



Growth of Solutions of Second Order Complex Linear Differential Equations with Entire Coefficients

Jianren Long

^aSchool of Mathematical Sciences, Guizhou Normal University, Guiyang, 550001, P.R. China.

^bSchool of Computer Science and School of Science, Beijing University of Posts and Telecommunications, Beijing, 100876, P.R. China.

Abstract. Some new conditions on the entire coefficients $A(z)$ and $B(z)$, which guarantee every nontrivial solution of $f'' + A(z)f' + B(z)f = 0$ is of infinite order, are given in this paper. Two classes of entire functions are involved in these conditions, the one is entire functions having Fabry gaps, the another is function extremal for Yang's inequality. Moreover, a kind of entire function having finite Borel exception value is considered.

1. Introduction and Main Results

For a meromorphic function f in the complex plane \mathbb{C} , the order of growth and the lower order of growth are defined as

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r}, \quad \mu(f) = \liminf_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r},$$

respectively. If f is entire function, then the Nevanlinna characteristic $T(r, f)$ can be replaced with $\log M(r, f)$, where $M(r, f) = \max_{|z|=r} |f(z)|$ is the usual maximum modulus of f , see [26, p. 10].

We consider the order of growth of solutions of complex linear differential equations of the form

$$f'' + A(z)f' + B(z)f = 0, \tag{1}$$

where $A(z)$ and $B(z) (\neq 0)$ are entire functions. We focus on looking for the conditions of coefficients, which guarantee every nontrivial solution of (1) is of infinite order. There are many results in the literature concerning this problem; see, for example, [11] and [12]. The following result is a summary of results derived from Gundersen [5], Hellerstein, Miles and Rossi [8], and Ozawa [21].

Theorem 1.1. *Suppose that $A(z)$ and $B(z)$ are entire functions satisfying one of the following conditions.*

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Email address: longjianren2004@163.com; jrlong@gznu.edu.cn (Jianren Long)

- (i) $\rho(A) < \rho(B)$;
- (ii) $A(z)$ is a polynomial and $B(z)$ is a transcendental entire function;
- (iii) $\rho(B) < \rho(A) \leq \frac{1}{2}$.

Then every nontrivial solution of (1) is of infinite order.

Motivated by Theorem 1.1, many parallel results written thereafter focus on the case $\rho(A) \geq \rho(B)$; see, for example, [1, 13–16, 24]. However, in general, the conclusions are false for the case $\rho(A) \geq \rho(B)$. For example, (i) The case of $\rho(A) = \rho(B)$: $f(z) = e^{P(z)}$ solves (1) for arbitrary $A(z)$ with $B(z) = -P'' - (P')^2 - A(z)P'$, where $P(z)$ is a polynomial. (ii) The case of $\rho(A) > \rho(B)$: We can find some examples to show that there exist finite order solution of (1) in [5].

In [13], Laine and Wu studied the growth of solutions of (1) by considering when the coefficient $A(z)$ has Fejér gaps. An entire function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ is said to have Fejér gaps if

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n} < \infty. \tag{2}$$

Theorem 1.2 ([13]). Suppose that $A(z)$ and $B(z)$ are entire functions with $\rho(B) < \rho(A) < \infty$ and $A(z)$ has Fejér gaps. Then every nontrivial solution of (1) is of infinite order.

Now we consider another gap series to study the growth of solutions of (1), namely Fabry gaps, if the gap condition (2) is replaced with

$$\frac{\lambda_n}{n} \rightarrow \infty \text{ as } n \rightarrow \infty. \tag{3}$$

The condition (3) is a weaker gap condition than (2), and an entire function with Fabry gaps has positive order; see, [7, p. 651].

Here we mention an improvement of Theorem 1.2 in which $A(z)$ has Fejér gaps is replaced with $A(z)$ has Fabry gaps, which can be found in [20, Corollary 1]. In [17, 24], we study the growth of solutions of (1) by assuming the coefficient $A(z)$ itself is a solution of another differential equation. Here we will prove the following result by combining this approach with the properties of Fabry gap series.

Theorem 1.3. Let $A(z)$ be a nontrivial solution of $w'' + P(z)w = 0$, where $P(z) = a_n z^n + \dots + a_0, a_n \neq 0$. Let $B(z)$ be an entire function with Fabry gaps such that $\rho(B) \neq \rho(A)$. Then every nontrivial solution of (1) is of infinite order.

An analogue of Theorem 1.3 in which the assumption $B(z)$ having Fabry gaps is replaced with $\mu(B) < \frac{1}{2} + \frac{1}{2(n+1)}$ can be proved in [24].

Next, we consider a kind of entire functions in studying the growth of solutions of (1), namely functions extremal for Yang’s inequality. To this end, we first recall the definition of Borel direction as follows [25].

Definition 1.4. Let f be a meromorphic function in \mathbb{C} with $0 < \mu(f) < \infty$. A ray $\arg z = \theta \in [0, 2\pi)$ from the origin is called a Borel direction of order $\geq \mu(f)$ of f , if for any positive number ε and for any complex number $a \in \mathbb{C} \cup \{\infty\}$, possibly with two exceptions, the following inequality holds

$$\limsup_{r \rightarrow \infty} \frac{\log n(S(\theta - \varepsilon, \theta + \varepsilon, r), a, f)}{\log r} \geq \mu(f), \tag{4}$$

where $n(S(\theta - \varepsilon, \theta + \varepsilon, r), a, f)$ denotes the number of zeros, counting the multiplicities, of $f - a$ in the region $S(\theta - \varepsilon, \theta + \varepsilon, r) = \{z : \theta - \varepsilon < \arg z < \theta + \varepsilon, |z| < r\}$.

The definition of Borel direction of order $\rho(f)$ of f can be found in [28, p. 78], it is defined similarly with the only exception that “ $\geq \mu(f)$ ” in (4) is to be replaced with “ $= \rho(f)$ ”.

In the sequel we will require the following result, known as Yang’s inequality, on general value distribution theory.

Theorem 1.5 ([25]). *Suppose that f is an entire function of finite lower order $\mu > 0$. Let $q < \infty$ denote the number of Borel directions of order $\geq \mu$, and let p denote the number of finite deficient values of f . Then $p \leq \frac{q}{2}$.*

An entire function f is called extremal for Yang’s inequality if f satisfies the assumptions of Theorem 1.5 with $p = \frac{q}{2}$. These functions were introduced in [23]. The example of functions extremal for Yang’s inequality can be found in [26, pp. 210-211].

The functions extremal of Yang’s inequality is used firstly to study the growth of solution of (1) in [15], and the following result is proved.

Theorem 1.6 ([15]). *Let $A(z)$ be an entire function extremal for Yang’s inequality, and let $B(z)$ be a transcendental entire function such that $\rho(B) \neq \rho(A)$. Then every nontrivial solution of (1) is of infinite order.*

Now we will prove the following result by combining the functions extremal for Yang’s inequality with the properties of entire function with Fabry gaps.

Theorem 1.7. *Let $A(z)$ be an entire function extremal for Yang’s inequality, and let $B(z)$ be an entire function with Fabry gaps. Then every nontrivial solution of (1) is of infinite order.*

Comparing the condition of Theorem 1.6, we know that the case of $\rho(A) = \rho(B)$ is contained in the conclusion of Theorem 1.7.

Next we remind Wu and Zhu’s result in [22], if $A(z)$ is an entire function having a finite deficient value while $B(z)$ is a transcendental entire function with $\mu(B) < \frac{1}{2}$, then every nontrivial solution of (1) is of infinite order. Motivated by this result, we want to ask: is it possible the condition of $\mu(B) < \frac{1}{2}$ be deleted? We study the problem in the last result in which the concept of Borel exceptional value is involved.

Theorem 1.8. *Let $A(z)$ be an entire function having a finite Borel exceptional value, and let $B(z)$ be an entire function with Fabry gaps. Then every nontrivial solution of (1) is of infinite order.*

2. Auxiliary Results

The Lebesgue linear measure of a set $E \subset [0, \infty)$ is $m(E) = \int_E dt$, and the logarithmic measure of a set $F \subset [1, \infty)$ is $m_1(F) = \int_F \frac{dt}{t}$. The upper and lower logarithmic densities of $F \subset [1, \infty)$ are given by

$$\overline{\log \text{dens}}(F) = \limsup_{r \rightarrow \infty} \frac{m_1(F \cap [1, r])}{\log r}$$

and

$$\underline{\log \text{dens}}(F) = \liminf_{r \rightarrow \infty} \frac{m_1(F \cap [1, r])}{\log r},$$

respectively. We say F has logarithmic density if $\overline{\log \text{dens}}(F) = \underline{\log \text{dens}}(F)$.

A lemma on logarithmic derivatives due to Gundersen [4] play an important role in proving our results.

Lemma 2.1. *Let f be a transcendental meromorphic function of finite order $\rho(f)$. Let $\varepsilon > 0$ be a given real constant, and let k and j be two integers such that $k > j \geq 0$. Then there exists a set $E \subset (1, \infty)$ with $m_1(E) < \infty$, such that for all z satisfying $|z| \notin (E \cup [0, 1])$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho(f)-1+\varepsilon)}.$$

Next, we state the properties of entire function with Fabry gaps. It can be found in [2, Theorem 1], we find also another similar version of Lemma 2.2 in [6, Theorem 3].

Lemma 2.2 ([2, Theorem 1]). Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of finite order with Fabry gaps. Then, for any given $\varepsilon > 0$,

$$\log L(r, f) > (1 - \varepsilon) \log M(r, f)$$

holds outside a set of logarithmic density 0, where $L(r, f) = \min_{|z|=r} |f(z)|$, $M(r, f) = \max_{|z|=r} |f(z)|$.

Lemma 2.3 ([10, Lemma 2.2]). Let $\varphi(r)$ be a non-decreasing, continuous function on \mathbb{R}^+ . Suppose that

$$0 < \rho < \limsup_{r \rightarrow \infty} \frac{\log \varphi(r)}{\log r},$$

and set

$$G = \{r \in \mathbb{R}^+ : \varphi(r) \geq r^\rho\}.$$

Then we have $\overline{\text{logdens}}(G) > 0$.

Combing Lemma 2.2 and Lemma 2.3, we have the following result.

Lemma 2.4. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of finite order with Fabry gaps, and let g be an entire function with order $\rho(g) \in (0, \infty)$. Then, for any given $\varepsilon \in (0, \varsigma)$, where $\varsigma = \min\{1, \rho(g)\}$, there exists a set $F \subset (1, \infty)$ satisfying $\overline{\text{logdens}}(F) \geq \eta$, where $\eta \in (0, 1)$ is a constant, such that for all z satisfying $|z| = r \in F$,

$$\log L(r, f) > (1 - \varepsilon) \log M(r, f), \quad \log M(r, g) > r^{\rho(g) - \varepsilon}.$$

Proof. By using Lemma 2.3, for any given $\varepsilon \in (0, \varsigma)$, there exists a set $F_1 = \{r \in (1, \infty) : \log M(r, g) > r^{\rho(g) - \varepsilon}\}$, such that $\overline{\text{logdens}}(F_1) = \sigma \in (0, 1]$. On the other hand, by using Lemma 2.2, for $\eta = \frac{\sigma}{2}$, there exists a set $F_2 \subset (1, \infty)$ with $\overline{\text{logdens}}(F_2) > 1 - \eta$, such that for all $|z| = r \in F_2$, we have

$$\log L(r, f) > (1 - \varepsilon) \log M(r, f).$$

Set $F = F_1 \cap F_2$. Then

$$\begin{aligned} & \overline{\text{logdens}}(F) + \overline{\text{logdens}}(F_1^c \cap F_2) \\ &= \overline{\text{logdens}}(F_1 \cap F_2) + \overline{\text{logdens}}(F_1^c \cap F_2) \\ &\geq \overline{\text{logdens}}(F_2) \\ &> 1 - \eta. \end{aligned}$$

It follows from $\overline{\text{logdens}}(F_1) + \overline{\text{logdens}}(F_1^c) = 1$ that

$$\begin{aligned} \overline{\text{logdens}}(F) &\geq 1 - \eta - \overline{\text{logdens}}(F_1^c \cap F_2) \\ &\geq 1 - \eta - \overline{\text{logdens}}(F_1^c) \\ &= \overline{\text{logdens}}(F_1) - \eta = \eta. \end{aligned}$$

This completes the proof. \square

From Lemma 2.4, we deduce immediately that if $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ is an entire function of order $\rho(f) \in (0, \infty)$ with Fabry gaps, then for any given $\varepsilon \in (0, \frac{\rho(f)}{2})$, there exists a set $F \subset (1, \infty)$ satisfying $\overline{\log \text{dens}}(F) > 0$, such that for all z satisfying $|z| = r \in F$,

$$|f(z)| > M(r, f)^{1-\varepsilon} > \exp\left((1-\varepsilon)r^{\rho(f)-\varepsilon}\right) > \exp(r^{\rho(f)-2\varepsilon}).$$

The next two lemmas are related with Borel exceptional value. The Lemma 2.6 can be found in [14], in order to reader’s convenience, here a proof is given.

Lemma 2.5 ([27, Theorem 2.11]). *Let f be a meromorphic function of order $\rho(f) > 0$. If f has two distinct Borel exceptional values, then f is of regular growth and its (lower) order is a positive integer or ∞ .*

Lemma 2.6. *Let f be an entire function of finite order having a finite Borel exceptional value c . Then*

$$f(z) = h(z)e^{Q(z)} + c,$$

where $h(z)$ is an entire function with $\rho(h) < \rho(f)$, $Q(z)$ is a polynomial of degree $\deg(Q) = \rho(f)$.

Proof. By the condition of Lemma 2.6, suppose that c is a finite Borel exceptional value of $f(z)$. Set

$$g(z) = f(z) - c.$$

Then $g(z)$ has two Borel exceptional values 0 and ∞ . Applying Lemma 2.5 and Hadamard’s factorization theory, $g(z)$ takes the form

$$g(z) = h(z)e^{Q(z)},$$

where $h(z)$ is an entire function satisfying

$$\rho(h) = \lambda(g) < \rho(g),$$

and $Q(z)$ is a polynomial satisfying

$$\deg(Q) = \rho(g) = \rho(f),$$

where $\lambda(g)$ denotes the exponent of convergence of zeros of $g(z)$. Therefore,

$$f(z) = h(z)e^{Q(z)} + c.$$

This completes the proof. \square

The following lemma due to Markushevich [18].

Lemma 2.7. *Let $P(z) = b_n z^n + b_{n-1} z^{n-1} + \dots + b_0$, where n is a positive integer and $b_n = \alpha_n e^{i\theta_n}$, $\alpha_n > 0$, $\theta_n \in [0, 2\pi)$. For any given $\varepsilon \in (0, \frac{\pi}{4n})$, we introduce $2n$ open angles*

$$S_j = \{z : -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon < \arg z < -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon\},$$

where $j = 0, 1, \dots, 2n-1$. Then there exists a positive number $R = R(\varepsilon)$ such that for $|z| = r > R$,

$$\text{Re}\{P(z)\} > \alpha_n(1-\varepsilon)\sin(n\varepsilon)r^n$$

if $z \in S_j$ when j is even; while

$$\text{Re}\{P(z)\} < -\alpha_n(1-\varepsilon)\sin(n\varepsilon)r^n$$

if $z \in S_j$ when j is odd.

Now for any given $\theta \in [0, 2\pi)$, if $\theta \neq -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n}$, $j = 0, 1, \dots, 2n-1$, then we take ε sufficiently small, there exists some S_j such that $z \in S_j$, where $\arg z = \theta$ and $j \in \{0, 1, \dots, 2n-1\}$.

3. Proof of Theorem 1.3

In order to prove the Theorem 1.3, an auxiliary result is also needed, in which the properties of solutions of $w'' + P(z)w = 0$ is described. To this end, some notations are stated. Let $\alpha < \beta$ be such that $\beta - \alpha < 2\pi$, and let $r > 0$. Denote

$$S(\alpha, \beta) = \{z : \alpha < \arg z < \beta\},$$

$$S(\alpha, \beta, r) = \{z : \alpha < \arg z < \beta\} \cap \{z : |z| < r\}.$$

Let \bar{F} denote the closure of F . Let A be an entire function of order $\rho(A) \in (0, \infty)$. For simplicity, set $\rho = \rho(A)$ and $S = S(\alpha, \beta)$. We say that A blows up exponentially in \bar{S} if for any $\theta \in (\alpha, \beta)$

$$\lim_{r \rightarrow \infty} \frac{\log \log |A(re^{i\theta})|}{\log r} = \rho$$

holds. We also say that A decays to zero exponentially in \bar{S} if for any $\theta \in (\alpha, \beta)$

$$\lim_{r \rightarrow \infty} \frac{\log \log |A(re^{i\theta})|^{-1}}{\log r} = \rho$$

holds.

The following lemma, originally due to Hille [9, Chapter 7.4], see also [3] and [19], plays an important role in proving Theorem 1.3. The method used in proving the lemma is typically referred to as the method of asymptotic integration.

Lemma 3.1. *Let A be a nontrivial solution of $w'' + P(z)w = 0$, where $P(z) = a_n z^n + \dots + a_0, a_n \neq 0$. Set $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ and $S_j = S(\theta_j, \theta_{j+1})$, where $j = 0, 1, 2, \dots, n + 1$ and $\theta_{n+2} = \theta_0 + 2\pi$. Then A has the following properties.*

- (i) *In each sector S_j , A either blows up or decays to zero exponentially.*
- (ii) *If, for some j , A decays to zero in S_j , then it must blow up in S_{j-1} and S_{j+1} . However, it is possible for A to blow up in many adjacent sectors.*
- (iii) *If A decays to zero in S_j , then A has at most finitely many zeros in any closed sub-sector within $S_{j-1} \cup \bar{S}_j \cup S_{j+1}$.*
- (iv) *If A blows up in S_{j-1} and S_j , then for each $\varepsilon > 0$, A has infinitely many zeros in each sector $\bar{S}(\theta_j - \varepsilon, \theta_j + \varepsilon)$, and furthermore, as $r \rightarrow \infty$,*

$$n(\bar{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r), 0, A) = (1 + o(1)) \frac{2\sqrt{|a_n|}}{\pi(n+2)} r^{\frac{n+2}{2}},$$

where $n(\bar{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r), 0, A)$ is the number of zeros of A in the region $\bar{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r)$.

If $\rho(A) < \rho(B)$, then the conclusion is proved by Gundersen [5, Theorem 2]. Therefore we may assume $\rho(A) > \rho(B)$. Suppose on the contrary to the assertion that there is a nontrivial solution f of (1) with $\rho(f) < \infty$. We aim for a contradiction. Set $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ and $S_j = \{z : \theta_j < \arg z < \theta_{j+1}\}$, where $j = 0, 1, 2, \dots, n + 1$ and $\theta_{n+2} = \theta_0 + 2\pi$. We divide into two cases by using Lemma 3.1.

Case 1: Suppose that $A(z)$ blows up exponentially in each sector $S_j, j = 0, 1, \dots, n + 1$. We get a contradiction by using the similar way in proving [24, Theorem 1.3].

Case 2: There exists at least one sector of the $n + 2$ sectors, such that $A(z)$ decays to zero exponentially, say $S_{j_0} = \{z : \theta_{j_0} < \arg z < \theta_{j_0+1}\}, 0 \leq j_0 \leq n + 1$. It is well known that $\rho(A) = \frac{n+2}{2}$, see also [11]. It implies that for any $\theta \in (\theta_{j_0}, \theta_{j_0+1})$, we have

$$\lim_{r \rightarrow \infty} \frac{\log \log \frac{1}{|A(re^{i\theta})|}}{\log r} = \frac{n+2}{2}. \tag{5}$$

By Lemma 2.4, for any given $\varepsilon \in (0, \frac{\rho(B)}{4})$, there exists a set $E_1 \subset (1, \infty)$ with $\overline{\text{logdens}}(E_1) > 0$, such that for all z satisfying $|z| = r \in E_1$,

$$|B(z)| > \exp(r^{\rho(B)-\varepsilon}). \tag{6}$$

By Lemma 2.1, there exists a set $E_2 \subset (1, \infty)$ with $m_1(E_2) < \infty$, such that for all z satisfying $|z| = r \notin (E_2 \cup [0, 1])$,

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq |z|^{2\rho(f)}, \quad k = 1, 2. \tag{7}$$

Thus, there exists a sequence of points $z_n = r_n e^{i\theta}$ with $r_n \rightarrow \infty$ as $n \rightarrow \infty$, where $r_n \in E_1 - (E_2 \cup [0, 1])$ and $\theta \in (\theta_{j_0}, \theta_{j_0+1})$, such that (5), (6) and (7) hold. It follows from (5), (6), (7) and (1) that, for every $n > n_0$,

$$\begin{aligned} \exp(r_n^{\rho(B)-\varepsilon}) &< |B(r_n e^{i\theta})| \\ &\leq \left| \frac{f''(r_n e^{i\theta})}{f(r_n e^{i\theta})} \right| + |A(r_n e^{i\theta})| \left| \frac{f'(r_n e^{i\theta})}{f(r_n e^{i\theta})} \right| \\ &\leq r_n^{2\rho(f)} (1 + o(1)). \end{aligned}$$

Obviously, this is a contradiction for sufficiently large n and arbitrary small ε . This completes the proof.

4. Proof of Theorem 1.7

We begin by recalling some basic properties satisfied by entire functions that are extremal for Yang’s inequality. To this end, if A is a function extremal for Yang’s inequality, then let the rays $\arg z = \theta_k$, denote the q distinct Borel directions of order $\geq \mu(A)$ of A , where $k = 1, 2, \dots, q$ and $0 \leq \theta_1 < \theta_2 < \dots < \theta_q < \theta_{q+1} = \theta_1 + 2\pi$.

Lemma 4.1 ([23]). *Suppose that A is a function extremal for Yang’s inequality. Then $\mu(A) = \rho(A)$. Moreover, for every deficient value $a_i, i = 1, 2, \dots, \frac{q}{2}$, there exists a corresponding sector domain $S(\theta_{k_i}, \theta_{k_i+1}) = \{z : \theta_{k_i} < \arg z < \theta_{k_i+1}\}$ such that for every $\varepsilon > 0$ the inequality*

$$\log \frac{1}{|A(z) - a_i|} > C(\theta_{k_i}, \theta_{k_i+1}, \varepsilon, \delta(a_i, A)) T(|z|, A) \tag{8}$$

holds for $z \in S(\theta_{k_i} + \varepsilon, \theta_{k_i+1} - \varepsilon; r, \infty) = \{z : \theta_{k_i} + \varepsilon < \arg z < \theta_{k_i+1} - \varepsilon, r < |z| < \infty\}$, where $C(\theta_{k_i}, \theta_{k_i+1}, \varepsilon, \delta(a_i, A))$ is a positive constant depending only on $\theta_{k_i}, \theta_{k_i+1}, \varepsilon$ and $\delta(a_i, A)$.

In the sequel, we shall say that A decays to a_i exponentially in $S(\theta_{k_i}, \theta_{k_i+1})$, if (8) holds in $S(\theta_{k_i}, \theta_{k_i+1})$. Note that if A is a function extremal for Yang’s inequality, then $\mu(A) = \rho(A)$. Thus, for these functions, we need only to consider the Borel directions of order $\rho(A)$.

Lemma 4.2 ([15]). *Let A be an entire function extremal for Yang’s inequality. Suppose that there exists $\arg z = \theta$ with $\theta_j < \theta < \theta_{j+1}, 1 \leq j \leq q$, such that*

$$\limsup_{r \rightarrow \infty} \frac{\log \log |A(re^{i\theta})|}{\log r} = \rho(A).$$

Then $\theta_{j+1} - \theta_j = \frac{\pi}{\rho(A)}$.

Suppose on the contrary to the assertion that there is a nontrivial solution f of (1) with $\rho(f) < \infty$. We aim for a contradiction. Since $B(z)$ is entire function with Fabry gaps, by Lemma 2.4, then for any given $\varepsilon \in (0, \frac{\rho(B)}{4})$, there exists a set $E_1 \subset (1, \infty)$ with $\overline{\log \text{dens}}(E_1) > 0$, such that for all z satisfies $|z| = r \in E_1$, the (6) holds. By Lemma 2.1, there exists a set $E_2 \subset (1, \infty)$ with $m_1(E_2) < \infty$, such that for all z satisfying $|z| = r \notin (E_2 \cup [0, 1])$, the (7) holds.

Suppose that $a_i, i = 1, 2, \dots, p$, are all the finite deficient values of $A(z)$. Thus we have $2p$ sectors $S_j = \{z : \theta_j < \arg z < \theta_{j+1}\}, j = 1, 2, \dots, 2p$, such that $A(z)$ has the following properties. In each sector S_j , either there exists some a_i such that

$$\log \frac{1}{|A(z) - a_i|} > C(\theta_j, \theta_{j+1}, \varepsilon, \delta(a_i, A))T(|z|, A) \tag{9}$$

holds for $z \in S(\theta_j + \varepsilon, \theta_{j+1} - \varepsilon; r, \infty)$, where $C(\theta_j, \theta_{j+1}, \varepsilon, \delta(a_i, A))$ is a positive constant depending only on $\theta_j, \theta_{j+1}, \varepsilon$ and $\delta(a_i, A)$, or there exists $\arg z = \theta \in (\theta_j, \theta_{j+1})$ such that

$$\limsup_{r \rightarrow \infty} \frac{\log \log |A(re^{i\theta})|}{\log r} = \rho(A) \tag{10}$$

holds. For the sake of simplicity, in the sequel we use C to represent $C(\theta_j, \theta_{j+1}, \varepsilon, \delta(a_i, A))$. Note that if there exists some a_i such that (9) holds in S_j , then there exists $\arg z = \theta$ such that (10) holds in S_{j-1} and S_{j+1} . If there exists $\theta \in (\theta_j, \theta_{j+1})$ such that (10) holds, then there are a_i ($a_{i'}$) such that (9) holds in S_{j-1} and S_{j+1} , respectively.

Without loss of generality, we assume that there is a ray $\arg z = \theta$ in S_1 such that (10) holds. Therefore, there exists a ray in each sector $S_3, S_5, \dots, S_{2p-1}$, such that (10) holds. By using Lemma 4.2, we know that all the sectors have the same magnitude $\frac{\pi}{\rho(A)}$. It is not hard to see that there exists a sequence of points $z_n = r_n e^{i\theta}$ with $r_n \rightarrow \infty$ as $n \rightarrow \infty$, and finite deficient value a_{j_0} , where $r_n \in E_1 - (E_2 \cup [0, 1])$ and $\theta \in (\theta_j, \theta_{j+1}), j = 2, 4, \dots, 2p$, such that

$$\log \frac{1}{|A(r_n e^{i\theta}) - a_{j_0}|} > CT(r_n, A), \tag{11}$$

$$|B(r_n e^{i\theta})| > \exp(r_n^{\rho(B)-\varepsilon}) \tag{12}$$

and

$$\left| \frac{f^{(k)}(r_n e^{i\theta})}{f(r_n e^{i\theta})} \right| \leq r_n^{2\rho(f)}, \quad k = 1, 2. \tag{13}$$

Combining (11), (12), (13) and (1), we get

$$\begin{aligned} \exp(r_n^{\rho(B)-\varepsilon}) &< |B(r_n e^{i\theta})| \\ &\leq \left| \frac{f''(r_n e^{i\theta})}{f(r_n e^{i\theta})} \right| + (|A(r_n e^{i\theta}) - a_{j_0}| + |a_{j_0}|) \left| \frac{f'(r_n e^{i\theta})}{f(r_n e^{i\theta})} \right| \\ &\leq r_n^{2\rho(f)} (1 + |a_{j_0}| + \exp(-CT(r_n, A))) \end{aligned}$$

holds for all sufficiently large n . Obviously this is a contradiction. This completes the proof.

5. Proof of Theorem 1.8

If $\rho(A) = \infty$, then it is clear that $\rho(f) = \infty$ for every nontrivial solution f of (1). Hence we may assume $\rho(A) < \infty$. Suppose on the contrary to the assertion that there is a nontrivial solution f of (1) with $\rho(f) < \infty$.

We aim for a contradiction. By our assumption, suppose that a is a finite Borel exceptional value of $A(z)$. By Lemma 2.6, we have

$$A(z) = h(z)e^{Q(z)} + a,$$

where $h(z)$ is an entire function satisfying

$$\rho(h) < \rho(A),$$

and $Q(z)$ is a polynomial satisfying

$$\deg(Q) = \rho(A).$$

Let $Q(z) = b_d z^d + b_{d-1} z^{d-1} + \dots + b_0$, where $b_d = \alpha_d e^{i\theta_d}$, $\alpha_d > 0$, $\theta_d \in [0, 2\pi)$. For any given $\varepsilon \in (0, \min(\frac{\pi}{8d}, \frac{\rho(B)}{2}))$, let

$$S_j = \{z : -\frac{\theta_d}{d} + (2j - 1)\frac{\pi}{2d} + \varepsilon < \arg z < -\frac{\theta_d}{d} + (2j + 1)\frac{\pi}{2d} - \varepsilon\},$$

where $j = 0, 1, \dots, 2d - 1$. By Lemma 2.7 and $\rho(h) < d = \rho(A)$, for any $z = re^{i\theta} \in S_j$ and j is even,

$$|A(re^{i\theta}) - a| > \exp(Cr^d) \tag{14}$$

for all sufficiently large r , while for any $z = re^{i\theta} \in S_j$ and j is odd,

$$|A(re^{i\theta}) - a| < \exp(-Cr^d) \tag{15}$$

for all sufficiently large r , where C is a positive constant. We consider one of the n sectors S_i , $i = 1, 3, \dots, 2d - 1$. Without loss of generality, say S_1 . This implies (15) holds for all $z = re^{i\theta} \in S_1$, where r enough large.

Next, by our assumption, we get a contradiction by using similar way in proving Theorem 1.7, we omit the details. This completes the proof.

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