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On a Generalized Quarter Symmetric Metric Recurrent Connection

Wanxiao Tang[a], Tal Yun Ho[b], Kwang Il Ri[c], Fengyun Fu[d], Peibiao Zhao[e,1]

[a]*Dept. of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, P. R. China*
[b]*Faculty of Mathematics, Kim I1 Sung University, Pyongyang, DPRK*
[c]*Faculty of Education, Kim Hyong Jik Normal University, Pyongyang, DPRK*
[d]*School of Mathematics and Statistics, Guangdong University of Finance and Economics, Guangzhou 510320, P. R. China*
[e]*Dept. of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, P. R. China*

Abstract. We introduce a generalized quarter-symmetric metric recurrent connection and study its geometrical properties. We also derive the Schur's theorem for the generalized quarter-symmetric metric recurrent connection.

1. Introduction

The concept of the semi-symmetric connection was introduced by Friedman and Schouten in [4] for the first time. Hayden in [13] introduced the metric connection with torsion, and Yano in [15] defined a semi-symmetric metric connection and studied its properties. De, Han and Zhao in [1] recently studied the semi-symmetric no-metric connection. A quarter-symmetric connection in [5] was defined and studied. Afterwards, several types of a quarter-symmetric metric connection were studied ([3, 9, 14, 16–18]). On the other hand, the Schur's theorem of a semi-symmetric non-metric connection is well known ([10, 11]) based only on the second Bianchi identity. A semi-symmetric metric connection that is a geometrical model for scalar-tensor theories of gravitation was studied ([2]) and the Amari-Chentsov connection with metric recurrent property was also studied ([12]). Recently, Han, Fu and Zhao in [7, 8] further studied the similar topics in sub-Riemannian manifolds.

Based on the previous researches we define newly in this note the generalized quarter-symmetric metric recurrent connection and study its properties. And the Schur's theorem of the generalized quarter-symmetric metric recurrent connection is posed and several types of the generalized quarter-symmetric metric recurrent connections with constant curvature are discovered.

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[1] Corresponding author.
Email addresses: wanxiaotang92@163.com (Wanxiao Tang), ffy1984229@163.com (Fengyun Fu), pbzhao@njust.edu.cn (Peibiao Zhao)

2. A Generalized Quarter-symmetric Metric Recurrent Connection

Let (M, g) be a Riemannian manifold (dim $M \geq 2$), g be the Riemannian metric on M, and $\overset{\circ}{\nabla}$ be the Levi-Civita connection with respect to g. Let $T(M)$ denote the collection of all vector fields on M.

Definition 2.1. *A connection $\overset{R}{\nabla}$ is called a quarter-symmetric metric recurrent connection, if it satisfies*

$$(\overset{R}{\nabla}_Z g)(X, Y) = 2\omega(Z)g(X, Y), \overset{R}{T}(X, Y) = \pi(Y)\varphi(X) - \pi(X)\varphi(Y), \tag{1}$$

where φ is a $(1, 1)$-type tensor field, and ω, π are 1-form respectively. If $\varphi(X) = X$, then $\overset{R}{\nabla}$ is a semi-symmetric metric recurrent connection studied in [17].

Definition 2.2. *A linear connection ∇ is called a generalized quarter-symmetric metric recurrent connection, if it satisfies*

$$\begin{cases} (\nabla_Z g)(X, Y) = -2(t-1)\omega(Z)g(X, Y) - t\omega(X)g(Y, Z) - t\omega(Y)g(Z, X), \\ T(X, Y) = \pi(Y)\varphi(X) - \pi(X)\varphi(Y). \end{cases} \tag{2}$$

where $t \in R$.

Remark 2.3. *By (2), it is obvious that there holds the following*

When $t = 0$, then the generalized quarter-symmetric metric recurrent connection ∇ is a quarter-symmetric metric recurrent connection $\overset{R}{\nabla}$;

When $\omega = 0$, then ∇ is a quarter-symmetric metric connection([9]);

When $t = 1$ and $\varphi(X) = X$, then ∇ is a semi-symmetric non-metric connection;

When $t = 2$ and $\varphi(X) = 0$, then ∇ is a special type of the Amari-Chentsov connection([12]);

When $\omega = 0$ and $T = 0$, then ∇ is Levi-Civita connection $\overset{0}{\nabla}$.

Let (x^i) be the local coordinate, then $g, \overset{0}{\nabla}, \nabla, \omega, \varphi, \pi, T$ have the local expressions, $g_{ij}, \{^k_{ij}\}, \Gamma^k_{ij}, \omega_i, \pi_i, \varphi^j_i, T^j_i$, respectively. At the same time the expression (2) can be rewritten as

$$\begin{cases} \nabla_k g_{ij} = -2(t-1)\omega_k g_{ij} - t\omega_i g_{jk} - t\omega_j g_{ki}, \\ T^k_{ij} = \pi_j \varphi^k_i - \pi_i \varphi^k_j. \end{cases} \tag{3}$$

The coefficient of ∇ is given as

$$\Gamma^k_{ij} = \{^k_{ij}\} + (t-1)\omega_i \delta^k_j + (t-1)\omega_j \delta^k_i + g_{ij}\omega^k + \pi_j U^k_i - \pi_i V^k_j - U_{ij}\pi^k, \tag{4}$$

where $U_{ij} = \frac{1}{2}(\varphi_{ij} + \varphi_{ji})$, $V_{ij} = \frac{1}{2}(\varphi_{ij} - \varphi_{ji})$.

From (4), the curvature tensor of ∇, by a direct computation, is

$$\begin{aligned} R^l_{ijk} = \; & K^l_{ijk} + \delta^l_j a_{ik} - \delta^l_i a_{jk} + g_{jk} b^l_i - g_{ik} b^l_j + U^l_j c_{ik} - U^l_i c_{jk} + U_{ik} c^l_j - U_{jk} c^l_i \\ & + U^l_{ij}\pi_k - U_{ijk}\pi^l - V^l_k \pi_{ij} + V^l_{jk}\pi_i - V^l_{ik}\pi_j + (t-1)\delta^l_k \omega_{ij} + T^l_{ij}\omega_k \\ & + t(\delta^l_j \pi_i - \delta^l_i \pi_j) V^p_k \omega_p, \end{aligned} \tag{5}$$

where K^l_{ijk} is the curvature tensor of the Levi-Civita connection $\overset{\circ}{\nabla}$, and the other notations are given as follows

$$\begin{aligned}
a_{ik} &= (t-1)[\overset{0}{\nabla}_i\omega_k - (t-1)\omega_i\omega_k + g_{ik}\omega^p\omega_p + U_{ik}\omega_p\pi^p - U_i^p\omega_p\pi_k],\\
b_{ik} &= \overset{0}{\nabla}_i\omega_k + \omega_i\omega_k + U_{ik}\omega^p\omega^p - U_{ip}\omega^p\pi_k,\\
c_{ik} &= \overset{0}{\nabla}_i\omega_k + \pi_i\omega_k - U_k^p\pi_p\pi_k + \frac{1}{2}U_{ik}\pi^p\pi_p,\\
U_{ij}^l &= \overset{0}{\nabla}_iU_j^l - \overset{0}{\nabla}_jU_i^l,\\
\omega_{ij} &= \overset{0}{\nabla}_i\omega_j - \overset{0}{\nabla}_j\omega_i,\\
\pi_{ij} &= \overset{0}{\nabla}_i\pi_j - \overset{0}{\nabla}_j\pi_i,\\
V_{ik}^l &= \overset{0}{\nabla}_iV_k^l + V_i^l\omega_k - V_{ik}\omega^l + U_{ik}V_p^l\pi^p + U_i^lV_k^p\pi_p - U_i^pV_p^l\pi_k - U_{ip}V_k^p\pi^l\\
&\quad - \delta_i^lV_k^p\omega_p - g_{ik}V_p^l\omega^p.
\end{aligned} \tag{6}$$

From (4), the coefficient of dual connection $\overset{*}{\nabla}$([6]) of the generalized quarter-symmetric metric recurrent connection ∇ is

$$\overset{*}{\Gamma}{}_{ij}^k = \{^k_{ij}\} - (t-1)\omega_i\delta_j^k - \omega_j\delta_i^k - (t-1)g_{ij}\omega^k + \pi_jU_i^k - \pi_iV_j^k - U_{ij}\pi^k, \tag{7}$$

by using the expression (7), the curvature tensor of dual connection $\overset{*}{\nabla}$ is

$$\begin{aligned}
\overset{*}{R}{}_{ijk}^l = {} & K_{ijk}^l + \delta_i^lb_{jk} - \delta_j^lb_{ik} + g_{ik}a_i^l - g_{jk}a_i^l + U_j^lc_{ik} - U_i^lc_{jk} + U_{ik}c_j^l - U_{jk}c_i^l\\
& + U_{ij}^l\pi_k - U_{ijk}\pi^l - V_k^l\pi_{ij} + V_{jk}^l\pi_i - V_{ik}^l\pi_j - (t-1)\delta_k^l\omega_{ij} - tT_{ijk}\omega_l\\
& + t(g_{jk}\pi_i - g_{ik}\pi_j)V_p^l\omega^p.
\end{aligned} \tag{8}$$

Theorem 2.4. *For a Riemannian manifold (M, g), if a 1-form ω is a closed form, then the semi-Ricci curvature tensor R_{ji}^s of the generalized quarter-symmetric metric recurrent connection ∇ is zero, namely*

$$R_{ji}^s = 0, \tag{9}$$

where R_{ji}^s is said to be the semi-Ricci curvature tensor of ∇ defined by $R_{ji}^s = R_{ji\alpha}^{\alpha} = g^{\alpha\beta}R_{ji\alpha\beta}$, the (classical) Ricci curvature tensor of ∇ is defined as $R_{ji} = R_{\alpha ji}^{\alpha} = g^{\alpha\beta}R_{\alpha ji\beta}$.

Proof. Contracting the indices k and l of the expression (5), then we obtain

$$\begin{aligned}
R_{ij}^s = {} & \overset{0}{K}{}_{ij}^s + a_{ij} - a_{ji} + b_{ij} - b_{ji} + U_j^kc_{ik} - U_i^kc_{jk} + U_{ik}c_k^j - U_{jk}c_i^k\\
& + U_{ij}^k\pi_k - U_{ijk}\pi^k - V_k^k\pi_{ij} + V_{jk}^k\pi_i - V_{ik}^k\pi_j + (t-1)n\omega_{ij} + tT_{ij}^k\omega_k\\
& + t(V_j^p\pi_i - V_i^p\pi_j)\omega_p,
\end{aligned} \tag{10}$$

where $\overset{0}{K}{}_{ij}^s$ is a semi-Ricci curvature tensor of Levi-Civita connection $\overset{0}{\nabla}$. Notice that $\overset{0}{K}{}_{ij}^s = 0$ and using the expression (6), we obtain

$$a_{ij} - a_{ji} + b_{ij} - b_{ji} + t(V_j^p\pi_i - V_i^p\pi_j)\omega_p = t(\omega_{ij} - T_{ij}^p\omega_p),$$

$$U_j^kc_{ik} - U_i^kc_{jk} + U_{ik}c_j^k - U_{jk}c_i^k = 0, U_{ij}^k\pi_k - U_{ijk}\pi^k = 0, V_k^k = 0, V_{jk}^k = 0.$$

Hence from the expression (10) we arrive at

$$R^s_{ij} = [(n+1)t - n]\omega_{ij}. \tag{11}$$

If a 1-form ω is a closed form, it is obvious that (9) is tenable. □

Remark 2.5. *Theorem 2.4 shows that the semi-Ricci flat condition of the generalized quarter-symmetric metric recurrent connection is independent of a quarter-symmetric component φ^j_i, and that it is dependent only on a metric recurrent component φ_i.*

It is well known that if a sectional curvature at a point P in a Riemannian manifold is independent of Π(a 2-dimensional subspace of $T_p(M)$), the curvature tensor is

$$R_{ijkl} = k(P)(g_{il}g_{jk} - g_{ik}g_{jl}). \tag{12}$$

In this case, if $k(P)$=const, then the Riemannian manifold is a constant curvature manifold.

Theorem 2.6. *Suppose that (M^n, g) $(n \geq 3)$ is a connected Riemannian manifold associated with a generalized isotropic quarter-symmetric metric recurrent connection. If there holds*

$$t\omega_h = 2(\omega_h + s_h), \tag{13}$$

then (M^n, g, ∇) is a constant curvature manifold, where $s_h = \frac{1}{n-1}T^p_{hp}$ (generalized Schur's theorem).

Proof. Substituting the expression (12) into the second Bianchi identity of the curvature tensor of the generalized quarter-symmetric metric recurrent connection, we get

$$\nabla_h R^l_{ijk} + \nabla_i R^l_{jhk} + \nabla_j R^l_{hik} = T^m_{hi}R^l_{jmk} + T^m_{ij}R^l_{hmk} + T^m_{jh}R^l_{imk},$$

then we have

$$\begin{aligned}
&[\nabla_h K - K(t-2)\omega_h](g_{il}g_{jk} - g_{ik}g_{jl}) + [\nabla_i K - K(t-2)\omega_i](g_{jl}g_{hk} - g_{jk}g_{hl})\\
&+[\nabla_j K - K(t-2)\omega_j](g_{hl}g_{ik} - g_{hk}g_{il}) = K[\pi_h(g_{il}\varphi_{jk} - g_{ik}\varphi_{jl} + \varphi_{il}g_{jk} - \varphi_{ik}g_{jk})\\
&+\pi_i(g_{jl}\varphi_{hk} - g_{jk}\varphi_{hl} + \varphi_{jl}g_{hk} - \varphi_{jk}g_{hl}) + \pi_j(g_{hl}\varphi_{ik} - g_{hk}\varphi_{il} + \varphi_{hl}g_{ik} - \varphi_{hk}g_{il})].
\end{aligned}$$

Multiplying both sides of this equation above by g^{jk} and contracting the indices j, k, then we obtain

$$\begin{aligned}
&(n-1)[\nabla_h K - K(t-2)\omega_h]g_{il} - (n-1)[\nabla_i K - K(t-2)\omega_i]g_{hl}\\
&+[\nabla_j K - K(t-2)\omega_j](\delta^j_i g_{hl} - \delta^j_h g_{il}) = K\{\pi_h((n-2)\varphi_{il} + g_{il}\varphi^p_p)\\
&-\pi_i((n-2)\varphi_{hl} - g_{hl}\varphi^p_p) + \pi_j(g_{hl}\varphi^j_i - \delta^j_h\varphi_{il} + \delta^j_i\varphi_{hl} - g_{il}\varphi^j_h)\}.
\end{aligned}$$

Multiplying both sides of this expression again by g^{il} and contracting the indices i, l, then we have

$$(n-1)(n-2)[\nabla_h K - K(t-2)\omega_h] = 2(n-2)K(\pi_h\varphi^p_p - \pi_p\varphi^p_h).$$

From this equation above we obtain

$$\nabla_h K = K((t-2)\omega_h - 2s_h).$$

Consequently, we know from that K=const if and only if $t\omega_h = 2(\omega_h + s_h)$. □

By Theorem 2.6 and using (13), the expression (3) for the generalized quarter-symmetric recurrent connection shows

$$\nabla_k g_{ij} = -2(\omega_k + 2s_k)g_{ij} - 2(\omega_i + s_i)g_{jk} - 2(\omega_j + s_j)g_{ki}, T^k_{ij} = \pi_j\varphi^k_i - \pi_i\varphi^k_j. \tag{14}$$

Similarly, the formula (4) for ∇ shows

$$\Gamma^k_{ij} = \{^k_{ij}\} + (\omega_i + 2s_i)\delta^k_j + (\omega_j + 2s_j)\delta^k_i + g_{ij}\omega^k + \pi_j U^k_i - \pi_i V^k_j - U_{ij}\pi^k. \tag{15}$$

3. Quarter-symmetric Metric Recurrent Connection

The local expression of the relation (1) is

$$\overset{R}{\nabla}_k g_{ij} = 2\omega_k g_{ij},\ T^k_{ij} = \pi_j\varphi^k_i - \pi_j\varphi^k_j, \tag{16}$$

and its coefficient is

$$\overset{R}{\Gamma}{}^k_{ij} = \{^{\,k}_{ij}\} - \omega_i\delta^k_j - \omega_j\delta^k_i + g_{ij}\omega^k + \pi_j U^k_i - \pi_i V^k_j - U_{ij}\pi^k. \tag{17}$$

From the expression (17) we know that the curvature tensor of $\overset{R}{\nabla}$ is

$$\begin{aligned}\overset{R}{R}{}^l_{ijk} &= K^l_{ijk} + \delta^l_i h_{jk} - \delta^l_j h_{ik} + g_{jk}h^l_i - g_{ik}h^l_j + U^l_j c_{ik} - U^l_i c_{jk} + U_{ik}c^l_j - U_{jk}c^l_i \\ &+ U^l_{ij}\pi_k - U_{ijk}\pi^l - \delta^l_k\omega_{ij} - V^l_k\pi_{ij} + V^l_{jk}\pi_i - V^l_{ik}\pi_j,\end{aligned} \tag{18}$$

where $h_{ik} = \overset{R}{\nabla}_i\omega_k + \omega_i\omega_k + U_{ik}\omega_p\pi^p - U^p_i\omega_p\pi_k - \frac{1}{2}g_{ik}\omega_p\pi^p$. From the expression (17) the connection coefficient of dual connection $\overset{R}{\nabla}{}^*$ of the quarter-symmetric metric recurrent connection $\overset{R}{\nabla}$ is

$$\overset{R}{\Gamma}{}^{*k}_{ij} = \{^{\,k}_{ij}\} + \omega_i\delta^k_j - \omega_j\delta^k_i + g_{ij}\omega^k + \pi_j U^k_i - \pi_i V^k_j - U_{ij}\pi^k,$$

and the curvature tensor of $\overset{R}{\nabla}{}^*$ is

$$\begin{aligned}\overset{R}{R}{}^{*k}_{ij} &= K^l_{ijk} + \delta^l_i h_{jk} - \delta^l_j h_{ik} + g_{jk}h^l_i - g_{ik}h^l_j + U^l_j c_{ik} - U^l_i c_{jk} + U_{ik}c^l_j - U_{jk}c^l_i \\ &+ U^l_{ij}\pi_k - U_{ijk}\pi^l + \delta^l_k\omega_{ij} - V^l_k\pi_{ij} + V^l_{jk}\pi_i - V^l_{ik}\pi_j.\end{aligned} \tag{19}$$

Theorem 3.1. *If a 1-form ω is a closed form, then the curvature tensor of the quarter-symmetric metric recurrent connection $\overset{R}{\nabla}$ on a Riemannian manifold (M, g) is a conjugate symmetric.*

Proof. From the expression (18) and (19), we obtain

$$\overset{*}{R}{}^l_{ijk} = R^l_{ijk} + 2\delta^l_k\omega_{ij}. \tag{20}$$

If a 1-form ω is a closed form, then $\omega_{ij} = 0$. Hence from the expression (20), we have $\overset{*}{R}{}^l_{ijk} = R^l_{ijk}$. Consequently, the quarter-symmetric metric recurrent connection $\overset{R}{\nabla}$ is a conjugate symmetry. □

Remark 3.2. *According to Theorem 2.6, for the quarter-symmetric metric recurrent connection $\overset{R}{\nabla}$, the formula (13) is*

$$\omega_h = -s_h. \tag{21}$$

Using the expression (21), the quarter-symmetric metric recurrent connection $\overset{R}{\nabla}$ satisfying the generalized Schur's theorem satisfies the relation

$$\overset{R}{\nabla}_k g_{ij} = -2s_k g_{ij},\ T^k_{ij} = \pi_j\varphi^k_i - \pi_i\varphi^k_j. \tag{22}$$

From (15), the connection coefficient of $\overset{R}{\nabla}$ *is*

$$\overset{R}{\Gamma}{}^k_{ij} = \{^{\,k}_{ij}\} + s_i\delta^k_j - s_j\delta^k_i + g_{ij}s^k + \pi_j U^k_i - \pi_i V^k_j - U_{ij}\pi^k, \tag{23}$$

it is easy to see that by Theorem 2.6, for the quarter-symmetric metric recurrent connection $\overset{R}{D}$*, the expression (13) is*

$$\omega_h = f\pi_h.$$

Example 3.3. *The quarter-symmetric metric recurrent connection* $\overset{R}{D}$ *satisfying the generalized Schur's theorem implies the following*

$$\overset{R}{D}_k g_{ij} = 2f\pi_k g_{ij},\ T^k_{ij} = f(\pi_j\delta^k_i - \pi_i\delta^k_j).$$

For a 1-form π *is a closed form, it was pointed out in [2] that this connection is a geometrical model for scalar-tensor theories of gravitation.*

4. Special Type of the Generalized Quarter-Symmetric Metric Recurrent Connection

In this subsection we study the geometrical characteristics of a manifold associated with a generalized quarter-symmetric metric recurrent connection ∇ satisfying the condition $\varphi(X) = fX(f \in C^\infty(M))$. This connection is denoted as D. The connection D is a special type of the generalized quarter-symmetric metric recurrent connection ∇.

From the expression (3), the local expression of the generalized quarter-symmetric metric recurrent connection D is

$$D_k g_{ij} = -2(t-1)\omega_k g_{ij} - t\omega_i g_{jk} - t\omega_j g_{ik},\ T^k_{ij} = f(\pi_j\delta^k_i - \pi_i\delta^k_j), \tag{24}$$

and from (4) the coefficient of D is

$$\Gamma^k_{ij} = \{^{\,k}_{ij}\} + (t-1)\omega_i\delta^k_j + \Big((t-1)\omega_j + f\pi_j\Big)\delta^k_i + g_{ij}(\omega^k - f\pi^k). \tag{25}$$

By the equation (25), it is easy to see that the curvature tensor of D is

$$R^l_{ijk} = K^l_{ijk} + \delta^l_j d_{ik} - \delta^l_i d_{jk} + g_{jk}e^l_i - g_{ik}e^l_j + (t-1)\delta^l_k\omega_{ij}, \tag{26}$$

where d_{ik} and e_{ik} are denoted by

$$\begin{aligned} d_{ik} &= \overset{0}{\nabla}_i\Big[(t-1)\omega_k + f\pi_k\Big] - \Big[(t-1)\omega_i + f\pi_i\Big]\Big[(t-1)\omega_k + f\pi_k\Big] \\ &- g_{ik}\Big[(t-1)\omega_p + f\pi_p\Big](\omega_p - f\pi^p), \\ e_{ik} &= \overset{0}{\nabla}_i(\omega_k + f\pi_k) + (\omega_i - f\pi_i)(\omega_k - f\pi_k). \end{aligned}$$

From the expression (25), the coefficient of $\overset{*}{D}$ of dual connection of the connection D is

$$\overset{*}{\Gamma}{}^k_{ij} = \{^{\,k}_{ij}\} - (t-1)\omega_i\delta^k_j - (\omega_j - f\pi_j)\delta^k_i - g_{ij}\Big[(t-1)\omega^k + f\pi^k\Big],$$

and the curvature tensor of $\overset{*}{D}$ is

$$\overset{*}{R}{}^l_{ijk} = K^l_{ijk} - \delta^l_j e_{ik} + \delta^l_i e_{jk} - g_{jk}d^l_i + g_{ik}d^l_j - (t-1)\delta^l_k\omega_{ij}. \tag{27}$$

Theorem 4.1. *A Riemannian manifold* (M^n, g) $(n \geq 3)$ *associated with a generalized quarter-symmetric metric recurrent connection* D *with a constant curvature is conformally flat.*

Proof. Adding the expressions (26) and (27), we obtain

$$R^{l}_{ijk} + \overset{*}{R}{}^{l}_{ijk} = 2K^{l}_{ijk} - \delta^{l}_{i}\beta_{jk} + \delta^{l}_{j}\beta_{ik} - g_{jk}\beta^{l}_{i} + g_{ik}\beta^{l}_{j}, \tag{28}$$

where $\beta_{jk} = d_{jk} - e_{jk}$. Contracting the indices i and l of (28), we get

$$R_{jk} + \overset{*}{R}_{jk} = 2K_{jk} - (n-2)\beta_{jk} - g_{jk}\beta^{i}_{i}. \tag{29}$$

Multiplying both sides of (29) by g^{jk}, then we arrive at

$$R + \overset{*}{R} = 2K - 2(n-1)\beta^{i}_{i}.$$

From this expression above we have

$$\beta^{i}_{i} = \frac{1}{2(n-1)}\Big[2K - (R + \overset{*}{R})\Big].$$

Using the expression from (29), we have

$$\beta_{jk} = \frac{1}{n-2}\Big\{2K_{jk} - (R_{jk} + \overset{*}{R}_{jk}) - \frac{g_{jk}}{2(n-1)}[2K - (R + \overset{*}{R})]\Big\}.$$

Substituting this expression into (28) and putting

$$\begin{aligned}
C^{l}_{ijk} &= R^{l}_{ijk} - \frac{1}{n-2}(\delta^{l}_{i}R_{jk} - \delta^{l}_{j}R_{ik} + g_{jk}R^{l}_{i} - g_{ik}R^{l}_{j}) - \frac{R}{(n-1)(n-2)}(\delta^{l}_{i}g_{jk} - \delta^{l}_{j}g_{ik}),\\
\overset{*}{C}{}^{l}_{ijk} &= \overset{*}{R}{}^{l}_{ijk} - \frac{1}{n-2}(\delta^{l}_{i}\overset{*}{R}_{jk} - \delta^{l}_{j}\overset{*}{R}_{ik} + g_{jk}\overset{*}{R}{}^{l}_{i} - g_{ik}\overset{*}{R}{}^{l}_{j}) - \frac{\overset{*}{R}}{(n-1)(n-2)}(\delta^{l}_{i}g_{jk} - \delta^{l}_{j}g_{ik}),\\
\overset{0}{C}{}^{l}_{ijk} &= K^{l}_{ijk} - \frac{1}{n-2}(\delta^{l}_{i}K_{jk} - \delta^{l}_{j}K_{ik} + g_{jk}K^{l}_{i} - g_{ik}K^{l}_{j}) - \frac{K}{(n-1)(n-2)}(\delta^{l}_{i}g_{jk} - \delta^{l}_{j}g_{ik}).
\end{aligned}$$

then by a direct computation, we obtain

$$C^{l}_{ijk} + \overset{*}{C}{}^{l}_{ijk} = 2\overset{0}{C}{}^{l}_{ijk}. \tag{30}$$

By using the constant curvature assumption in Theorem 4.1, we have $C^{l}_{ijk} = \overset{*}{C}{}^{l}_{ijk} = 0$, hence it holds

$$\overset{0}{C}{}^{l}_{ijk} = 0.$$

This means consequently that the Riemannian manifold is conformally flat. □

Theorem 4.2. *The generalized quarter-symmetric metric recurrent connection D on a Riemannian manifold* (M, g) *is a conjugate symmetry if and only if its Ricci curvature tensor is equal to that of its dual connection.*

Proof. From the expressions (26) and (27) we obtain

$$\overset{*}{R}{}^{l}_{ijk} = R^{l}_{ijk} + \delta^{l}_{i}\gamma_{jk} - \delta^{l}_{j}\gamma_{ik} + g_{ik}\gamma^{l}_{j} - g_{jk}\gamma^{l}_{i} - 2(t-1)\delta^{l}_{k}\omega_{ij}, \tag{31}$$

where $\gamma_{jk} = d_{jk} + e_{jk}$. By using the contraction of the indices i and l in (31), we have

$$\overset{*}{R}_{jk} = R_{jk} + n\gamma_{jk} - g_{jk}\gamma^{l}_{l} + 2(t-1)\omega_{jk}. \tag{32}$$

Alternating the indices j and k in this expression and using $\gamma_{jk} - \gamma_{kj} = t\omega_{jk}$, we arrive at

$$\omega_{jk} = \frac{1}{nt + 4(t-1)}\Big[(\overset{*}{R}_{jk} - \overset{*}{R}_{kj}) - (R_{jk} - R_{kj})\Big].$$

Substituting this expression above into (32) and by directly computation, one gets the following

$$\gamma_{jk} = \frac{1}{n}\Big\{\overset{*}{R}_{jk} - R_{jk} + g_{jk}\gamma^l_l - \frac{2(t-1)}{nt+4(t-1)}[(\overset{*}{R}_{jk} - \overset{*}{R}_{kj}) - (R_{jk} - R_{kj})]\Big\}.$$

Substituting this expression into (31) and putting

$$\begin{aligned} V^l_{ijk} &= R^l_{ijk} - \frac{1}{n}(\delta^l_i R_{jk} - \delta^l_j R_{ik} - g_{jk}R^l_i + g_{ik}R^l_j) \\ &+ \frac{2(t-1)}{n(nt+4(t-1))}\Big[\delta^l_i(R_{jk} - R_{kj}) - \delta^l_j(R_{ik} - R_{ki}) + g_{ik}(R^l_j - R^l_{\cdot j}) - g_{jk}(R^l_i - R^l_{\cdot i}) \\ &+ n\delta^l_k(R_{ij} - R_{ji})\Big], \\ \overset{*}{V}{}^l_{ijk} &= \overset{*}{R}{}^l_{ijk} - \frac{1}{n}(\delta^l_i \overset{*}{R}_{jk} - \delta^l_j \overset{*}{R}_{ik} - g_{jk}\overset{*}{R}{}^l_i + g_{ik}\overset{*}{R}{}^l_j) \\ &+ \frac{2(t-1)}{n(nt+4(t-1))}\Big[\delta^l_i(\overset{*}{R}_{jk} - \overset{*}{R}_{kj}) - \delta^l_j(\overset{*}{R}_{ik} - \overset{*}{R}_{ki}) + g_{ik}(\overset{*}{R}{}^l_j - \overset{*}{R}{}^l_{\cdot j}) - g_{jk}(\overset{*}{R}{}^l_i - \overset{*}{R}{}^l_{\cdot i}) \\ &+ n\delta^l_k(R_{ij} - R_{ji})\Big]. \end{aligned}$$

where $R^l_j = R_{js}g^{sl}$, $\overset{*}{R}{}^l_j = R_{js}g^{sl}$, $R^l_{\cdot j} = R_{sj}g^{sl}$, $\overset{*}{R}{}^l_{\cdot j} = R_{sj}g^{sl}$.

Then we have

$$V^l_{ijk} = \overset{*}{V}{}^l_{ijk}. \tag{33}$$

From the equation (33), it is easy to show that $R^l_{ijk} = \overset{*}{R}{}^l_{ijk}$ if and only if $R_{jk} = \overset{*}{R}_{jk}$. □

By Theorem 4.2 with $s_h = -f\pi_h$, the expression (13) is

$$t\omega_h = 2(\omega_h - f\pi_h). \tag{34}$$

Using the expression (34) and the expression (14) the generalized quarter-symmetric metric recurrent connection D satisfying the generalized Schur's theorem satisfies

$$D_k g_{ij} = -2(\omega_k - 2f\pi_k)g_{ij} - 2(\omega_i - f\pi_i)g_{jk} - 2(\omega_j - f\pi_j)g_{ik}, T^k_{ij} = f(\pi_j\delta^k_i - \pi_i\delta^k_j), \tag{35}$$

and from the expression (15) its connection coefficient is

$$\Gamma^k_{ij} = \{^{\,k}_{ij}\} + (\omega_i - 2f\pi_i)\delta^k_j + (\omega_j - f\pi_j)\delta^k_i + g_{ij}(\omega^k - f\pi^k). \tag{36}$$

Example 4.3. *Let (M, g) be a Riemannian manifold (dim $M \geq 2$), g be the Riemannian metric on M, and $\overset{\circ}{\nabla}$ be the Levi-Civita connection with respect to g. Let f_1, f_2 be functions in M, then the connection $\overline{\nabla}$ is given by*

$$\begin{aligned} \overline{\nabla}_X Y &= \overset{\circ}{\nabla}_X Y + \pi(Y)\varphi_1 X - \pi(X)\varphi_2 Y - g(\varphi_1 X, Y)U \\ &- f_1\{\omega(X)Y + \omega(Y)X - g(X,Y)V\} \\ &- f_2 g(X,Y)V. \end{aligned}$$

is a generalized quarter-symmetric metric recurrent connection, which satisfies

$$\overline{T}(X,Y) = \pi(Y)\varphi(X) - \pi(X)\varphi(Y),$$

and

$$(\overline{\nabla}_X g)(Y, Z) = 2f_1\omega(X)g(Y, Z) + f_2\omega(Y)g(X, Z) + f_2\omega(Z)g(X, Y).$$

where π, ω *are 1-form such that*

$$\pi(X) = g(U, X), \omega(X) = g(V, X),$$

where φ *is a* $(1, 1)$ *tensor field such that*

$$g(\varphi X, Y) = \Phi(X, Y) = \Phi_1(X, Y) + \Phi_2(X, Y),$$

where Φ_1 *and* Φ_2 *are symmetric and skew-symmetric parts of the* $(0, 2)$ *tensor* Φ, *which satisfies* $\Phi_1(X, Y) = g(\varphi_1 X, Y)$, $\Phi_2(X, Y) = g(\varphi_2 X, Y)$.

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