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Approximate Analytical Solution of the Nonlinear System of Differential Equations Having Asymptotically Stable Equilibrium

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Abstract. The present paper is concerned with the purely analytic solutions of the highly nonlinear systems of differential equations possessing an asymptotically stable equilibrium. A methodology combined with the homotopy analysis method is proposed. The methodology involves proper introduction of an auxiliary linear operator and an auxiliary function during the implementation of the homotopy method so that it can yield uniformly valid solutions, not affected from the existing parameters or initial conditions. The technique is applied to the systems particularly appearing in mathematical biology. The obtained explicit analytical expressions for the solution generate results that compare excellently with the numerically computed ones.

1. Introduction

Finding analytical representation of the solutions of nonlinear differential equations has been a challenging problem over centuries. If the physical phenomenon at hand is modeled by a nonlinear system of differential equations, obtaining the exact solution becomes even a worse problem. Although several numerical techniques are available to serve in today's computer world, looking for analytical means for the solution has been an active research area in recent years.

There are some known simple nonlinear systems of differential equations that can be solved exactly [1]. On the other hand, specifically when the nonlinear equations exhibit a naturel strong nonlinearity, it is real hard to achieve an analytical solution. This brought attention to the researchers to seek alternative methods to find approximate solutions. One such method is the recently favorable homotopy analysis method first published in 1992 by Liao [2]. Since then it has been tested on many nonlinear problems, see amongst them [3, 4], [5] and the recent book by Liao [6]. The convergence issue of the homotopy method was successfully outlined in this book, see Chapter 5 and in the papers [7, 8].

We in the present paper use homotopy analysis technique for the solution of nonlinear system of differential equations having an asymptotically stable equilibrium solution. We particularly direct our attention to the systems arising from the mathematical modeling of the real world problems in biology, such as the Lotka-Volterra equations for the logistic determination of population in ecological system [9, 10], the nonlinear systems representing the epidemiological diseases in a community [11], [12], [13], [14] [15]

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and equations modeling the plant-herbivore interactions in ecology [16] and [17]. We in priority admit that such models were generally solved by means of numerical methods, like the Runge-Kutta integration or finite-difference schemes. However, instead, the present interest is to locate uniformly valid approximate analytic solutions regardless of the values of initial conditions, or the coefficients existing in the equation. We are also aware of the fact that such systems have already been treated before by the homotopy method, but without any methodology as presented here. The proposed homotopy analysis technique, when applied to the popular equations in mathematical biology, results in approximate analytic formulas regarding the solutions which asymptotically approach the numerical or full solutions.

2. The Homotopy Analysis Method

In this section we propose a methodology for solving the systems of highly nonlinear differential equations within the perspective of homotopy analysis method, see [2, 18]. To serve to this purpose, let us consider the nonlinear system of differential equations

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \quad = \quad N(\mathbf{u}),\ \mathbf{u}(0) = \mathbf{A}, \tag{1}$$

where $\mathbf{u}(t)$ is an unknown vector function, N is a given nonlinear function of $\mathbf{u}(t)$ and $\mathbf{A}$ is a prescribed constant vector. Equation of the form (1) appears in many engineering applications as well as in mathematical biology and dynamical systems.

We initially assume that system (1) has at least an asymptotically stable critical point, say,

$$\mathbf{u} = \mathbf{u}^*.$$

A homotopy for system (1) is then constructed via the homotopy analysis method in the following form

$$(1-p)L[\mathbf{u}-\mathbf{u}_0] + phH(t)[\mathbf{u}' - N(\mathbf{u})] \quad = \quad 0, \tag{2}$$

where $p \in [0, 1]$ is an embedding parameter, h is a constant to control the convergence rate of the subsequent iterations, $H(t)$ is an auxiliary function to adjust the solution in a required shape (if, for instance, iterative solutions contain secular terms, then $H(t)$ might be chosen to avoid the appearance of such terms). The homotopy system (2) is subject to the initial conditions

$$\mathbf{u}(0, p) - \mathbf{A} = 0. \tag{3}$$

Moreover, the auxiliary linear differential operator L in (2) is assigned to be in the form

$$L \quad = \quad \frac{\mathrm{d}}{\mathrm{d}t} + c, \tag{4}$$

where c is a positive constant to be determined later, so that L has the property

$$L(\mathbf{C}_1 e^{-ct}) = 0,$$

with $\mathbf{C}_1$ being an arbitrary constant vector. Furthermore, $\mathbf{u}_0$ in (2) is an initial approximation to the solution of (1), whose appropriate form, taking into account of the mathematical structure of the equilibrium point $\mathbf{u}^*$, may be given by

$$\mathbf{u}_0 = (A - \mathbf{u}^*)e^{-ct} + \mathbf{u}^*. \tag{5}$$

We should remark here that, as Liao in [18] emphasized, the flexibility and freedom in the selection of auxiliary variables h, $H(t)$, L and u_0 are great advantages of the homotopy approach in order for adjusting the solution and controlling the rate of convergence. However, it should be stressed that the freedom does not mean that the resulting solutions to system (1) are different. On the contrary, as Liao proved in [18], alternative presentation of the unique solution is made possible within the approach. In addition to this, since the system (1) has an asymptotically stable critical point at $\mathbf{u}*$, the corresponding linearized system has a Jacobian matrix J whose real eigenvalues are all negative. Thus, despite the fact that it is not obligatory, the value of c can be chosen in the manner (note that the value of c is not unique in general)

$$c \approx max\{\text{Eigenvalues of } J\}, \tag{6}$$

or around this point. Such a choice of c plays a crucial role in the determination of the solutions as will be readily justified later.

We observe that $p = 0$ gives the initial approximation $\mathbf{u}_0(t) = \mathbf{u}(t, 0)$, whereas $p = 1$ results in the exact solution $\mathbf{u}(t) = \mathbf{u}(t, 1)$ to equation (1). The kth-order equations regarding the deformation are obtained by successive differentiation of (2) in the following manner

$$L(\mathbf{u}_k - \kappa_k \mathbf{u}_{k-1}) = -hH(t)\mathbf{R}_k, \tag{7}$$

complemented with the initial conditions

$$\mathbf{u}_k(0) = 0. \tag{8}$$

κ_k in (7) is defined by

$$\kappa_k = \begin{cases} 0, & k \le 1, \\ 1, & k > 1, \end{cases}$$

and $\mathbf{R}_k$ is given depending upon the shape of the nonlinear term N in (1), which will be calculated separately for each equation to be treated soon.

By means of the Taylor expansion, it is straightforward to get

$$\mathbf{u}(t) = \mathbf{u}_0(t) + \sum_{k=1}^{\infty} \mathbf{u}_k(t), \tag{9}$$

where $\mathbf{u}_k$ are defined by $\mathbf{u}_k = \frac{1}{k!}\frac{\partial \mathbf{u}}{\partial p}|_{p=0}$. A close inspection of (9) reveals the fact that solution of system (1) was obtained in an analytic form which can be improved by computing as many terms in the solution series as required.

3. Application to Systems

Three popular systems in mathematical biology are treated in this section by means of the technique as outlined in §2. The accuracy is measured via

$$\int_0^{\infty} |\mathbf{u}(t) - \mathbf{u}_e(t)| dt, \tag{10}$$

where $\mathbf{u}_e(t)$ stands for the numerical solution. Moreover, the auxiliary parameter h is fixed as -1 for the examples considered, although this is not an obligation and an optimal value of h can always be found as described in [19] that was recently implemented in [20–25] amongst many others. We instead concentrate on the methodology combined with the homotopy analysis method.

3.1. A Lotka-Volterra system

Here we consider a specific case of the well-known Lotka-Volterra system

$$\begin{aligned} x' &= 14x - 2x^2 - xy, \quad x(0) = a, \\ y' &= 16y - 2y^2 - xy, \quad y(0) = b, \end{aligned} \tag{11}$$

that may physically represent the logistic population interaction and the predator-prey model of a competing species $x(t)$ and $y(t)$ within time t in a population. It is straightforward to see that system (11) has four critical points; (0,0), (0,8), (7,8) and (4,6) respectively. Moreover, the point (4,6) constitutes an asymptotically stable equilibrium solution of (11), that will balance the population without none of being extinct. The eigenvalues of the Jacobian matrix around this point are simply $-2(5 \pm \sqrt{7})$, so the value of c for the linear operator in (4) can be fixed as $c = 6$.

Now let's look for the solution of (11) together with the prescribed initial conditions $(a, b) = (8, 1)$. Taking all these into consideration, the initial approximations (5) take the form

$$\begin{aligned} x_0 &= 4e^{-6t} + 4, \\ y_0 &= -5e^{-6t} + 6. \end{aligned} \tag{12}$$

Moreover, $\mathbf{R}_k = (R_k^x, R_k^y)$ in equation (7) can be written as

$$\begin{aligned} R_k^x &= x'_{k-1} - 14x_{k-1} + \sum_{j=0}^{k-1} x_j(2x_{k-1-j} + y_{k-1-j}), \\ R_k^y &= y'_{k-1} - 16y_{k-1} + \sum_{j=0}^{k-1} y_j(2y_{k-1-j} + x_{k-1-j}). \end{aligned} \tag{13}$$

In addition to this, if the solution sought for (11) is preferred in purely exponential form, then it is better to choose the auxiliary function $H(t) = e^{-t}$ so that no secular terms will appear in the next-order solutions. The homotopy analysis method (2-9) then generates the subsequent result at the approximate level $k = 3$

$$x = 4 + \tfrac{318e^{-27t}}{343} - \tfrac{5204e^{-21t}}{245} + \tfrac{9042e^{-20t}}{343} + \tfrac{349e^{-15t}}{21} - \tfrac{3426e^{-14t}}{49} + \tfrac{20064e^{-13t}}{343} - 68e^{-9t} + \tfrac{726e^{-8t}}{7} - \tfrac{3579e^{-7t}}{49} + \tfrac{154774e^{-6t}}{5145},$$

$$y = 6 + \tfrac{1950e^{-27t}}{343} - \tfrac{5503e^{-21t}}{245} - \tfrac{1482e^{-20t}}{343} + \tfrac{4195e^{-15t}}{42} - \tfrac{19905e^{-14t}}{196} + \tfrac{15783e^{-13t}}{343} - 104e^{-9t} + \tfrac{765e^{-8t}}{7} - \tfrac{1287e^{-7t}}{98} - \tfrac{419743e^{-6t}}{20580},$$

Figure 1 demonstrates this and other homotopy solutions that graphically match perfectly with the numerical solution.

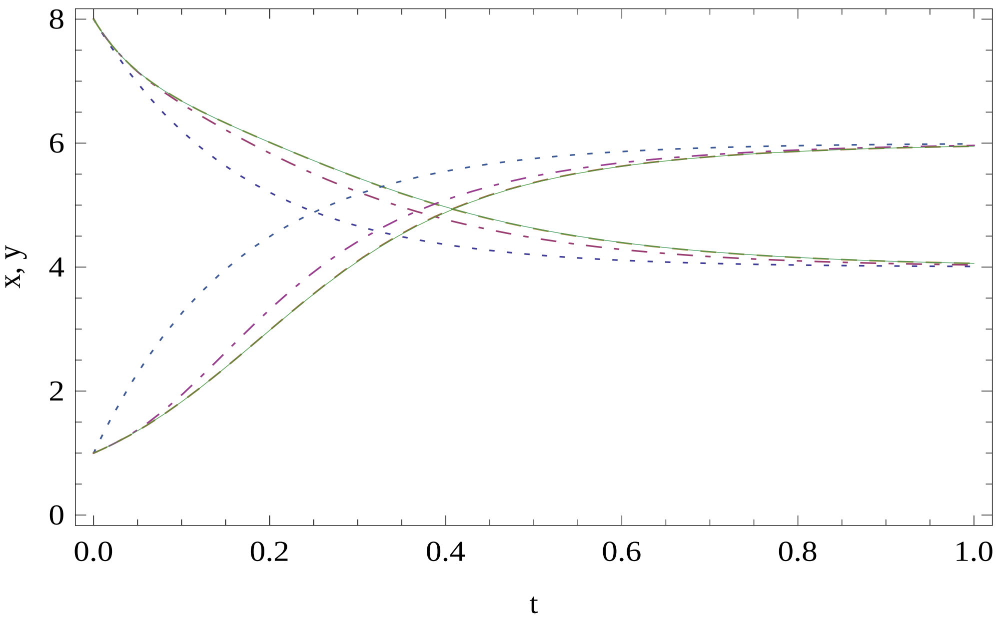

Figure 1: Solution $(x(t), y(t))$ of Lotka-Volterra equation (11): the numerical solution (straight curve), the homotopy solution of $k = 15$ (dashed curve), the homotopy solution of $k = 3$ (dot-dashed curve) and the initial approximation (dotted curve).

Table 1 gives error (10) occurred at different orders. As expected, increasing iteration level in the homotopy method helps the solution quickly to recover the numerical one.

$k=3$	$k=5$	$k=10$	$k=15$	$k=20$	$k=25$
0.176608	0.085831	0.0186456	0.00431826	0.00118063	0.00051083

Table 1: The errors (10) for problem (11) at different truncation levels.

3.2. *A plant-herbivore population dynamic system*

As a second example we consider the initial-value system

$$\begin{aligned} x' &= x(1-2x) - x^2 y, \quad x(0) = a, \\ y' &= x^2 y - y, \qquad\quad y(0) = b, \end{aligned} \tag{14}$$

that may represent physically the dynamics of a plant-herbivore interaction, such as that of a mammalian browser and its plant forage species, see for instance [17]. Among the four critical points, (1/2,0) is the asymptotically stable one. The eigenvalues of the Jacobian matrix around this point can be computed as $(-\frac{3}{4}, -1)$, hence allowing c to be chosen as $c = 3/4$ for this example.

If we explore the solution of (14) together with the specified initial conditions $(a, b) = (1, 1/2)$, the initial guesses (5) are written by

$$\begin{aligned} x_0 &= \frac{1}{2}e^{-3t/4} + \frac{1}{2}, \\ y_0 &= \frac{1}{2}e^{-3t/4}. \end{aligned} \tag{15}$$

Moreover, distinct from the Lotka-Volterra system (11), $\mathbf{R}_k = (R_k^x, R_k^y)$ in equation (7) should take the form

$$\begin{aligned} R_k^x &= x'_{k-1} - x_{k-1} + \sum_{j=0}^{k-1} 2x_j x_{k-1-j} + \sum_{j=0}^{k-1}\sum_{i=0}^{j} x_i x_{j-i} y_{k-1-j}, \\ R_k^y &= y'_{k-1} + y_{k-1} + \sum_{j=0}^{k-1}\sum_{i=0}^{j} x_i x_{j-i} y_{k-1-j}. \end{aligned} \tag{16}$$

As for the Lotka-Volterra system (11), if the solution to (14) is sought in purely exponential form, then it is better to choose the auxiliary function $H(t) = e^{-3t/4}$ so that no secular terms will appear in the next-order solutions. The homotopy analysis method (2-9) then generates an approximate analytic solution, whose third-order correspondence is

$$x = \frac{1}{2} - \frac{e^{-15t/2}}{8748} + \frac{31e^{-27t/4}}{6480} + \frac{407e^{-6t}}{4536} + \frac{2107e^{-21t/4}}{7290} - \frac{8}{45}e^{-9t/2} - \frac{311}{648}e^{-15t/4} + \frac{10253e^{-3t}}{29160} + \frac{7}{81}e^{-9t/4} + \frac{92}{135}e^{-3t/2} - \frac{422797e^{-3t/4}}{1224720}$$

$$y = \frac{e^{-15t/2}}{8748} - \frac{53e^{-27t/4}}{19440} - \frac{13e^{-6t}}{360} - \frac{4}{729}e^{-21t/4} + \frac{23}{243}e^{-9t/2} - \frac{1}{8}e^{-15t/4} - \frac{653e^{-3t}}{29160} - \frac{121e^{-9t/4}}{2430} + \frac{22631e^{-3t/4}}{34992}.$$

This third-order solution, fifteenth-order solution, initial approximation (15) and the numerical solution are compared in Figure 2.

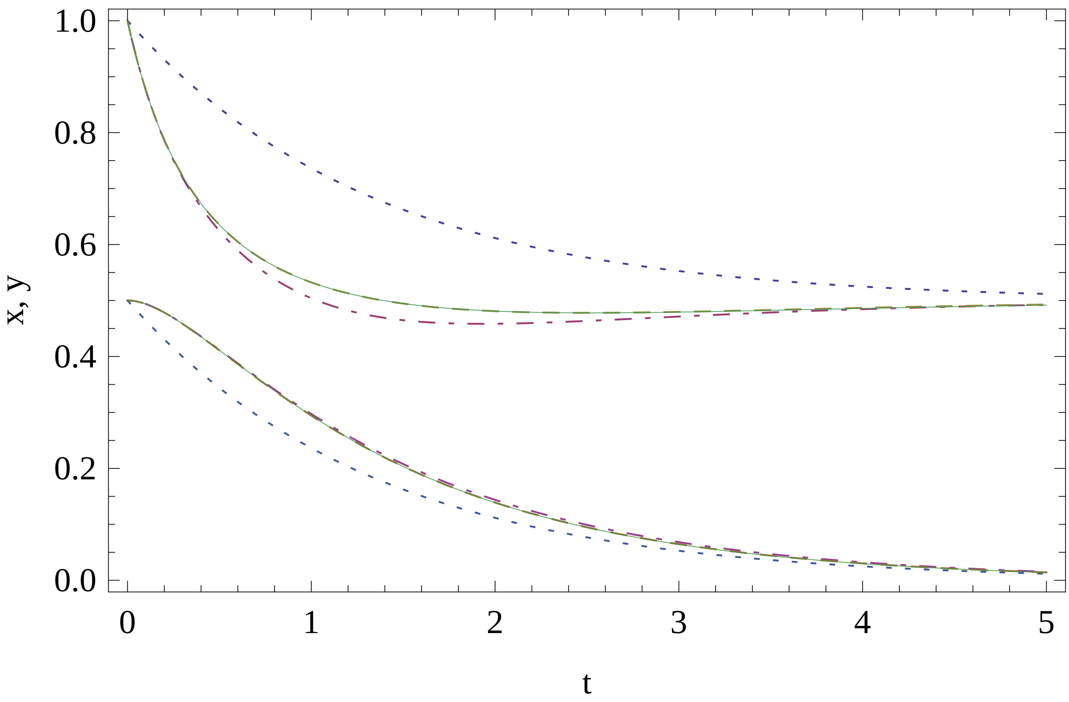

Figure 2: Solution $(x(t), y(t))$ of population equation (14): the numerical solution (straight curve), the homotopy solution of $k = 15$ (dashed curve), the homotopy solution of $k = 3$ (dot-dashed curve) and the initial approximation (dotted curve).

A fair agreement between the fifteenth-order homotopy solution and the numerical solution can be evidently observed, valuing such homotopy approximations. The total errors accumulated at various orders of homotopy approximations are tabulated in Table 2. Again the error decays, but slower than the previous case.

$k = 3$	$k = 5$	$k = 10$	$k = 15$	$k = 20$	$k = 25$
0.0660502	0.0281901	0.0130761	0.00488995	0.00360455	0.00258002

Table 2: The errors (10) for problem (14) at different truncation levels.

3.3. *A system for some infectious epidemiological diseases*

In this final demonstration, the following third-order system is considered

$$\begin{aligned} y' &= y(1-\nu-\theta-y-z+\theta y-(\nu+\gamma)q), \quad & y(0)=a, \\ q' &= (1+q)(\theta y-(\nu+\gamma)q), & q(0)=b, \\ z' &= \gamma q-\nu z+z(\theta y-(\nu+\gamma)q), & z(0)=c, \end{aligned} \tag{17}$$

that may represent physically recurrent outbreaks of some epidemic diseases such as influenza and measles, see for instance [11] and [12]. Within this concept, in the above system, $y(t)$, $q(t)$ and $z(t)$ denote the infected (non isolated) individuals, the isolated individuals (in quarantine) and the recovered (immune) individuals in a population respectively. The appearing parameters are due to interaction and contact rate among the portions. The asymptotically stable critical point of the system (17) can be worked out exactly as

$$(1-\theta-\nu-\frac{\gamma\theta-\gamma\theta^2-\gamma\theta\nu}{\gamma\theta+\gamma\nu+\nu^2},-\frac{\theta\nu(-1+\theta+\nu)}{\gamma\theta+\gamma\nu+\nu^2},\frac{\gamma\theta-\gamma\theta^2-\gamma\theta\nu}{\gamma\theta+\gamma\nu+\nu^2}).$$

The values of the parameters are taken as respectively, $\nu = 1/5$, $\theta = 1/5$ and $\gamma = 1/2$. The real eigenvalue of the Jacobian matrix around this point is -0.7529, so c for L in (4) can be fixed as $c = 3/4$.

Now, if we seek the solution of (17) together with the initial conditions $(a, b, c) = (7/10, 0, 0)$, the initial approximations (5) take the form

$$\begin{aligned} y_0 &= 7(1+e^{-3t/4})/20, \\ q_0 &= (1-e^{-3t/4})/10, \\ z_0 &= (1-e^{-3t/4})/4. \end{aligned} \tag{18}$$

Moreover, $\mathbf{R}_k = (R_k^y, R_k^q, R_k^z)$ in equation (7) can be written as

$$\begin{aligned} R_k^y &= y'_{k-1}+(-1+\nu+\theta)y_{k-1}+\sum_{j=0}^{k-1} y_j(z_{k-1-j}+(1-\theta)y_{k-1-j}+(\nu+\gamma)q_{k-1-j}), \\ R_k^q &= q'_{k-1}-\theta y_{k-1}+(\nu+\gamma)q_{k-1}+\sum_{j=0}^{k-1} q_j(-\theta y_{k-1-j}+(\nu+\gamma)q_{k-1-j}), \\ R_k^z &= z'_{k-1}-\gamma q_{k-1}+\nu z_{k-1}+\sum_{j=0}^{k-1} z_j(-\theta y_{k-1-j}+(\nu+\gamma)q_{k-1-j}). \end{aligned} \tag{19}$$

Taking $H(t) = 1$ in for the current problem, the homotopy analysis method (2-9) then produces an approximate analytic solution, whose third-order correspondence is

$$\begin{aligned} y &= \tfrac{7}{20}-\tfrac{3976e^{-3t}}{6328125}+\tfrac{66941e^{-9t/4}}{2109375}+\tfrac{403354e^{-3t/2}}{2109375}+\tfrac{3231739e^{-3t/4}}{25312500}+\tfrac{50197e^{-9t/4}t}{2812500}+\tfrac{139069e^{-3t/2}t}{703125} \\ &\quad +\tfrac{2975273e^{-3t/4}t}{11250000}+\tfrac{35637e^{-3t/2}t^2}{500000}+\tfrac{187607e^{-3t/4}t^2}{2000000}+\tfrac{104111e^{-3t/4}t^3}{8000000}, \\ q &= \tfrac{6377e^{-3t}}{12656250}-\tfrac{322e^{-9t/4}}{703125}-\tfrac{9149e^{-3t/2}}{156250}+\tfrac{370244e^{-3t/4}}{6328125}+\tfrac{1}{10}\left(1-e^{-3t/4}\right)+\tfrac{826e^{-9t/4}t}{234375} \\ &\quad -\tfrac{130249e^{-3t/2}t}{2812500}+\tfrac{361789e^{-3t/4}t}{5625000}-\tfrac{9163e^{-3t/2}t^2}{1500000}+\tfrac{6569e^{-3t/4}t^2}{200000}+\tfrac{69779e^{-3t/4}t^3}{12000000}, \\ z &= \tfrac{6377e^{-3t}}{5062500}-\tfrac{6307e^{-9t/4}}{1687500}+\tfrac{23359e^{-3t/2}}{562500}-\tfrac{197687e^{-3t/4}}{5062500}+\tfrac{1}{4}\left(1-e^{-3t/4}\right)-\tfrac{679e^{-9t/4}t}{562500} \\ &\quad +\tfrac{3549e^{-3t/2}t}{125000}-\tfrac{419201e^{-3t/4}t}{2250000}+\tfrac{539e^{-3t/2}t^2}{200000}-\tfrac{467e^{-3t/4}t^2}{16000}+\tfrac{813e^{-3t/4}t^3}{1600000} \end{aligned}$$

Figure 3 and Table 3 are to display how the homotopy results match well with the numerical one. Indeed, the error decays considerably for the Lotka-Volterra system (17) with increasing truncation level of homotopy series.

$k=3$	$k=5$	$k=10$	$k=15$	$k=20$	$k=25$
0.279821	0.24643	0.0634668	0.0186309	0.00621743	0.00214597

Table 3: The errors (10) for problem (17) at different truncation levels.

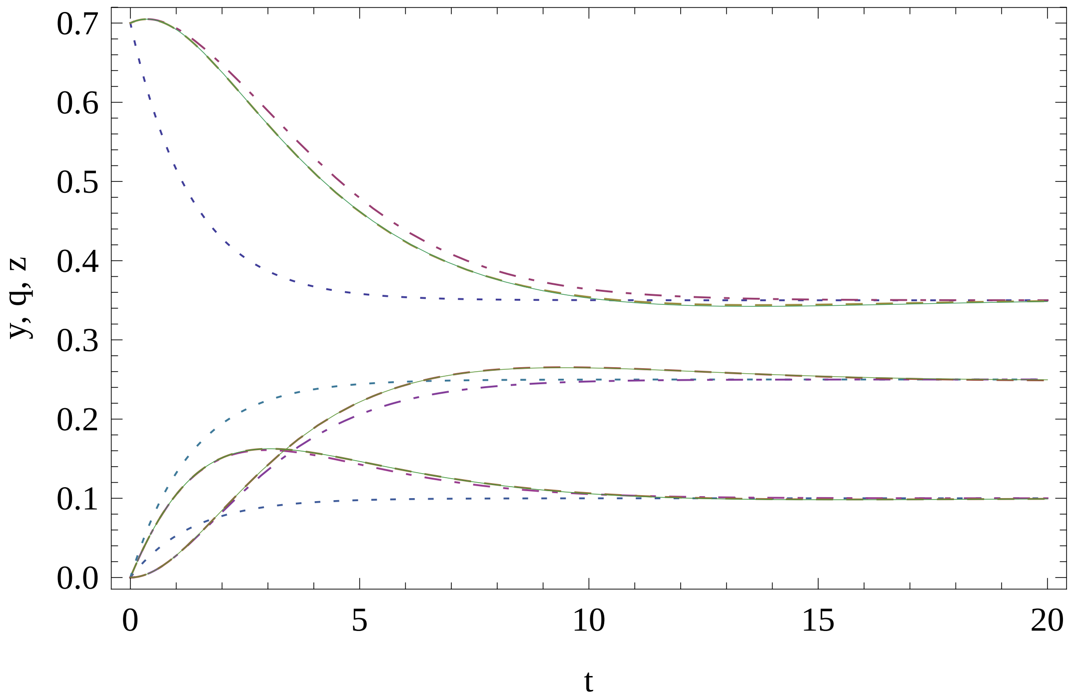

Figure 3: Solution $(y(t), q(t), z(t))$ of population equation (17): the numerical solution (straight curve), the homotopy solution of $k = 15$ (dashed curve), the homotopy solution of $k = 3$ (dot-dashed curve) and the initial approximation (dotted curve).

4. Concluding Remarks

The homotopy analysis method is taken into account in the present study to obtain approximate analytic solutions of highly nonlinear system of differential equations. It is systematically shown that proper selection of the auxiliary linear operator and the other auxiliary parameters is essential to get the exponentially decaying type of solutions for the systems possessing asymptotically stable equilibrium point. Examples from the open literature in mathematical biology justify the success of the adopted approach. Hence, similar nonlinear systems in different fields can also be safely treated via the procedure outlined here. However, whenever the nonlinear dynamic systems are chaotic, the present approach may not be applicable.

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