\varphi : \Omega \to \mathbb{R}$ with compact support in the space variable x such that

$$\begin{cases} \varphi(t,s;x) &= \varphi_t(0,s;x) = 0, \quad t \geq T, \quad (s;x) \in \Omega, \\ \varphi(t,s;x) &= \varphi_s(t,0;x) = 0, \quad s \geq T, \quad (t;x) \in \Omega. \end{cases} \tag{8}$$

We conclude this introduction with a short remark: In the course of the proofs, we frequently use the fact that if φ is as above and $\gamma > 1$, then it is always possible to select φ so that

$$\int_{\mathcal{D}} \varphi^{1-\gamma} |D\varphi|^{\gamma}\,\mathrm{d}P < +\infty,$$

where $D = \mathrm{d}^2/\,\mathrm{d}t^2$ or $D = \Delta$. A justification of this fact is contained for instance in [17].

2. Results

2.1. *Blowing–up for two–time hyperbolic equations*

In this section we will show that under certain conditions on the initial data, p, and N, the solution of (1) does not exist globally in time.

We set

$$\mathcal{U}_{0,\varphi} := \int_{\Omega} u_{02}(s;x)\varphi(0,s;x)\,\mathrm{d}P_0 + \int_{\Omega} u_{20}(s;x)\varphi(t,0;x)\,\mathrm{d}P_0,$$

and

$$\mathcal{U}_0 := \int_{\Omega} u_{02}(s;x)\,\mathrm{d}P_0 + \int_{\Omega} u_{20}(s;x)\,\mathrm{d}P_0,$$

Definition 2.1. *Let $p > 1$ be a real number. A function $u := u(P)$ such that $u \in L^p_{loc}(\mathcal{D})$ is a weak solution of (1) if,*

$$\int_{\mathcal{D}} |u(P)|^p \varphi(P)\,\mathrm{d}P + \mathcal{U}_{0,\varphi} = \int_{\mathcal{D}} u(P)\left\{\varphi_{tt}(P) + \varphi_{ss}(P) - \Delta\varphi(P)\right\}\,\mathrm{d}P,$$

for every test function $\varphi \in C^2_0(\mathcal{D})$.

Let us assume that

$$u_{10}(\cdot;x), \quad u_{20}(\cdot;x), \quad u_{01}(\cdot;x), \quad u_{02}(\cdot;x) \in L^1(\Omega^0). \tag{9}$$

We have the following theorem.

Theorem 2.2. *If $p \leq 1 + 2/N$, and $0 < \mathcal{U}_0$, then no global non-trivial weak solution of (1) exists.*

Proof. The proof is by contradiction. Let us assume that the solution is global. Let $\chi : [0, +\infty) \to \mathbb{R}$, with, $0 \leq \chi \leq 1$, be a regular function defined by

$$\chi(\xi) = \begin{cases} 1, & \text{for} \quad 0 \leq \xi \leq 1, \\ \searrow, & \text{for} \quad 1 \leq \xi \leq 2, \\ 0, & \text{for} \quad \xi \geq 2. \end{cases}$$

We define φ to be the function

$$\varphi(P) := \chi\left(\frac{t^2 + s^2 + |x|^2}{R^2}\right), \quad P \in \mathcal{D}. \tag{10}$$

Then

$$\varphi_t(P) = \frac{2t}{R^2}\chi'\left(\frac{t^2+s^2+|x|^2}{R^2}\right),$$

so $\varphi_t(0,s;x) = 0$; we also have $\varphi_s(t,0;x) = 0$.

We are going to distinguish two cases:

Case 1: $p < 1 + 2/N$. Using ϵ–Young's inequality, for $t_j = t$ or s, we have

$$|u\varphi_{t_jt_j}| = |u\varphi^{1/p}\varphi_{t_jt_j}\varphi^{-1/p}| \le \epsilon|u|^p\varphi + C(\epsilon)|\varphi_{t_jt_j}|^{p'}\varphi^{-p'/p}, \tag{11}$$

$$|u\Delta\varphi| = |u\varphi^{1/p}\Delta\varphi\varphi^{-1/p}| \le \epsilon|u|^p\varphi + C(\epsilon)|\Delta\varphi|^{p'}\varphi^{-p'/p}, \tag{12}$$

where $p + p' = pp'$. Using (11) and (12), we may, for ϵ small enough, write

$$\int_{\mathcal{D}} |u|^p\varphi\,\mathrm{d}P + \mathcal{U}_{0,\varphi} \le C\int_{\mathcal{D}} \varphi^{-p'/p}\left\{|\varphi_{tt}|^{p'} + |\varphi_{ss}|^{p'} + |\Delta\varphi|^{p'}\right\}\mathrm{d}P. \tag{13}$$

Now, scaling the variables

$$t = R\tau, \quad s = R\sigma, \quad x = Ry.$$

we obtain

$$\int_{\mathcal{D}} |u|^p\varphi\,\mathrm{d}P + \mathcal{U}_{0,\varphi} \le CR^{-2p'+2+N}\int_{\Omega_1} \tilde{\varphi}^{-p'/p}\left\{|\tilde{\varphi}_{\tau\tau}|^{p'} + |\tilde{\varphi}_{\sigma\sigma}|^{p'} + |\Delta\tilde{\varphi}|^{p'}\right\}\mathrm{d}\tilde{P}, \tag{14}$$

where

$$\tilde{\varphi}(\tau,\sigma;y) := \chi(\tau^2+\sigma^2+|y|^2).$$

In the case $p < 1 + 2/N$, that is, $-2p' + 2 + N < 0$, if $R \to +\infty$, then from (14) we have

$$0 < \int_{\mathcal{D}} |u|^p\,\mathrm{d}P + \mathcal{U}_0 \le 0;$$

this contradicts our assumption. Therefore problem (1) admits no global nontrivial weak solution.

Case 2: $p = 1 + 2/N$. In this case, we have

$$0 < \int_{\mathcal{D}} |u|^p\varphi\,\mathrm{d}P \le C. \tag{15}$$

Using Hölder's inequality, we obtain

$$\int_{\mathcal{D}} |u|^p\varphi\,\mathrm{d}P + \mathcal{U}_{0,\varphi} \le \left(\int_{C_R} |u|^p\varphi\,\mathrm{d}P\right)^{\frac{1}{p}}\mathcal{H}(\varphi) \tag{16}$$

where $C_R = \left\{(P) \in \mathcal{D} : R^2 \le t^2 + s^2 + |x|^2 \le 2R^2\right\}$, and

$$\mathcal{H}(\varphi) := \left(\int_{C_R} |\varphi_{tt}|^{p'}\varphi^{-p'/p}\,\mathrm{d}P\right)^{1/p'} + \left(\int_{C_R} |\varphi_{ss}|^{p'}\varphi^{-p'/p}\,\mathrm{d}P\right)^{1/p'} + \left(\int_{C_R} |\Delta\varphi|^{p'}\varphi^{-p'/p}\,\mathrm{d}P\right)^{1/p'} \le C < +\infty.$$

Observe that (15) implies that

$$\lim_{R\to+\infty}\int_{C_R} |u|^p\varphi\,\mathrm{d}P = 0.$$

Passing onto the limit when $R \to +\infty$ in (16), we obtain

$$0 < \int_{\mathcal{D}} |u|^p\,\mathrm{d}P + \mathcal{U}_0 \le 0; \tag{17}$$

a contradiction. Hence non–trivial global weak solutions of problem (1) do not exist. □

2.2. Blowing–up for a system of two–time hyperbolic equations

Let us consider the system of equations (2)–(3).

We set

$$\mathcal{U}_{0,\varphi} := \int_\Omega w_{02}(s;x)\varphi(0,s;x)\,\mathrm{d}P_0 + \int_\Omega w_{10}(t;x)\varphi(t,0;x)\,\mathrm{d}P_0,$$

$$\mathcal{V}_{0,\varphi} := \int_\Omega r_{02}(s;x)\varphi(0,s;x)\,\mathrm{d}P_0 + \int_\Omega r_{10}(t;x)\varphi(t,0;x)\,\mathrm{d}P_0.$$

Definition 2.3. *Let $p, q > 1$ be two real numbers, and let $u := u(P)$, $v := v(P)$, be two functions such that $u \in L^q_{loc}(\mathcal{D})$, $v \in L^p_{loc}(\mathcal{D})$. We say that (u, v) is a weak solution of (2)–(3) if, for every $\varphi \in \Sigma$,*

$$\int_{\mathcal{D}} |v(P)|^p \varphi(P)\,\mathrm{d}P + \mathcal{U}_{0,\varphi} = \int_{\mathcal{D}} u(P)\{\varphi_{tt}(P) + \varphi_{ss}(P) - \Delta\varphi(P)\}\,\mathrm{d}P, \tag{18}$$

and

$$\int_{\mathcal{D}} |u(P)|^q \varphi(P)\,\mathrm{d}P + \mathcal{V}_{0,\varphi} = \int_{\mathcal{D}} v(P)\{\varphi_{tt}(P) + \varphi_{ss}(P) - \Delta\varphi(P)\}\,\mathrm{d}P, \tag{19}$$

for every test function $\varphi \in C^2_0(\mathcal{D})$.

Now we assume the following conditions

$$\mathcal{U}_{0,\varphi} > 0, \quad \text{and} \quad \mathcal{V}_{0,\varphi} > 0, \quad \text{for every} \quad \varphi \in \Sigma, \tag{20}$$

and also that all initial data belong to $L^1(\Omega^0)$ in a similar way as in (9).

Theorem 2.4. *Let us consider system (2)–(3) under the assumption (20). If*

$$N(pq-1) \le 2(p+1) \quad \text{or} \quad N(pq-1) \le 2(q+1),$$

then system (2)–(3) does not admit a global non–trivial weak solution.

Proof. We have from (18)

$$\int_{\mathcal{D}} |v|^p \varphi\,\mathrm{d}P + \mathcal{U}_{0,\varphi} = \int_{\mathcal{D}} u\{\varphi_{tt} + \varphi_{ss} - \Delta\varphi\}\,\mathrm{d}P.$$

Let $p', q' \in \mathbb{R}^+$ be the conjugates of p and q, respectively. Using Hölder's inequality, we have

$$\int_{\mathcal{D}} \left|u\varphi_{t_j t_j}\right|\,\mathrm{d}P \le \left(\int_{\mathcal{D}} |u|^q \varphi\,\mathrm{d}P\right)^{1/q} \left(\int_{\mathcal{D}} \varphi^{-q'/q} |\varphi_{t_j t_j}|^{q'}\,\mathrm{d}P\right)^{1/q'},$$

for $j = 1, 2$; $t_1 = t$, $t_2 = s$, and

$$\int_{\mathcal{D}} |u\Delta\varphi|\,\mathrm{d}P \le \left(\int_{\mathcal{D}} |u|^q \varphi\,\mathrm{d}P\right)^{1/q} \left(\int_{\mathcal{D}} \varphi^{-q'/q} |\Delta\varphi|^{q'}\,\mathrm{d}P\right)^{1/q'}.$$

We proceed analogously for the second equation of (2).

Let us set

$$\mathcal{B}_j(\vartheta) := \left(\int_{\mathcal{D}} \varphi^{-\vartheta'/\vartheta} |\varphi_{t_j t_j}|^{\vartheta'}\,\mathrm{d}P\right)^{1/\vartheta'},$$

and

$$\mathcal{A} := \left(\int_{\mathcal{D}} \varphi^{-\vartheta'/\vartheta} |\Delta\varphi|^{\vartheta'}\,\mathrm{d}P\right)^{1/\vartheta'},$$

where for $j = 1, 2$, $t_1 = t$, $t_2 = s$, $\vartheta = q$ if $j = 1$, $\vartheta = p$ if $j = 2$. Using the same change of variables as in Subsection 2.1, $t = R\tau$, $s = R\sigma$, and $x = Ry$, in the integrals of $\mathcal{B}_j(\vartheta)$, $j = 1, 2$, and $\mathcal{A}$, we obtain

$$\mathcal{B}_j(\vartheta) = R^{\frac{N+2-2\vartheta'}{\vartheta'}} \left(\int_\Omega \tilde{\varphi}^{-\vartheta'/\vartheta} |\tilde{\varphi}_{\tau_j\tau_j}|^{\vartheta'} \, \mathrm{d}\tilde{P} \right)^{1/\vartheta'},$$

and

$$\mathcal{A} = R^{\frac{N+2-2\vartheta'}{s'}} \left(\int_\Omega \tilde{\varphi}^{-\vartheta'/\vartheta} |\Delta_y \tilde{\varphi}|^{\vartheta'} \, \mathrm{d}\tilde{P} \right)^{1/\vartheta'}.$$

Whereupon,

$$\begin{aligned} \left(\int_{\Omega_R} |v|^p \varphi \, \mathrm{d}P \right)^{1-1/pq} &\leq \left(\mathcal{B}^1(p) + \mathcal{B}^2(p) + \mathcal{A}(p) \right)^{1/q} \cdot \left(\mathcal{B}^1(q) + \mathcal{B}^2(q) + \mathcal{A}(q) \right) \\ &\leq CR^{\frac{N+2-2p'}{qp'} + \frac{N+2-2q'}{q'}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \left(\int_{\Omega_R} |u|^q \varphi \, \mathrm{d}P \right)^{1-1/pq} &\leq \left(\mathcal{B}^1(q) + \mathcal{B}^2(q) + \mathcal{A}(q) \right)^{1/p} \cdot \left(\mathcal{B}^1(p) + \mathcal{B}^2(p) + \mathcal{A}(p) \right) \\ &\leq CR^{\frac{N+2-2q'}{pq'} + \frac{N+2-2p'}{p'}}. \end{aligned}$$

If

$$\frac{N+2-2p'}{qp'} + \frac{N+2-2q'}{q'} < 0, \quad \text{or} \quad \frac{N+2-2q'}{pq'} + \frac{N+2-2p'}{p'} < 0,$$

then taking the limit as $R \to +\infty$, we obtain the contradiction

$$0 < \int_\Omega |v|^p \, \mathrm{d}P \leq 0, \quad \text{or} \quad 0 < \int_\Omega |u|^q \, \mathrm{d}P \leq 0,$$

respectively; this ends the proof. □

In the case,

$$\frac{N+2-2p'}{qp'} + \frac{N+2-2q'}{q'} = 0, \quad \text{or} \quad \frac{N+2-2q'}{pq'} + \frac{N+2-2p'}{p'} = 0,$$

we conclude like in the case of a single equation.

2.3. Fractional two–time hyperbolic equations

· Basic definitions and properties on fractional calculus.

For the convenience of the reader, we recall some basic definitions and properties which will be useful in the sequel.

Definition 2.5. *The left– and right–sided Riemann–Liouville integrals of order α are defined as*

$$\left(I^\alpha_{0|t} f \right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, \mathrm{d}s, \quad t > 0, \tag{21}$$

$$\left(I^\alpha_{t|T} f \right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s) \, \mathrm{d}s, \quad t < s, \tag{22}$$

where Γ is the Euler gamma function.

Definition 2.6. *The left– and right–handed Riemann–Liouville fractional derivatives of order $n-1<\gamma<n$ for a function $f\in AC^n[0,T]:=\left\{f:[0,T]\to\mathbb{R}, D^{n-1}f\in AC[0,T]\right\}$, $n\in\mathbb{N}$ are defined by (see [20])*

$$(D^{\gamma}_{0|t}f)(t):=D^n(I^{n-\gamma}_{0|t}f)(t),\quad t>0; \tag{23}$$

$$(D^{\gamma}_{t|T}f)(t):=(-1)^nD^n\left(I^{n-\gamma}_{t|T}f\right)(t), \tag{24}$$

where $D=\frac{\mathrm{d}}{\mathrm{d}t}$. The analogous formulas for the left– and right–handed Caputo fractional derivative of order $n-1<\gamma<n$, for a function $f\in C^n[0,T]$ are:

$$\left({}^cD^{\gamma}_{0|t}f\right)(t):=(-1)^n(I^{n-\gamma}_{t|T}D^nf)(t),\quad t>0. \tag{25}$$

$$\left({}^cD^{\gamma}_{t|T}f\right)(t):=(-1)^n(I^{n-\gamma}_{0|t}D^nf)(t). \tag{26}$$

Furthermore, for every $f,g\in C([0,T])$, such that $D^{\alpha}_{0|t}f(t), D^{\alpha}_{t|T}g(t)$ exist and are continuous, for all $t\in[0,T]$, $0<\alpha<1$, we have the formula of integration by parts due to Love and Young [20]

$$\int_0^T\left(D^{\alpha}_{0|t}f\right)(t)g(t)\,\mathrm{d}t=\int_0^T f(t)\left(D^{\alpha}_{t|T}g\right)(t)\,\mathrm{d}t. \tag{27}$$

Note also that, for all $f\in AC^2[0,T]$, we have (see (2.30) and (2.31) in [20])

$$D^{1+\alpha}_{0|t}f=DD^{\alpha}_{0|t}f,\qquad -D.D^{\alpha}_{t|T}f=D^{1+\alpha}_{t|T}f, \tag{28}$$

where D is the usual time derivative.

We also have the formulas (see [20])

$$D^{\alpha}_{t|T}f(t)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{f(T)}{(T-t)^{\alpha}}-\int_t^T(T-t)^{-\alpha}f'(t)\,\mathrm{d}t\right]\quad\text{and}\quad {}^cD^{\alpha}_{0|t}f(t)=D^{\alpha}_{0|t}(f(t)-f(0)-tf'(0)) \tag{29}$$

linking the Riemann–Liouville derivative to the Caputo derivative.

Later on, we will use the following results, see [12].

If $\Phi_1(t)=\left(1-\frac{t^2}{T^2}\right)^l$, $t\geq0$, $T>0$, $l>>1$, then

$$D^{\gamma}_{t|T}\Phi_1(t)=-\frac{T^{-2l}}{\Gamma(1-\gamma)}\Sigma^l_{k=0}2^{l-k}C^l_kM_{lk}t^{l-k-1}(T-t)^{l+k-\gamma}[(l-k)T-(2l+1-\gamma)t], \tag{30}$$

where $M_{lk}=\Gamma(l+1)\Sigma^k_{n=0}C^k_n\dfrac{\Gamma(n+1-\beta)}{\Gamma(l+n+2-\beta)}$ and $C^k_n=\dfrac{l(l-1)(l-2)\cdots(l-k+1)}{k!}$.

$$\begin{aligned}D^{\alpha+1}_{t|T}\Phi_1(t)\ &=\ \frac{T^{-2l}}{\Gamma(1-\alpha)}\Sigma^l_{k=0}2^{l-k}C^l_kM_{lk}t^{l-k-2}(T-t)^{l+k-\alpha-1}\\ &\quad\times[(l-k)(l-k-1)T^2-2tT(l-k)(2l-\alpha)+(2l-\alpha)(2l-\alpha+1)t^2],\end{aligned} \tag{31}$$

$$\int_0^T tD^{\alpha+1}_{t|T}\left(1-\frac{t^2}{T^2}\right)^l\mathrm{d}t=\frac{T^{1-\alpha}}{\Gamma(1-\alpha)}\Sigma^l_{k=0}L_{\alpha k}C^l_k, \tag{32}$$

$$\int_0^T D^{\beta}_{t|T}\left(1-\frac{t^2}{T^2}\right)^l\mathrm{d}t=\frac{T^{1-\beta}}{\Gamma(1-\beta)}\Sigma^l_{k=0}L_{\beta k}C^l_k, \tag{33}$$

and

$$\int_0^T D^{\beta}_{t|T}\left(1-\frac{t^2}{T^2}\right)^l\mathrm{d}t\geq0, \tag{34}$$

where $L_{\gamma k}=\dfrac{\Gamma(l+1)\Gamma(k+1-\gamma)}{\Gamma(l-\gamma+k+2)}$.

2.4. Blowing–up solutions for a two–time fractional hyperbolic equation

In this section we consider problem (4).

We begin with the definition of a weak solution for (4).

We set the space Σ_f of functions $\Phi : \mathcal{D} \to (0, +\infty)$, such that Φ compactly supported in the space variable x and $\Phi(t, s; x) = 0$, $D^{\alpha_1}_{t|T}\varphi(t, s; x) = 0$, $t \geq T$, $\varphi(t, s; x) = D^{\alpha_2}_{s|T}\varphi(t, s; x) = 0$, $s \geq T$.

Definition 2.7. *Let $p > 1$ be a real number, and $0 < \alpha_1, \alpha_2 < 1$. A function $u := u(P)$ such that $u \in L^p_{loc}(\mathcal{D})$), is said to be a weak solution of (4)–(5) if*

$$\int_{\mathcal{D}} u(P)\Delta\varphi(P)\,\mathrm{d}P + \int_{\mathcal{D}} (u - u_{01}(s; x) - tu_{02}(s; x))D^{1+\alpha_1}_{t|T}\varphi(s; x)\,\mathrm{d}P_0$$
$$+ \int_{\mathcal{D}} (u - u_{10}(t; x) - su_{20}(t; x))D^{1+\alpha_2}_{s|T}\varphi(s; x)\,\mathrm{d}P_0 = \int_{\mathcal{D}} |u(P)|^p\varphi(P)\,\mathrm{d}P, \tag{35}$$

for every $\varphi \in \Sigma_f$.

Theorem 2.8. *If $p \leq 1 + \alpha/(N + 2 - \alpha)$, where $\alpha = \min\{\alpha_1, \alpha_2\}$, and*

$$\int_{\Omega} (u_{01}(s; x) + u_{02}(s; x))\Phi_0(x)\Phi_1(s)\,\mathrm{d}P_0 > 0, \tag{36}$$

and

$$\int_{\Omega} (u_{10}(t; x) + u_{20}(t; x))\Phi_0(x)\Phi_1(t)\,\mathrm{d}P_0 > 0, \tag{37}$$

are satisfied for every $\Phi_0, \Phi_1 \in \Sigma_f$, then there is no nontrivial global weak solution of problem (4)–(5).

Let us highlight that in the case $\alpha_1 = \alpha_2 = 2$, the result in Theorem 3 is coherent with that of Theorem 1.

Proof. Suppose, on the contrary, that some solution exists for all time $t > 0$. Let us suppose that φ is such that

$$\int_{\mathcal{D}} \varphi^{-p'/p}\left\{|D^{1+\alpha_1}_{t|T}\varphi|^{p'} + |D^{1+\alpha_2}_{s|T}\varphi|^{p'} + |\Delta\varphi|^{p'}\right\}\mathrm{d}P < \infty,$$

where $p + p' = pp'$.

Now, taking

$$\varphi(t, s; x) = \Phi_0(x)\Phi_1(t)\Phi_1(s),$$

and using the ϵ–Young inequality, we obtain the estimates

$$\int_{\mathcal{D}} |u|^p\varphi\,\mathrm{d}P + CT^{1-\alpha_1}\int_{\Omega} (u_{01}(s; x) + u_{02}(s; x))\,\Phi_0(x)\Phi_1(s)\,\mathrm{d}P_0 + CT^{1-\alpha_2}\int_{\Omega} (u_{10}(t; x) + u_{20}(t; x))\Phi_0(x)\Phi_1(t)\,\mathrm{d}P_0$$
$$\leq C\int_{\mathcal{D}} \varphi^{-p'/p}\left\{|D^{1+\alpha_1}_{t|T}\varphi|^{p'} + |D^{1+\alpha_2}_{s|T}\varphi|^{p'} + |\Delta\varphi|^{p'}\right\}\mathrm{d}P, \tag{38}$$

where $p + p' = pp'$, and C is a positive constant. Taking $\Phi_0(x) = \chi\left(|x|/T^{\alpha/2}\right)$, changing the variables $t = T\tau$, $s = T\sigma$, $x = T^{\alpha/2}y$, and taking account of the constraints (36) and (37), we obtain the estimate

$$\int_{\mathcal{D}} |u|^p\varphi\,\mathrm{d}P \leq CT^{-\alpha p' + N + 2}. \tag{39}$$

The remainder of proof is similar as in the previous situation and hence it is omitted. □

2.5. 2×2–Fractional Differential two–times Systems

In this section we only formulate the main result as its proof is similar to the previous ones.

Now we consider the system of equations (6)–(7).

Definition 2.9. *Let $p,q>1$ be two real numbers, and $u\in L^q_{loc}(\mathcal{D})$, $v\in L^p_{loc}(\mathcal{D})$. We say that (u,v) is a weak solution of (6)–(7) if, for every $\varphi\in\Sigma_f$,*

$$\int_{\mathcal{D}}|v(P)|^p\varphi(P)\,\mathrm{d}P+\mathcal{U}_{0,\phi}=\int_{\mathcal{D}}u(P)\left\{D^{1+\alpha_1}_{t|T}\varphi(P)+D^{1+\alpha_2}_{s|T}\varphi(P)-\Delta\varphi(P)\right\}\mathrm{d}P,$$

and

$$\int_{\mathcal{D}}|u(P)|^q\varphi(P)\,\mathrm{d}P+\mathcal{V}_{0,\varphi}=\int_{\mathcal{D}}v(P)\left\{D^{1+\beta_1}_{t|T}\varphi(P)+D^{1+\beta_2}_{s|T}\varphi(P)-\Delta\varphi(P)\right\}\mathrm{d}P,$$

where

$$\mathcal{U}_{0,\varphi}:=\int_\Omega u_{02}(s;x)\Phi_0(x)\Phi_1(s)\,\mathrm{d}P_0+\int_\Omega u_{10}(t;x)\Phi_0(x)\Phi_1(t)\,\mathrm{d}P_0,$$
$$\mathcal{V}_{0,\varphi}:=\int_\Omega v_{02}(s;x)\Phi_0(x)\Phi_1(s)\,\mathrm{d}P_0+\int_\Omega v_{10}(t;x)\Phi_0(x)\Phi_1(t)\,\mathrm{d}P_0.$$

We assume that

$$\mathcal{U}_{0,\varphi}>0,\quad\text{and}\quad\mathcal{V}_{0,\varphi}>0,\quad\text{for every}\quad\varphi\in\Sigma_f,\tag{40}$$

and also that all initial data belong to $L^1(\Omega)$.

Theorem 2.10. *Consider system (6)–(7) subject to the conditions (40). If*

$$\frac{N+2-\alpha p'}{qp'}+\frac{N+2-\alpha q'}{q'}\le 0\quad or\quad\frac{N+2-\beta q'}{pq'}+\frac{N+2-\beta p'}{p'}\le 0,$$

where $p+p'=pp'$ and $q+q'=qq'$, and $\alpha=\min\{\alpha_1,\alpha_2\}$, $\beta=\min\{\beta_1,\beta_2\}$, then there is no nontrivial global weak solution of (6)–(7).

Proof. We have

$$\int_{\mathcal{D}}|v(P)|^p\varphi(P)\,\mathrm{d}P+\mathcal{U}_{0,\phi}=\int_{\mathcal{D}}u(P)\left\{D^{1+\alpha_1}_{t|T}\varphi(P)+D^{1+\alpha_2}_{s|T}\varphi(P)-\Delta\varphi(P)\right\}\mathrm{d}P,$$

and

$$\int_{\mathcal{D}}|u(P)|^q\varphi(P)\,\mathrm{d}P+\mathcal{V}_{0,\varphi}=\int_{\mathcal{D}}v(P)\left\{D^{1+\beta_1}_{t|T}\varphi(P)+D^{1+\beta_2}_{s|T}\varphi(P)-\Delta\varphi(P)\right\}\mathrm{d}P.$$

Using Hölder's inequality, there exists $C>0$ such that

$$\int_{\mathcal{D}}|v(P)|^p\varphi(P)\,\mathrm{d}P+\mathcal{U}_{0,\phi}\le C\left(\int_{\mathcal{D}}u(P)^q\varphi(P)\,\mathrm{d}P\right)^{1/q}\left(\int_{\mathcal{D}}\varphi(P)^{-q'/q}\left\{|D^{1+\alpha_1}_{t|T}\varphi(P)|^{q'}+|D^{1+\alpha_2}_{s|T}\varphi(P)|^{q'}-|\Delta\varphi(P)|^{q'}\right\}\mathrm{d}P\right)^{1/q'},$$

and

$$\int_{\mathcal{D}}|u(P)|^p\varphi(P)\,\mathrm{d}P+\mathcal{V}_{0,\phi}\le C\left(\int_{\mathcal{D}}v(P)^q\varphi(P)\,\mathrm{d}P\right)^{1/q}\left(\int_{\mathcal{D}}\varphi(P)^{-q'/q}\left\{|D^{1+\alpha_1}_{t|T}\varphi(P)|^{q'}+|D^{1+\alpha_2}_{s|T}\varphi(P)|^{q'}-|\Delta\varphi(P)|^{q'}\right\}\mathrm{d}P\right)^{1/q'}.$$

Let us denote

$$\mathcal{A}(\alpha_1,\alpha_2,q)=\int_{\mathcal{D}}\varphi(P)^{-q'/q}\left\{|D^{1+\alpha_1}_{t|T}\varphi(P)|^{q'}+|D^{1+\alpha_2}_{s|T}\varphi(P)|^{q'}-|\Delta\varphi(P)|^{q'}\right\}\mathrm{d}P,$$

and

$$\mathcal{B}(\beta_1,\beta_2,p)=\int_{\mathcal{D}}\varphi(P)^{-q'/q}\left\{|D^{1+\alpha_1}_{t|T}\varphi(P)|^{q'}+|D^{1+\alpha_2}_{s|T}\varphi(P)|^{q'}-|\Delta\varphi(P)|^{q'}\right\}\,\mathrm{d}P.$$

If we set

$$I=\int_{\mathcal{D}}v(P)^q\varphi(P)\,\mathrm{d}P\qquad\text{and}\qquad J=\int_{\mathcal{D}}u(P)^q\varphi(P)\,\mathrm{d}P,$$

then we may write

$$J^q\le C\,I\,\mathcal{A}^{q/q'}(\alpha_1,\alpha_2,q),$$

and

$$I^p\le C\,J\,\mathcal{B}^{p/p'}(\beta_1,\beta_2,p),$$

as $\mathcal{U}_{0,\phi}\ge 0$, and $\mathcal{V}_{0,\phi}\ge 0$ by hypotheses.

Whereupon

$$J^{pq-1}\le C\,\mathcal{A}^{pq/q'}(\alpha_1,\alpha_2,q)\,\mathcal{B}^{p/p'}(\beta_1,\beta_2,p),$$

and

$$I^{pq-1}\le C\,\mathcal{A}^{q/q'}(\alpha_1,\alpha_2,q)\,\mathcal{B}^{pq/p'}(\beta_1,\beta_2,p).$$

Without loss of generality, we may assume

$$\beta_1<\beta_2,\qquad\text{and}\qquad\alpha_1<\alpha_2<\beta_2.$$

Choosing $\Phi_0=\chi(|x|^2/T^{2\sigma})$ where $2\sigma=\alpha_2$, $\Phi_1(t)$ and $\Phi_2(t)$ as before, we obtain the estimates

$$J^{pq-1}\le C\,T^{-2\sigma(pq+1)+(N+2)(pq-1)},$$

and analogously I^{pq-1}. Here also, we require $-2\sigma(pq+1)+(N+2)(pq-1)\le 0$, which is equivalent to

$$N+2\le\frac{\alpha_2(pq+1)}{pq-1},$$

to obtain a contradiction when we let $T\to+\infty$. □

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