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Some Approximation Results For (p, q)-Lupaş-Schurer Operators

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Abstract. In this paper, we introduce Lupaş-Schurer operators based on (p, q)-integers. Then, we deal with the approximation properties for (p, q)-Lupaş-Schurer operators based on Korovkin type approximation theorem. Moreover, we compute rate of convergence by using modulus of continuity, with the help of functions of Lipschitz class and Peetre's K-functionals.

1. Introduction

In 1912, Bernstein [2] defined the following sequences of linear and positive operators $B_n : C[0, 1] \to C[0, 1]$

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\left(\begin{array}{c} n \\ k \end{array}\right) x^k(1 - x)^{n-k}, \tag{1}$$

where $n \in \mathbb{N}$ and $f \in C[0, 1]$. In 1987, Lupaş [3] introduced q-calculus for Bernstein operators. He defined q-analogue of the Bernstein operators in the following form

$$L_{n,q}(f; x) = \sum_{k=0}^{n} \frac{f\left(\frac{[k]_q}{[n]_q}\right)\left[\begin{array}{c} n \\ k \end{array}\right]_q q^{\frac{k(k-1)}{2}} x^k(1 - x)^{n-k}}{\prod_{j=1}^{n}\{(1 - x) + q^{j-1}x\}}. \tag{2}$$

The operators $L_{n,q}(f; x)$ generate positive linear operators for all $q > 0$. As we see above, there are two kinds of q-analogue of Bernstein operators i.e Phillips and Lupaş. In 2015, Mursaleen, Ansari and Khan [4] first introduced the concept of (p, q)-calculus in approximation theory. They defined a generalisation of q-Bernstein operator and called it as (p, q)-Bernstein operators. The form of the (p, q)-Bernstein operators are as follows

$$B_{n,p,q}(f; x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right)\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{k(k-1)}{2}} \prod_{j=0}^{n-k-1}\{p^j - q^j x\}. \tag{3}$$

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In 2017, (p,q)-analogue of Lupaş Bernstein operators are defined by Khalid et al [13].
For each $p>0$ and $q>0$, $L^n_{p,q}:C[0,1]\to C[0,1]$

$$L^n_{p,q}(f;x)=\sum_{k=0}^{n}\frac{f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right)\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^k(1-x)^{n-k}}{\prod_{j=1}^{n}\{p^{j-1}(1-x)+q^{j-1}x\}} \tag{4}$$

are (p,q)-analogue of Lupaş Bernstein operators. Note that (p,q)-analogue of Lupaş Bernstein operators generate positive linear operators for all $p>0$ and $q>0$. Consequently, the (p,q)-counterparts introduced by Mursaleen et al [4] is generalisation of q-analogue of Bernstein operators given by Phillips [19] whereas Khalid et al [13] generalised q-Lupaş Bernstein operators. The novelty of (p,q)-calculus in computer aided geometric design (CAGD) given by Khalid et al [13] will help readers to understand the application. Another advantage of using parameter p has been shown in [9]. Besides this, we also refer to the reader some recent papers on (p,q)-calculus in approximation theory: e.g. [1], [6], [7], [8], [10], [11], [12], [16], [17] and [18].
Before proceeding further, we recall significant definitions and notations on the concept of (p,q)-calculus. For any non-negative p anq q, the (p,q)-integers of the number n is defined by

$$[n]_{p,q}:=p^{n-1}+p^{n-2}q+\cdots+pq^{n-2}+q^{n-1}=\begin{cases} \frac{p^n-q^n}{p-q} & \text{if } p\neq q\neq 1 \\ np^{n-1} & \text{if } p=q\neq 1 \\ n & \text{if } p=q=1 \\ [\mathrm{n}]_q & \text{if } p=1 \end{cases},$$

where $[n]_q$ denotes q-integers for $n=0,1,2,\ldots$. The (p,q)-binomial expansion is defined as

$$(ax+by)^n_{p,q}:=\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} a^{n-k}b^k x^{n-k}y^k, \tag{5}$$

where

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}=\frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}$$

are the (p,q)-binomial coefficients. By using (5) we obtain

$$(x+y)^n_{p,q}=(x+y)(px+qy)(p^2x+q^2y)\ldots(p^{n-1}x+q^{n-1}y)$$

and

$$(1-x)^n_{p,q}=(1-x)(p-qx)(p^2-q^2x)\ldots(p^{n-1}-q^{n-1}x).$$

More information about (p,q)-calculus can be read from [4] and [15].

2. Construction of The Operator

Schurer type generalization of linear positive operators has been studied in several years. In this part, we construct the class of the (p,q)-analogue of Lupaş Schurer operators.

Definition 2.1. *We consider for each $p>0$, $q>0$ and for any $m\in\mathbb{N}$, $x\in[0,1]$ and $f\in C[0,1+l]$, fixed $l\in\mathbb{N}^+\cup\{0\}$. We construct the (p,q)-analogue of Lupaş Schurer operators by*

$$L^{p,q}_{m,l}(f;x)=\sum_{k=0}^{m+l}\frac{f\left(\frac{p^{m+l-k}[k]_{p,q}}{[m]_{p,q}}\right)\left[\begin{array}{c} m+l \\ k \end{array}\right]_{p,q} p^{\frac{(m+l-k)(m+l-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^k(1-x)^{m+l-k}}{\prod_{j=1}^{m+l}\{p^{j-1}(1-x)+q^{j-1}x\}}. \tag{6}$$

Note that if we take $p = q = 1$ (p, q)-Lupaş-Schurer operators reduce to be Schurer-Bernstein operators which are defined in the article of Schurer [14] in 1962. We have the following lemma to give some equalities for the operators (6).

Lemma 2.2. *Let $L^{p,q}_{m,l}(.;.)$ is given by (6). The following equalities*

$$L^{p,q}_{m,l}(1;x) = 1, \tag{7}$$

$$L^{p,q}_{m,l}(t;x) = \frac{[m+l]_{p,q}}{[m]_{p,q}}x, \tag{8}$$

$$L^{p,q}_{m,l}(t^2;x) = \frac{p^{m+l-1}[m+l]_{p,q}}{[m]^2_{p,q}}x + \frac{q^2[m+l]_{p,q}[m+l-1]_{p,q}}{[m]^2_{p,q}(p(1-x)+qx)}x^2 \tag{9}$$

$$L^{p,q}_{m,l}(t-x;x) = \left(\frac{[m+l]_{p,q}}{[m]_{p,q}} - 1\right)x, \tag{10}$$

$$\begin{aligned} L^{p,q}_{m,l}((t-x)^2;x) &= \frac{p^{m+l-1}[m+l]_{p,q}}{[m]^2_{p,q}}x \\ &+ \left(\frac{q^2[m+l]_{p,q}[m+l-1]_{p,q}}{[m]^2_{p,q}(p(1-x)+qx)} - \frac{2[m+l]_{p,q}}{[m]_{p,q}} + 1\right)x^2 \end{aligned} \tag{11}$$

hold.

Proof. *(i)* Firstly, we begin with

$$\begin{aligned} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k} &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left(\frac{x}{1-x}\right)^k (1-x)^n \\ &= (p(1-x)+qx)(p^2(1-x)+q^2x)\dots(p^{n-1}(1-x)+q^{n-1}x) \\ &= \prod_{j=1}^{n}\{p^{j-1}(1-x)+q^{j-1}x\}. \end{aligned} \tag{12}$$

So, we have from (12)

$$\sum_{k=0}^{n} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k}}{\prod_{j=1}^{n}\{p^{j-1}(1-x)+q^{j-1}x\}} = 1. \tag{13}$$

In the (13) we choose $n := m + l$. Then we get

$$\sum_{k=0}^{m+l} \frac{\begin{bmatrix} m+l \\ k \end{bmatrix}_{p,q} p^{\frac{(m+l-k)(m+l-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^k (1-x)^{m+l-k}}{\prod_{j=1}^{m+l}\{p^{j-1}(1-x)+q^{j-1}x\}} = 1. \tag{14}$$

Now, let us write $L^{p,q}_{m,l}(1;x)$

$$L^{p,q}_{m,l}(1;x) = \sum_{k=0}^{m+l} \frac{\begin{bmatrix} m+l \\ k \end{bmatrix}_{p,q} p^{\frac{(m+l-k)(m+l-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^k (1-x)^{m+l-k}}{\prod_{j=1}^{m+l}\{p^{j-1}(1-x)+q^{j-1}x\}}. \tag{15}$$

By using (14) and (15) we obtain $L_{m,l}^{p,q}(1;x)=1$.
(ii) Secondly, we write $L_{m,l}^{p,q}(t;x)$ as follows

$$\begin{aligned}
L_{m,l}^{p,q}(t;x) &= \sum_{k=0}^{m+l}\frac{\frac{p^{m+l-k}[k]_{p,q}}{[m]_{p,q}}\begin{bmatrix} m+l \\ k\end{bmatrix}_{p,q} p^{\frac{(m+l-k)(m+l-k-1)}{2}}q^{\frac{k(k-1)}{2}}x^k(1-x)^{m+l-k}}{\prod_{j=1}^{m+l}\{p^{j-1}(1-x)+q^{j-1}x\}}\\
&= \sum_{k=0}^{m+l}\frac{\frac{p^{m+l-k}[k]_{p,q}}{[m]_{p,q}}\frac{[m+l]_{p,q}!}{[k]_{p,q}![m+l-k]_{p,q}!} p^{\frac{(m+l-k)(m+l-k-1)}{2}}q^{\frac{k(k-1)}{2}}x^k(1-x)^{m+l-k}}{\prod_{j=1}^{m+l}\{p^{j-1}(1-x)+q^{j-1}x\}}\\
&= \sum_{k=1}^{m+l}\frac{\frac{p^{m+l-k}}{[m]_{p,q}}\frac{[m+l]_{p,q}!}{[k-1]_{p,q}![m+l-k]_{p,q}!} p^{\frac{(m+l-k)(m+l-k-1)}{2}}q^{\frac{k(k-1)}{2}}x^k(1-x)^{m+l-k}}{\prod_{j=1}^{m+l}\{p^{j-1}(1-x)+q^{j-1}x\}}\\
&= \sum_{k=0}^{m+l-1}\frac{\frac{p^{m+l-k-1}}{[m]_{p,q}}\frac{[m+l]_{p,q}!}{[k]_{p,q}![m+l-k-1]_{p,q}!} p^{\frac{(m+l-k-1)(m+l-k-2)}{2}}q^{\frac{k(k+1)}{2}}x^{k+1}(1-x)^{m+l-k-1}}{\prod_{j=1}^{m+l}\{p^{j-1}(1-x)+q^{j-1}x\}}\\
&= \sum_{k=0}^{m+l-1}\frac{\frac{p^{m+l-k-1}[m+l]_{p,q}}{[m]_{p,q}}\begin{bmatrix} m+l-1 \\ k\end{bmatrix}_{p,q} p^{\frac{(m+l-k-1)(m+l-k-2)}{2}}q^{\frac{k(k+1)}{2}}x^{k+1}(1-x)^{m+l-k-1}}{\prod_{j=1}^{m+l}\{p^{j-1}(1-x)+q^{j-1}x\}}
\end{aligned}$$

direct calculations yield,

$$\begin{aligned}
L_{m,l}^{p,q}(t;x) &= \frac{[m+l]_{p,q}}{[m]_{p,q}}x\sum_{k=0}^{m+l-1}\frac{p^{m+l-k-1}\begin{bmatrix} m+l-1 \\ k\end{bmatrix}_{p,q} p^{\frac{(m+l-k-1)(m+l-k-2)}{2}}q^{\frac{k(k+1)}{2}}x^k(1-x)^{m+l-k-1}}{\prod_{j=0}^{m+l-1}\{p^j(1-x)+q^jx\}}\\
&= \frac{[m+l]_{p,q}}{[m]_{p,q}}x\sum_{k=0}^{m+l-1}\frac{p^{m+l-k-1}\begin{bmatrix} m+l-1 \\ k\end{bmatrix}_{p,q} p^{\frac{(m+l-k-1)(m+l-k-2)}{2}}q^{\frac{k(k+1)}{2}}(\frac{x}{1-x})^k(1-x)^{m+l-1}}{\prod_{j=1}^{m+l-1}\{p^j(1-x)+q^jx\}}\\
&= \frac{[m+l]_{p,q}}{[m]_{p,q}}x\sum_{k=0}^{m+l-1}\frac{p^{m+l-k-1}\begin{bmatrix} m+l-1 \\ k\end{bmatrix}_{p,q} p^{\frac{(m+l-k-1)(m+l-k-2)}{2}}q^{\frac{k(k+1)}{2}}\left(\frac{x}{1-x}\right)^k(1-x)^{m+l-1}}{\prod_{j=0}^{m+l-2}\{p^{j+1}(1-x)+q^{j+1}x\}}.
\end{aligned}$$

Now, suppose that $x=\frac{u}{u+1}$, or equivalently, $u=\frac{x}{1-x}$

$$\begin{aligned}
L_{m,l}^{p,q}\left(t;\frac{u}{u+1}\right) &= \frac{[m+l]_{p,q}}{[m]_{p,q}}\frac{u}{u+1}\sum_{k=0}^{m+l-1}\frac{\begin{bmatrix} m+l-1 \\ k\end{bmatrix}_{p,q} p^{\frac{(m+l-k-1)(m+l-k-2)}{2}}q^{\frac{k(k-1)}{2}}\left(\frac{q}{p}u\right)^k}{\frac{1}{p^{m+l-1}}\prod_{j=0}^{m+l-2}\{p^{j+1}+q^{j+1}u\}}\\
&= \frac{[m+l]_{p,q}}{[m]_{p,q}}\frac{u}{u+1}\sum_{k=0}^{m+l-1}\frac{\begin{bmatrix} m+l-1 \\ k\end{bmatrix}_{p,q} p^{\frac{(m+l-k-1)(m+l-k-2)}{2}}q^{\frac{k(k-1)}{2}}\left(\frac{q}{p}u\right)^k}{\prod_{j=0}^{m+l-2}\left\{p^j+q^j\left(\frac{q}{p}u\right)\right\}}\\
&= \frac{[m+l]_{p,q}}{[m]_{p,q}}\frac{u}{u+1}L_{m,l}^{p,q}(1;x).
\end{aligned}$$

Finally, we obtain

$$L_{m,l}^{p,q}(t;x) = \frac{[m+l]_{p,q}}{[m]_{p,q}}x.$$

(iii) Thirdly, let us write $L_{m,l}^{p,q}(t^2;x)$

$$\begin{aligned}
L_{m,l}^{p,q}(t^2;x) &= \sum_{k=0}^{m+l} \frac{\left(\frac{p^{m+l-k}[k]_{p,q}}{[m]_{p,q}}\right)^2 \left[\begin{array}{c} m+l \\ k \end{array}\right]_{p,q} p^{\frac{(m+l-k)(m+l-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^k (1-x)^{m+l-k}}{\prod_{j=1}^{m+l}\{p^{j-1}(1-x)+q^{j-1}x\}} \\
&= \sum_{k=0}^{m+l} \frac{\frac{p^{2(m+l-k)}[k]_{p,q}^2}{[m]_{p,q}^2} \frac{[m+l]_{p,q}!}{[k]_{p,q}![m+l-k]_{p,q}!} p^{\frac{(m+l-k)(m+l-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^k (1-x)^{m+l-k}}{\prod_{j=1}^{m+l}\{p^{j-1}(1-x)+q^{j-1}x\}} \\
&= \sum_{k=1}^{m+l} \frac{\frac{p^{2(m+l-k)}[k]_{p,q}}{[m]_{p,q}^2} \frac{[m+l]_{p,q}!}{[k-1]_{p,q}![m+l-k]_{p,q}!} p^{\frac{(m+l-k)(m+l-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^k (1-x)^{m+l-k}}{\prod_{j=1}^{m+l}\{p^{j-1}(1-x)+q^{j-1}x\}} \\
&= \sum_{k=0}^{m+l-1} \frac{\frac{p^{2(m+l-k-1)}[k+1]_{p,q}}{[m]_{p,q}^2} \frac{[m+l]_{p,q}!}{[k]_{p,q}![m+l-k-1]_{p,q}!} p^{\frac{(m+l-k-1)(m+l-k-2)}{2}} q^{\frac{k(k+1)}{2}} x^{k+1} (1-x)^{m+l-k-1}}{\prod_{j=1}^{m+l}\{p^{j-1}(1-x)+q^{j-1}x\}}.
\end{aligned}$$

Take $[k+1]_{p,q} = p^k + q[k]_{p,q}$, then we get

$$\begin{aligned}
L_{m,l}^{p,q}(t^2;x) &= \sum_{k=0}^{m+l-1} \frac{\frac{p^{2(m+l-k-1)}(p^k+q[k]_{p,q})}{[m]_{p,q}^2} \frac{[m+l]_{p,q}!}{[k]_{p,q}![m+l-k-1]_{p,q}!} p^{\frac{(m+l-k-1)(m+l-k-2)}{2}} q^{\frac{k(k+1)}{2}} x^{k+1} (1-x)^{m+l-k-1}}{\prod_{j=1}^{m+l}\{p^{j-1}(1-x)+q^{j-1}x\}} \\
&= \frac{[m+l]_{p,q}}{[m]_{p,q}^2} \sum_{k=0}^{m+l-1} \frac{p^{2(m+l-k-1)} p^k \left[\begin{array}{c} m+l-1 \\ k \end{array}\right]_{p,q} p^{\frac{(m+l-k-1)(m+l-k-2)}{2}} q^{\frac{k(k+1)}{2}} x^{k+1} (1-x)^{m+l-k-1}}{\prod_{j=0}^{m+l-1}\{p^{j}(1-x)+q^{j}x\}} \\
&\quad + q\frac{[m+l]_{p,q}}{[m]_{p,q}^2} x \\
&\quad \times \sum_{k=0}^{m+l-1} \frac{p^{2(m+l-k-1)}[k]_{p,q} \left[\begin{array}{c} m+l-1 \\ k \end{array}\right]_{p,q} p^{\frac{(m+l-k-1)(m+l-k-2)}{2}} q^{\frac{k(k+1)}{2}} x^{k} (1-x)^{m+l-k-1}}{\prod_{j=0}^{m+l-1}\{p^{j}(1-x)+q^{j}x\}}.
\end{aligned}$$

$$\begin{aligned}
L_{m,l}^{p,q}(t^2;x) &= \frac{p^{m+l-1}[m+l]_{p,q}x}{[m]_{p,q}^2}\sum_{k=0}^{m+l-1}\frac{\begin{bmatrix} m+l-1 \\ k \end{bmatrix}_{p,q} p^{\frac{(m+l-k-1)(m+l-k-2)}{2}} q^{\frac{k(k-1)}{2}}\left(\frac{qx}{p(1-x)}\right)}{\prod_{j=1}^{m+l-1}\{p^j(1-x)+q^jx\}\left(\frac{1}{p(1-x)}\right)^{m+l-1}} \\
&+\frac{q[m+l]_{p,q}[m+l-1]_{p,q}x^2}{[m]_{p,q}^2} \\
&\times\sum_{k=0}^{m+l-2}\frac{p^{2(m+l-k-1)}\begin{bmatrix} m+l-2 \\ k \end{bmatrix}_{p,q} p^{\frac{(m+l-k-2)(m+l-k-3)}{2}} q^{\frac{k(k-1)}{2}} q^{2k+1}x^k(1-x)^{m+l-k-2}}{\prod_{j=0}^{m+l-1}\{p^j(1-x)+q^jx\}} \\
&= \frac{p^{m+l-1}[m+l]_{p,q}x}{[m]_{p,q}^2}L_{m,l}^{p,q}(1;x) \\
&+\frac{q^2[m+l]_{p,q}[m+l-1]_{p,q}x^2}{[m]_{p,q}^2(p(1-x)+qx)}\sum_{k=0}^{m+l-2}\frac{\begin{bmatrix} m+l-2 \\ k \end{bmatrix}_{p,q} p^{\frac{(m+l-k-2)(m+l-k-3)}{2}} q^{\frac{k(k-1)}{2}}\left(\frac{q^2x}{p^2(1-x)}\right)^k}{\prod_{j=2}^{m+l-1}\{p^j(1-x)+q^jx\}\left(\frac{1}{p^2(1-x)}\right)^{m+l-2}} \\
&= \frac{p^{m+l-1}[m+l]_{p,q}x}{[m]_{p,q}^2}L_{m,l}^{p,q}(1;x) \\
&+\frac{q^2[m+l]_{p,q}[m+l-1]_{p,q}x^2}{[m]_{p,q}^2(p(1-x)+qx)}\sum_{k=0}^{m+l-2}\frac{\begin{bmatrix} m+l-2 \\ k \end{bmatrix}_{p,q} p^{\frac{(m+l-k-2)(m+l-k-3)}{2}} q^{\frac{k(k-1)}{2}}\left(\frac{q^2x}{p^2(1-x)}\right)^k}{\prod_{j=2}^{m+l-1}\{p^{j-2}+q^{j-2}\left(\frac{q^2x}{p^2(1-x)}\right)\}} \\
&= \frac{p^{m+l-1}[m+l]_{p,q}x}{[m]_{p,q}^2}L_{m,l}^{p,q}(1;x)+\frac{q^2[m+l]_{p,q}[m+l-1]_{p,q}x^2}{[m]_{p,q}^2(p(1-x)+qx)}L_{m,l}^{p,q}(1;x) \\
&= \frac{p^{m+l-1}[m+l]_{p,q}}{[m]_{p,q}^2}x+\frac{q^2[m+l]_{p,q}[m+l-1]_{p,q}}{[m]_{p,q}^2}\frac{1}{p(1-x)+qx}x^2.
\end{aligned}$$

As a result, we obtain

$$L_{m,l}^{p,q}(t^2;x)=\frac{p^{m+l-1}[m+l]_{p,q}}{[m]_{p,q}^2}x+\frac{q^2[m+l]_{p,q}[m+l-1]_{p,q}}{[m]_{p,q}^2(p(1-x)+qx)}x^2.$$

(iv) In order to show the equality of the first central moment $L_{m,l}^{p,q}(t-x;x)$, we will use the linearity of the operator $L_{m,l}^{p,q}$:

$$\begin{aligned}
L_{m,l}^{p,q}(t-x;x) &= L_{m,l}^{p,q}(t;x)-xL_{m,l}^{p,q}(1;x) \\
&= \frac{[m+l]_{p,q}}{[m]_{p,q}}x-x \\
&= \left(\frac{[m+l]_{p,q}}{[m]_{p,q}}-1\right)x.
\end{aligned}$$

(v) For the second central moment $L^{p,q}_{m,l}((t-x)^2;x)$, we will again use the linearity of the operator $L^{p,q}_{m,l}$:

$$\begin{aligned}
L^{p,q}_{m,l}((t-x)^2;x) &= L^{p,q}_{m,l}(t^2;x) - 2xL^{p,q}_{m,l}(t;x) + x^2L^{p,q}_{m,l}(1;x)\\
&= \frac{p^{m+l-1}[m+l]_{p,q}}{[m]^2_{p,q}}x + \frac{q^2[m+l]_{p,q}[m+l-1]_{p,q}}{[m]^2_{p,q}(p(1-x)+qx)}x^2 - 2\frac{[m+l]_{p,q}}{[m]_{p,q}}x^2 + x^2\\
&= \frac{p^{m+l-1}[m+l]_{p,q}}{[m]^2_{p,q}}x + \left(\frac{q^2[m+l]_{p,q}[m+l-1]_{p,q}}{[m]^2_{p,q}(p(1-x)+qx)} - 2\frac{[m+l]_{p,q}}{[m]_{p,q}} + 1\right)x^2.
\end{aligned}$$

As a consequence, the proof is completed. □

3. Main Results

It is obvious that operator $L^{p,q}_{m,l}(f;x)$ is linear and positive.

Remark 3.1. *[13] For $q \in (0,1)$ and $p \in (q,1]$, $lim_{n\to\infty}[n]_{p,q} = 0$ or $\frac{1}{p-q}$. In order to reach convergence results of our operator $L^{p,q}_{m,l}(f;x)$, we take the sequences $q_m \in (0,1)$ and $p_m \in (q_m,1])$ such that $lim_{m\to\infty}p_m = 1$, $lim_{m\to\infty}q_m = 1$, $lim_{m\to\infty}p_m^m = 1$ and $lim_{m\to\infty}q_m^m = 1$. Thus, we have $lim_{m\to\infty}[m]_{p_m,q_m} = \infty$.*

Here, we can give the following theorem which guarantees the approximation process based on Korovkin's type approximation theorem.

Theorem 3.2. *Let $L^{p,q}_{m,l}(f;x)$ satisfy the conditions in Remark (3.1) for $0 < q_m < p_m \le 1$. Then for each monotone increasing function $f \in C[0,l+1]$, $L^{p,q}_{m,l}(f;x)$ converges uniformly to f on $[0,1]$.*

Proof. To prove this theorem, it is sufficient by the Korovkin theorem to show that

$$\lim_{m\longrightarrow\infty}\left\|L^{p,q}_{m,l}(t^k;x) - x^k\right\|_{C[0,l+1]} = 0, \quad k = 0,1,2.$$

i) By Lemma (2.2), Equation (7), it is clear that

$$\lim_{m\longrightarrow\infty}\left\|L^{p,q}_{m,l}(1;x) - 1\right\|_{C[0,l+1]} = \lim_{m\longrightarrow\infty}\sup_{x\in[0,1]}|L^{p,q}_{m,l}(1;x) - 1| = 0.$$

ii) By Lemma (2.2), Equation (8), we have

$$\begin{aligned}
\lim_{m\longrightarrow\infty}\left\|L^{p,q}_{m,l}(t;x) - x\right\|_{C[0,l+1]} &= \lim_{m\longrightarrow\infty}\sup_{x\in[0,1]}|L^{p,q}_{m,l}(t;x) - x|\\
&= \lim_{m\longrightarrow\infty}\sup_{x\in[0,1]}\left|\frac{[m+l]_{p,q}}{[m]_{p,q}}x - x\right|\\
&\le \lim_{m\longrightarrow\infty}\left(\frac{[m+l]_{p,q}}{[m]_{p,q}} - 1\right)\\
&= 0.
\end{aligned}$$

iii) Using Lemma (2.2), Equation (9), we can write

$$\begin{aligned}
\lim_{m\longrightarrow\infty}\left\|L^{p,q}_{m,l}(t^2;x) - x^2\right\|_{C[0,l+1]} &= \lim_{m\longrightarrow\infty}\sup_{x\in[0,1]}|L^{p,q}_{m,l}(t^2;x) - x^2|\\
&\le \lim_{m\longrightarrow\infty}\frac{p^{m+l-1}[m+l]_{p,q}}{[m]^2_{p,q}} + \left(\frac{q^2[m+l]_{p,q}[m+l-1]_{p,q}}{[m]^2_{p,q}(p(1-x)+qx)} - 1\right)\\
&= 0.
\end{aligned}$$

Thus, because of the positivity and linearity of $L^{p,q}_{m,l}(f;x)$, the proof is completed by the classical Korovkin approximation theorem. □

Currently, we will give the following lemmas.

Lemma 3.3. *Let the function f be a monotone increasing function then $L_{m,l}^{p,q}(f;x)$ is a linear and positive operator.*

Proof. The proof is obvious. So we will omit it. □

Lemma 3.4. *(Hölder Inequality) Let $\frac{1}{\alpha}+\frac{1}{\beta}=1$. Then $L(f;x)$ satisfies the following inequality*

$$L_{m,l}^{p,q}\left(\left|fg\right|;x\right) \leq \left(L_{m,l}^{p,q}\left(\left|f\right|^{\alpha};x\right)\right)^{\frac{1}{\alpha}}\left(L_{m,l}^{p,q}\left(\left|g\right|^{\beta};x\right)\right)^{\frac{1}{\beta}}.$$

4. Rate of Convergence

In this section, we will give the approximation of order of the operator $L_{m,l}^{p,q}(f;x)$ by means of modulus of continuity, with the help of functions of Lipschitz class and Peetre's K-functionals. Let $f \in C[0,l+1]$. The modulus of continuity of f, denoted by $w(f,\delta)$, is defined

$$w(f,\delta) = \sup_{\substack{|y-x|\leq\delta \\ x,y\in[0,1]}} \left|f(y)-f(x)\right|. \tag{16}$$

Then it is known that $\lim_{\delta\to 0^+} w(f,\delta)=0$ for $f\in C[0,l+1]$; and also, for any $\delta>0$ and each $t,x\in[0,1]$, we have

$$\left|f(t)-f(x)\right| \leq w(f,\delta)\left(\frac{|t-x|}{\delta}+1\right). \tag{17}$$

Firstly, we will give the rate of convergence of $L_{m,l}^{p,q}(f;x)$ by means of modulus of continuity.

Theorem 4.1. *Let $p:=(p_m)$ and $q:=(q_m)$, $0<q_m<p_m\leq 1$, be sequences satisfying the conditions given in Remark (3.1). Then for all $f\in C[0,l+1]$,*

$$\|L_{m,l}^{p,q}(f;x)-f(x)\|_{C[0,l+1]} \leq 2\omega\left(f;\delta_m(x)\right),$$

where

$$\begin{aligned} \delta_m(x) &= \sqrt{L_{m,l}^{p,q}((t-x)^2;x)} \\ &= \left\{\frac{p^{m+l-1}[m+l]_{p,q}}{[m]_{p,q}^2}x+\left(\frac{q^2[m+l]_{p,q}[m+l-1]_{p,q}}{[m]_{p,q}^2(p(1-x)+qx)}-2\frac{[m+l]_{p,q}}{[m]_{p,q}}+1\right)x^2\right\}^{\frac{1}{2}}. \end{aligned} \tag{18}$$

Proof. In order to prove this theorem, we will use the linearity and positivity of the operator $L_{m,l}^{p,q}(f;x)$, we have

$$\begin{aligned} \left|L_{m,l}^{p,q}(f;x)-f(x)\right| &= \left|L_{m,l}^{p,q}(f(t)-f(x);x)\right| \\ &\leq L_{m,l}^{p,q}\left(\left|f(t)-f(x)\right|;q;x\right). \end{aligned}$$

Then applying (17), we have

$$\begin{aligned} \left|L_{m,l}^{p,q}(f;x)-f(x)\right| &\leq L_{m,l}^{p,q}\left(w(f,\delta_m)\left(\frac{|t-x|}{\delta_m}+1\right);x\right) \\ &= \frac{w(f,\delta_m)}{\delta_m}\sqrt{L_{m,l}^{p,q}((t-x)^2;x)}+w(f,\delta_m) \\ &= w(f,\delta_m)\left(1+\frac{1}{\delta_m}\sqrt{L_{m,l}^{p,q}((t-x)^2;x)}\right). \end{aligned} \tag{19}$$

$$\begin{aligned} \|L_{m,l}^{p,q}(f;x)-f(x)\|_{C[0,l+1]} &= \sup_{x\in[0,1]}\left|L_{m,l}^{p,q}(f;x)-f(x)\right| \\ &\leq w(f,\delta_m)\left(1+\frac{1}{\delta_m}\sqrt{L_{m,l}^{p,q}((t-x)^2;x)}\right). \end{aligned}$$

Choose

$$\delta_m(x)=\left\{\frac{p^{m+l-1}[m+l]_{p,q}}{[m]_{p,q}^2}x+\left(\frac{q^2[m+l]_{p,q}[m+l-1]_{p,q}}{[m]_{p,q}^2 p(1-x)+qx}-2\frac{[m+l]_{p,q}}{[m]_{p,q}}+1\right)x^2\right\}^{\frac{1}{2}}.$$

Thus, we get the desired result

$$\|L_{m,l}^{p,q}(f;x)-f(x)\|_{C[0,l+1]}\leq 2\omega\left(f;\delta_m(x)\right)$$

then the proof is completed. □

Now, we will give the rate of convergence of $L_{m,l}^{p,q}(f;x)$ with the help of functions of Lipschitz class. We recall that a function $f\in Lip_M(\alpha)$ on $[0,l+1]$ if the inequality

$$\left|f(t)-f(x)\right|\leq M\left|t-x\right|^{\alpha}\quad;\ \forall t,x\in[0,1] \tag{20}$$

holds.

Theorem 4.2. *Let $f\in Lip_M(\alpha)$. Let $p:=(p_m)$ and $q:=(q_m)$, $0<q_m<p_m\leq 1$, then we have*

$$\|L_{m,l}^{p,q}(f;x)-f(x)\|\leq M\delta_m^{\alpha}(x),$$

where

$$\begin{aligned} \delta_m(x) &= \sqrt{L_{m,l}^{p,q}((t-x)^2;x)} \qquad (21) \\ &= \left\{\frac{p^{m+l-1}[m+l]_{p,q}}{[m]_{p,q}^2}x+\left(\frac{q^2[m+l]_{p,q}[m+l-1]_{p,q}}{[m]_{p,q}^2 p(1-x)+qx}-2\frac{[m+l]_{p,q}}{[m]_{p,q}}+1\right)x^2\right\}^{\frac{1}{2}}. \end{aligned}$$

Proof. Let $f\in Lip_M(\alpha)$ and $0<\alpha\leq 1$. Since $L_{m,l}^{p,q}(f;x)$ is linear and monotone, by using (20), we have

$$\begin{aligned} \left|L_{m,l}^{p,q}(f;x)-f(x)\right| &\leq L_{m,l}^{p,q}(\left|f(t)-f(x)\right|;x) \\ &\leq ML_{m,l}^{p,q}(|t-x|^{\alpha};x). \end{aligned}$$

If we take $p=\frac{2}{\alpha}$, $q=\frac{2}{2-\alpha}$ and apply Hölder inequality, then we obtain

$$\begin{aligned} \left|L_{m,l}^{p,q}(f;x)-f(x)\right| &\leq M\left\{L_{m,l}^{p,q}\left((t-x)^2;x\right)\right\}^{\frac{\alpha}{2}} \\ &\leq M\delta_m^{\alpha}(x) \end{aligned}$$

immediately. If we choose

$$\delta_m(x)=\left\{\frac{p^{m+l-1}[m+l]_{p,q}}{[m]_{p,q}^2}x+\left(\frac{q^2[m+l]_{p,q}[m+l-1]_{p,q}}{[m]_{p,q}^2 p(1-x)+qx}-2\frac{[m+l]_{p,q}}{[m]_{p,q}}+1\right)x^2\right\}^{\frac{1}{2}},$$

the proof is completed. □

Lastly, we will give the rate of convergence of our operator $L_{m,l}^{p,q}(f;x)$ by means of Peetre-K functionals. First of all, we give the following lemma:

Lemma 4.3. *For $f \in C[0, 1 + l]$, we have*

$$|L_{m,l}^{p,q}(f; x)| \leq \|f\|. \tag{22}$$

Proof. By using the definition of (6), we obtain

$$\begin{aligned} |L_{m,l}^{p,q}(f; x)| &= \left| \sum_{k=0}^{m+l} \frac{f\left(\frac{p^{m+l-k}[k]_{p,q}}{[m]_{p,q}}\right) \begin{bmatrix} m+l \\ k \end{bmatrix}_{p,q} p^{\frac{(m+l-k)(m+l-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^k (1-x)^{m+l-k}}{\prod_{j=1}^{m+l} \{p^{j-1}(1-x) + q^{j-1}x\}} \right| \\ &\leq \sum_{k=0}^{m+l} \frac{\left|f\left(\frac{p^{m+l-k}[k]_{p,q}}{[m]_{p,q}}\right)\right| \begin{bmatrix} m+l \\ k \end{bmatrix}_{p,q} p^{\frac{(m+l-k)(m+l-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^k (1-x)^{m+l-k}}{\prod_{j=1}^{m+l} \{p^{j-1}(1-x) + q^{j-1}x\}} \\ &= \|f\| L_{m,l}^{p,q}(1; x) \\ &= \|f\| \end{aligned}$$

the desired result. □

And then, we recall the properties of Peetre's K-functionals. $C^2\,[0, l + 1]$ is the space of the functions f, for which f, f' and f'' are continuous on $[0, l + 1]$. We write the norm of function f in the space $C^2\,[0, l + 1]$

$$\|f\|_{C^2[0,l+1]} = \|f\|_{C[0,l+1]} + \|f'\|_{C[0,l+1]} + \|f''\|_{C[0,l+1]}.$$

Now, we define classical Peetre's K-functional as follows:

$$K(f, \delta) := \inf_{g \in C^2[0,1]} \left\{ \|f - g\|_{C[0,l+1]} + \delta \|g\|_{C^2[0,l+1]} \right\}$$

and second modulus of smoothness of the function is defined by

$$\omega_2(f, \delta) := \sup_{0<h<\delta} \sup_{x, x+h \in [0,l+1]} |f(x + 2h) - 2f(x + h) + f(x)|,$$

where $\delta > 0$. By [20], it is known that for $A > 0$

$$K(f, \delta) \leq A\omega_2(f, \sqrt{\delta}).$$

Theorem 4.4. *Let $f \in C[0, l + 1]$ and $0 < q_m < p_m \leq 1$. Then we have for all $n \in \mathbb{N}$, there exists a positive constant M such that,*

$$|L_{m,l}^{p,q}(f; x) - f(x)| \leq M\omega_2(f, \alpha_m(x)) + \omega(f, \beta_m(x)),$$

where

$$\alpha_m(x) = \sqrt{L_{m,l}^{p,q}((t - x)^2; x) + \frac{x^2}{2} \left(\frac{[m + l]_{p,q}}{[m]_{p,q}} - 1 \right)^2} \tag{23}$$

and

$$\beta_m(x) = \left(\frac{[m + l]_{p,q}}{[m]_{p,q}} - 1 \right) x. \tag{24}$$

Proof. Define an auxiliary operator $L^*_{m,l}(f;x) : C[0,l+1] \to C[0,1]$ by

$$L^*_{m,l}(f;x) = L^{p,q}_{m,l}(f;x) - f\left(\frac{[m+l]_{p,q}}{[m]_{p,q}}x\right) + f(x). \tag{25}$$

From Lemma (2.2), we have

$$\begin{aligned} L^*_{m,l}(1;x) &= 1, \\ L^*_{m,l}(t-x;x) &= L^{p,q}_{m,l}((t-x);x) - \left(\frac{[m+l]_{p,q}}{[m]_{p,q}}x - x\right) + x - x \\ &= \left(\frac{[m+l]_{p,q}}{[m]_{p,q}} - 1\right)x - \frac{[m+l]_{p,q}}{[m]_{p,q}}x + x + x - x \\ &= 0. \end{aligned} \tag{26}$$

This means that the operators $L^*_{m,l}(f;x)$ are linear. For a given function $g \in C^2[0,l+1]$, we have by the Taylor expansion that

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du, \quad t \in [0,1]. \tag{27}$$

Applying $L^*_{m,l}$ operator to both sides of the equation (27), we get

$$\begin{aligned} L^*_{m,l}(g;x) &= L^*_{m,l}\left(g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du\right) \\ &= g(x) + L^*_{m,l}\left((t-x)g'(x);x\right) + L^*_{m,l}\left(\int_x^t (t-u)g''(u)du\right). \end{aligned}$$

So,

$$L^*_{m,l}(g;x) - g(x) = g'(x)L^*_{m,l}\left((t-x);x\right) + L^*_{m,l}\left(\int_x^t (t-u)g''(u)du\right).$$

Using (25) and (26) we obtain

$$\begin{aligned} L^*_{m,l}(g;x) - g(x) &= L^*_{m,l}\left(\int_x^t (t-u)g''(u)du\right) \\ &= L^{p,q}_{m,l}\left(\int_x^t (t-u)g''(u)du\right) - \int_x^{\frac{[m+l]_{p,q}}{[m]_{p,q}}x}\left(\frac{[m+l]_{p,q}}{[m]_{p,q}}x - u\right)g''(u)du \\ &\quad + \int_x^x \left(\frac{[m+l]_{p,q}}{[m]_{p,q}}x - u\right)g''(u)du. \end{aligned} \tag{28}$$

Moreover,

$$\begin{aligned} \left|\int_x^t (t-u)g''(u)du\right| &\le \int_x^t |t-u||g''(u)|du \\ &\le \|g''\| \int_x^t |t-u|du \le (t-x)^2\|g''\| \end{aligned} \tag{29}$$

and

$$\begin{aligned}
\left|\int_x^{\frac{[m+l]_{p,q}}{[m]_{p,q}}x}\left(\frac{[m+l]_{p,q}}{[m]_{p,q}}x-u\right)g''(u)du\right| &\leq \|g''\|\int_x^{\frac{[m+l]_{p,q}}{[m]_{p,q}}x}\left(\frac{[m+l]_{p,q}}{[m]_{p,q}}x-u\right)du \\
&= \|g''\|\left(\frac{[m+l]_{p,q}^2x^2}{2[m]_{p,q}^2}-\frac{[m+l]_{p,q}x^2}{[m]_{p,q}}+\frac{x^2}{2}\right) \\
&= \frac{\|g''\|x^2}{2}\left(\frac{[m+l]_{p,q}}{[m]_{p,q}}-1\right)^2. \qquad (30)
\end{aligned}$$

When we rewrite (29) and (30) in the absolute value of (28), we obtain

$$\begin{aligned}
\left|L_{m,l}^*(g;x)-g(x)\right| &\leq \|g''\|L_{m,l}^{p,q}((t-x)^2;x)+\frac{\|g''\|x^2}{2}\left(\frac{[m+l]_{p,q}}{[m]_{p,q}}-1\right)^2 \\
&= \|g''\|\left(L_{m,l}^{p,q}((t-x)^2;x)+\frac{x^2}{2}\left(\frac{[m+l]_{p,q}}{[m]_{p,q}}-1\right)^2\right) \\
&= \|g''\|\alpha_m^2(x),
\end{aligned}$$

where

$$\alpha_m(x)=\sqrt{L_{m,l}^{p,q}((t-x)^2;x)+\frac{x^2}{2}\left(\frac{[m+l]_{p,q}}{[m]_{p,q}}-1\right)^2}.$$

Currently, we will find a bound for the auxiliary operator $L_{m,l}^*(f;x)$. In the light of the Lemma (4.3) we obtain

$$\begin{aligned}
|L_{m,l}^*(f;x)| &= \left|L_{m,l}^{p,q}(f;x)-f\left(\frac{[m+l]_{p,q}}{[m]_{p,q}}x\right)+f(x)\right| \\
&\leq |L_{m,l}^{p,q}(f;x)|+\left|f\left(\frac{[m+l]_{p,q}}{[m]_{p,q}}x\right)\right|+|f(x)| \\
&\leq 3\|f\|.
\end{aligned}$$

Accordingly,

$$\begin{aligned}
|L_{m,l}^{p,q}(f;x)-f(x)| &= \left|L_{m,l}^*(f;x)-f(x)+f\left(\frac{[m+l]_{p,q}}{[m]_{p,q}}x\right)-f(x)\mp g(x)\mp L_{m,l}^*(g;x)\right| \\
&\leq |L_{m,l}^*(f-g;x)-(f-g)(x)|+|L_{m,l}^*(g;x)-g(x)| \\
&\quad+\left|f\left(\frac{[m+l]_{p,q}}{[m]_{p,q}}x\right)-f(x)\right| \\
&\leq 4\|f-g\|+\|g''\|\alpha_m^2(x)+\omega\left(f,\left(\frac{[m+l]_{p,q}}{[m]_{p,q}}-1\right)x\right) \\
&\leq 4\left(\|f-g\|+\|g''\|\alpha_m^2(x)\right)+\omega\left(f,\beta_m(x)\right), \qquad (31)
\end{aligned}$$

where

$$\beta_m(x)=\left(\frac{[m+l]_{p,q}}{[m]_{p,q}}-1\right)x. \qquad (32)$$

Finally, for all $g\in C^2[0,l+1]$ take the infimum of the equation (31). We get

$$|L_{m,l}^{p,q}(f;x)-f(x)| \leq 4K(f,\alpha_m^2(x))+\omega\left(f,\beta_m(x)\right). \qquad (33)$$

As a result, using the property of Peetre's K-functional, we obtain

$$|L_{m,l}^{p,q}(f;x) - f(x)| \leq M\omega_2(f, \alpha_m(x)) + \omega\left(f, \beta_m(x)\right). \tag{34}$$

Thus the proof is completed. □

5. Conclusion

In this paper, we introduced (p,q)-analogue of Lupaş-Schurer operators by using (p,q)-integers. (p,q)-analogue of Lupaş-Schurer operators has an advantage to generate positive linear operators for all $p > 0$ and $q > 0$ whereas (p,q)-analogue of Bernstein-Schurer operators [5] generates positive linear operators only if $0 < q < p \leq 1$. We obtained some approximation properties of the constructed operators and dealed with the rate of convergence by using modulus of continuity, with the help of functions of Lipschitz class and Peetre's K-functionals.

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