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Horoball Packings related to the 4-Dimensional Hyperbolic 24 Cell Honeycomb $\{3,4,3,4\}$

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Dedicated to Professor Emil Molnár on the occasion of his 75th birthday

Abstract. In this paper we study the horoball packings related to the hyperbolic 24 cell honeycomb by Coxeter-Schläfli symbol $\{3,4,3,4\}$ in the extended hyperbolic 4-space $\overline{\mathbb{H}}^4$ where we allow *horoballs in different types* centered at the various vertices of the 24 cell.

Introducing the notion of the generalized polyhedral density function, we determine the locally densest horoball packing arrangement and its density with respect to the above regular tiling. The maximal density is ≈ 0.71645 which is equal to the known greatest horoball packing density in hyperbolic 4-space, given in [13].

1. Introduction

We consider horospheres and their bodies, the horoballs. A horoball packing $\mathcal{B}$ of $\overline{\mathbb{H}}^d$ is an arrangement of non-overlapping horoballs B in $\overline{\mathbb{H}}^d$ ($d = 3,4$ in this paper).

The definition of packing density of generalized ball packings (by balls, horoballs and hyperballs, respectively) is critical in hyperbolic space as shown by Böröczky [4]. For standard examples see also [19]. The most widely accepted notion of packing density considers first the local densities of balls with respect to their Dirichlet-Voronoi cells (cf. [4] and [9]). Then we consider the density infimum for every ball to get the density of the given packing. Then we can investigate the supremum of these infima for every packing of $\mathbb{H}^d$. In order to consider horoball packings in $\overline{\mathbb{H}}^d$ we suggest to use an extended notion of such local density.

Let B be a horoball in packing $\mathcal{B}$, and $P \in \overline{\mathbb{H}}^d$ be an arbitrary point. Define $d(P,B)$ to be the perpendicular distance from point P to the horosphere $S = \partial B$, where $d(P,B)$ is taken to be negative when $P \in B$. The Dirichlet–Voronoi cell $\mathcal{D}(B,\mathcal{B})$ of a horoball B of packing $\mathcal{B}$ is defined as the convex body

$$\mathcal{D}(B,\mathcal{B}) = \{P \in \mathbb{H}^n | d(P,B) \leq d(P,B'),\ \forall B' \in \mathcal{B}\}.$$

Both B and $\mathcal{D}$ are of infinite volume, so the usual notion of local density is modified as follows. Let $Q \in \partial\mathbb{H}^d$ denote the ideal center of B at infinity, and take its boundary S to be the one-point compactification of

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Euclidean $(d-1)$-space. Let $B_C^{d-1}(r) \subset S$ be an $(d-1)$-ball with center $C \in S \setminus \{Q\}$. Then $Q \in \partial \mathbb{H}^d$ and $B_C^{d-1}(r)$ determine a convex cone $C^d(r) = cone_Q(B_C^{d-1}(r)) \in \overline{\mathbb{H}}^d$ with apex Q consisting of all hyperbolic geodesics passing through $B_C^{d-1}(r)$ with limit point Q. The local density $\delta_d(B, \mathcal{B})$ of B to $\mathcal{D}$ is defined as

$$\delta_d(\mathcal{B}, B) = \overline{\lim_{r\to\infty}} \frac{vol(B \cap C^d(r))}{vol(\mathcal{D} \cap C^d(r))}.$$

This upper limit is independent of the choice of center C for $B_C^{d-1}(r)$.

For periodic ball or horoball packings the above local density can be extended to the entire hyperbolic space. This local density is related to the simplicial density function that was generalized in [28] and [29]. In this paper we will use the generalization of this local packing density.

In [28] we have refined the notion of the „congruent" horoballs in a horoball packing to the horoballs of the „same type" because the horoballs are always congruent in the hyperbolic space $\overline{\mathbb{H}}^d$, in general.

Two horoballs in a horoball packing are called of the same type, or „equipacked", if the local densities of the horoballs to the corresponding cell (e.g. D-V cell; or ideal regular polytope, later on) are equal.

If we assume that the „horoballs belong to the same type", then by analytical continuation, the well known simplicial density function on $\overline{\mathbb{H}}^d$ can be extended from d-balls of radius r to the case $r = \infty$, too. Namely, in this case consider $d+1$ horoballs which are mutually tangent and let B be one of them. The convex hull of their base points at infinity will be a totally asymptotic or ideal regular simplex $T_{reg}^\infty \in \overline{\mathbb{H}}^d$ of finite volume. Hence, in this case it is legitimated to write

$$\delta_d(\infty) = (d+1)\frac{vol(B \cap T_{reg}^\infty)}{vol(T_{reg}^\infty)}.$$

Then for a horoball packing $\mathcal{B}$, there is an analogue of ball packing, namely (cf. [4], Theorem 4 with modified notation)

$$\delta_d'(\mathcal{B}, B) \leq \delta_d(\infty), \ \forall B \in \mathcal{B}.$$

Remark 1.1. *In $\overline{\mathbb{H}}^3$ there is exactly one horoball packing with horoballs in the same type whose Dirichlet–Voronoi cells give rise to a regular honeycomb described by the Schläfli symbol $\{6, 3, 3\}$. Its dual $\{3,3,6\}$ consists of ideal regular simplices T_{reg}^∞ with dihedral angle $\frac{\pi}{3}$ building up a 6-cycle around each edge of the tessellation. The density of this packing is $\delta_3^\infty \approx 0.85328$*

If horoballs of different types at various ideal vertices are allowed i.e the horoballs are differently packed, then we generalized the notion of the simplicial density function [28]. In [12] we proved that the optimal horoball packing arrangement in $\mathbb{H}^3$ mentioned above is not unique. We gave several new examples of horoball packing arrangements based on totally asymptotic Coxeter tilings that yield the above Böröczky–Florian upper bound [5] 0.85328... (shortly B-F upper bound).

Furthermore, in [28], [29] we found that by admitting horoballs of different types at each vertex of a totally asymptotic simplex and generalizing the simplicial density function, the B-F-type density upper bound is no longer valid for the fully asymptotic simplices for $d \geq 3$. For example, in $\overline{\mathbb{H}}^4$ our locally optimal packing density 0.77038... is larger than the B-F-type density upper bound 0.73046.... However these horoball packing configurations are only locally optimal and cannot be extended to the entirety of $\mathbb{H}^d$.

In [13] we have continued our investigations on horoball packings in hyperbolic 4-space. Using horoball packings, and allowing horoballs of different types, we have found seven counterexamples to one of L. Fejes-Tóth's conjectures with density ≈ 0.71645 (which are realized by allowing up to three horoball types).

Several extremal properties relate to the so-called regular hyperbolic 24-cell and the corresponding Coxeter honeycomb (in the title) concerning the right angled polytopes and hyperbolic 4-manifolds.

A. Kolpakov in [11] has shown that the hyperbolic 24-cell has minimal volume and minimal facet number among all ideal right-angled polytopes in $\overline{\mathbb{H}}^4$.

J. G. Ratcliffe and S. T. Tschantz in [20] have constructed complete, open, hyperbolic 4-manifolds of smallest volume by gluing together the sides of a regular ideal 24-cell in hyperbolic 4-space. They also showed that the volume spectrum of hyperbolic 4-manifolds is the set of all positive integral multiples of $4\pi^2/3$.

Using the hyperbolic 24-cell L. Slavich has constructed in [22] two new examples of non-orientable, noncompact, hyperbolic 4-manifolds. The first has minimal volume $V_m = 4\pi^2/3$ and two cusps. This example has the lowest number of cusps among known minimal volume hyperbolic 4-manifolds. The second one has volume $2 \cdot V_m$ and one cusp. It has the smallest volume among known one-cusped hyperbolic 4-manifolds.

In this paper we study a new extremal property of the hyperbolic regular 24-cell and the corresponding regular 4-dimensional honeycomb described by the Coxeter-Schläfli symbol $\{3, 4, 3, 4\}$ *related to horoball packings.*

Introducing the notion of the generalized polyhedral density function, we determine the locally densest horoball packing arrangements and their densities with respect to the above 4-dimensional regular tiling. The maximal density is ≈ 0.71645 *that is equal to the known greatest horoball packing density in hyperbolic 4-space, given in [13].*

For our similar investigations in other Thurston geometries we mention only [15], [26], [27] where the Reader finds further references as well.

2. Formulas in the Projective Model

We use the projective model in Lorentzian $(d+1)$-space $\mathbb{E}^{1,d}$ of signature $(1, d)$, i.e. $\mathbb{E}^{1,d}$ is the real vector space $\mathbf{V}^{d+1}$ equipped with the bilinear form of signature $(1, d)$

$$\langle \mathbf{x},\ \mathbf{y}\rangle = -x^0y^0 + x^1y^1 + \dots + x^dy^d \tag{2.1}$$

where the non-zero real vectors $\mathbf{x} = (x^0, x^1, \dots, x^n) \in \mathbf{V}^{d+1}$ and $\mathbf{y} = (y^0, y^1, \dots, y^d) \in \mathbf{V}^{d+1}$ represent points in projective space $\mathcal{P}^d(\mathbb{R})$. $\mathbb{H}^d$ is represented as the interior of the absolute quadratic form

$$Q = \{[\mathbf{x}] \in \mathcal{P}^d | \langle \mathbf{x},\ \mathbf{x}\rangle = 0\} = \partial\mathbb{H}^d \tag{2.2}$$

in real projective space $\mathcal{P}^d(\mathbf{V}^{d+1}, \boldsymbol{V}_{d+1})$, i.e. by vectors and forms up to non-zero real factors. All proper interior points $\mathbf{x} \in \mathbb{H}^d$ are characterized by $\langle \mathbf{x},\ \mathbf{x}\rangle < 0$.

The boundary points $\partial\mathbb{H}^d$ in $\mathcal{P}^n$ represent the absolute, or points at infinity of $\mathbb{H}^d$. Points $\mathbf{y}$ with $\langle \mathbf{y},\ \mathbf{y}\rangle > 0$ lie outside $\partial\mathbb{H}^d$ and are called the outer points of $\mathbb{H}^d$. Take $P([\mathbf{x}]) \in \mathcal{P}^d$, point $[\mathbf{y}] \in \mathcal{P}^d$ is said to be conjugate to $[\mathbf{x}]$ relative to Q when $\langle \mathbf{x},\ \mathbf{y}\rangle = 0$. The set of all points conjugate to $P([\mathbf{x}])$ form a projective (polar) hyperplane

$$pol(P) := \{[\mathbf{y}] \in \mathcal{P}^d | \langle \mathbf{x},\ \mathbf{y}\rangle = 0\}. \tag{2.3}$$

Hence the bilinear form of Q in (2.1) induces a linear polarity $\mathbf{V}^{d+1} \to \boldsymbol{V}_{d+1}$ between the points and hyperplanes. Point $X[\mathbf{x}]$ and hyperplane $\alpha[\boldsymbol{a}]$ are incident $\mathbf{x}\boldsymbol{a} = 0$ where $\mathbf{x} \in \mathbf{V}^{n+1} \setminus \{\mathbf{0}\}$, and $\boldsymbol{a} \in \boldsymbol{V}_{d+1} \setminus \{\mathbf{0}\}$. Similarly, lines in $\mathcal{P}^d$ are characterized by 2-subspaces of $\mathbf{V}^{d+1}$ or $(d-1)$-spaces of $\boldsymbol{V}_{d+1}$ [14].

In this paper we set the so-called sectional curvature of $\mathbb{H}^d$, $K = -k^2$, with the natural distance unit $k = 1$. The distance $\mathrm{d}(\mathbf{x}, \mathbf{y})$ of two proper points $[\mathbf{x}]$ and $[\mathbf{y}]$ is calculated by the formula

$$\cosh \mathrm{d}(\mathbf{x}, \mathbf{y}) = \frac{-\langle \mathbf{x},\ \mathbf{y}\rangle}{\sqrt{\langle \mathbf{x},\ \mathbf{x}\rangle\langle \mathbf{y},\ \mathbf{y}\rangle}}. \tag{2.4}$$

The perpendicular foot $Y[\mathbf{y}]$ of point $X[\mathbf{x}]$ dropped onto hyperplane $[\boldsymbol{u}]$ is given by

$$\mathbf{y} = \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{u}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u}, \tag{2.5}$$

where $[\mathbf{u}]$ is the pole of the hyperplane $[\boldsymbol{u}]$.

A horosphere in $\mathbb{H}^d$ ($d \geq 3$) will be a generalized hyperbolic d-sphere with infinite radius centered at an ideal point on $\partial\mathbb{H}^n$. More precisely, a horosphere is an $(d-1)$-surface orthogonal to the set of parallel straight lines passing through a point of the absolute quadratic surface. A horoball is a horosphere together with its interior.

We consider the usual Euclidean Beltrami-Cayley-Klein ball model of $\mathbb{H}^d$ centered at $O = (1, 0, 0, \dots, 0)$ with a given vector basis $\mathbf{a}_i$ ($i = 0, 1, 2, \dots, d$) and set a point $T_0 = (1, 0, \dots, 0, 1)$ at the absolute sphere ($d + 1$ coordinates). The equation of a horosphere with center $T_0 = (1, 0, \dots, 1)$ passing through point $S = (1, 0, \dots, s)$ is derived from the equation of the the absolute sphere $-x^0x^0 + x^1x^1 + x^2x^2 + \dots + x^dx^d = 0$, and the plane $x^0 - x^d = 0$ tangent to the absolute sphere at T_0. The equation of the horosphere is in projective coordinates:

$$(s-1)\left(-x^0x^0 + \sum_{i=1}^{d}(x^i)^2\right) - (1+s)(x^0 - x^d)^2 = 0, \ (s \neq \pm 1) \tag{2.6}$$

and in Cartesian coordinates $h_i = \frac{x^i}{x^0}$ it becomes

$$\frac{2\left(\sum_{i=1}^{d-1} h_i^2\right)}{1-s} + \frac{4\left(h_d - \frac{s+1}{2}\right)^2}{(1-s)^2} = 1. \tag{2.7}$$

In order to compute volumes of horoball pieces, we use János Bolyai's classical formulas from the 19-th century:

1. The hyperbolic length $L(x)$ of a horospheric arc that belongs to a chord segment of length x is

$$L(x) = 2\sinh\left(\frac{x}{2}\right). \tag{2.8}$$

2. The intrinsic geometry of a horosphere is Euclidean, so the $(d-1)$-dimensional volume $\mathcal{A}$ of a polyhedron A on the surface of the horosphere can be calculated as in $\mathbb{E}^{d-1}$. The volume of the horoball piece $\mathcal{H}(A)$ determined by A and the aggregate of axes drawn from A to the center of the horoball is

$$Vol(\mathcal{H}(A)) = \frac{1}{d-1}\mathcal{A}. \tag{2.9}$$

3. On Hyperbolic 24 Cell

A d-dimensional honeycomb $\mathcal{P}$, also referred to as a solid tessellation or tiling, is an infinite collection of congruent polyhedra (polytopes) that fit together face-to-face to fill the entire geometric space (*at present* $\mathbb{H}^d$ ($d \geqq 3$)) exactly once. We take the cells to be congruent regular polytopes. A honeycomb with cells congruent to a given regular P exists if and only if the dihedral angle of P is a submultiple of 2π. A complete classification of honeycombs with bounded cells was first given by Schlegel in 1883. The classification was completed by including the polyhedra with unbounded cells, namely the fully asymptotic ones by Coxeter in 1954 [6]. *Such honeycombs (Coxeter tilings) exist only for* $d \leq 5$ in hyperbolic d-space $\mathbb{H}^d$.

A usual approach to describing honeycombs involves analysis of their symmetry groups. If $\mathcal{P}$ is a Coxeter honeycomb, then any rigid motion moving one cell into another maps the entire honeycomb onto itself. The symmetry group of a honeycomb is denoted by $Sym\mathcal{P}$. The characteristic simplex $\mathcal{F}$ of any cell $P \in \mathcal{P}$ is a fundamental domain of the symmetry group $Sym\mathcal{P}$ generated by reflections in its facets which are $(d-1)$-dimensional hyperfaces.

The scheme of a regular polytope P is a weighted graph (diagram) characterizing $\mathcal{F} \subset P \subset \mathbb{H}^d$ up to congruence. The nodes of the scheme, numbered by $0, 1, \dots, d$, correspond to the bounding hyperplanes $t^0, t^1, \dots t^d$ of $\mathcal{F}$ (and its linear forms $\boldsymbol{t}^0, \boldsymbol{t}^1, \dots \boldsymbol{t}^d$). Two nodes are joined by a weighted edge (or branch) if the corresponding hyperplanes are non-orthogonal. Let the set of weights $\{n_{01}, n_{12}, n_{23}, \dots, n_{d-1d}\}$ be the

Figure 1: Coxeter-Schläfli (C-Sch) simplex scheme (orthoscheme)

Coxeter-Schläfli symbol (or C-Sch symbol) of P, and n_{d-1d} be the weight describing the dihedral angle of P, such that the dihedral angle is equal to $\frac{2\pi}{n_{d-1d}}$. In this case $\mathcal{F}$ is the Coxeter simplex with the scheme in Fig. 1.

A $(d+1)\times(d+1)$ symmetric matrix (t^{ij}) is constructed for each scheme in the following manner: $t^{ii}=1$ and if $i \neq j \in \{0,1,2,\dots,d\}$ then $t^{ij} = -\cos\frac{\pi}{n_{ij}}$. Reversing the numbering of the nodes of scheme $\mathcal{P}$ while keeping the weights, leads to the scheme of the dual honeycomb $\mathcal{P}^*$ whose symmetry group coincides with $Sym\mathcal{P}$.

For example, $(t^{ij}) = (\langle \boldsymbol{t}^i, \boldsymbol{t}^j\rangle)$ below is the so called Coxeter-Schläfli matrix of the characteristic simplex $t^0t^1t^2t^3t^4 = T_0T_1T_2T_3T_4$ of the 4-dimensional hyperbolic 24 cell honeycomb $\{3,4,3,4\}$ with parameters (weights) $n_{01}=3, n_{12}=4, n_{23}=3, n_{34}=4$ (see Fig. 3):

$$(t^{ij}) := \begin{pmatrix} 1 & -\cos\frac{\pi}{3} & 0 & 0 & 0 \\ -\cos\frac{\pi}{3} & 1 & -\cos\frac{\pi}{4} & 0 & 0 \\ 0 & -\cos\frac{\pi}{4} & 1 & -\cos\frac{\pi}{3} & 0 \\ 0 & 0 & -\cos\frac{\pi}{3} & 1 & -\cos\frac{\pi}{4} \\ 0 & 0 & 0 & -\cos\frac{\pi}{4} & 1 \end{pmatrix}. \tag{3.1}$$

As we know, the inverse matrix $(t^{ij})^{-1} = (t_{ij}) = (\langle \mathbf{t}_i, \mathbf{t}_j\rangle)$ of the above (3.1) with scalar products of vectors $\mathbf{t}_i$ to the vertices T_i of the characteristic simplex $T_0T_1T_2T_3T_4$ above characterizes (by formula (2.4)) the distance metrics of $\mathbb{H}^d$. For instance $t_{00}=0$ means that T_0 lies on the absolute. We could also use this projective simplex coordinate system, but we prefer the easier Cartesian one, equivalently in the following.

Every d-dimensional totally asymptotic regular polytope P has a hyperbolic presentation obtained by normalising the normalising the coordinates of its ideal vertices so that they lie on the unit sphere $\mathbb{S}^{n-1}$ as the ideal boundary of $\overline{\mathbb{H}}^d$ in Beltrami-Cayley-Klein's ball model. Therefore the ideal regular hyperbolic 24-cell tiling $\mathcal{P}^{24}$ can be derived from an Euclidean 24-cell as the convex hull of the points (see [18]).

$$\begin{aligned}
&A_1(1,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0); && A_{13}(1,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0,0); \\
&A_2(1,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0,0); && A_{14}(1,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0); \\
&A_3(1,\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}},0); && A_{15}(1,-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}},0); \\
&A_4(1,-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}},0); && A_{16}(1,\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}},0); \\
&A_5(1,\frac{1}{\sqrt{2}},0,0,\frac{1}{\sqrt{2}}); && A_{17}(1,-\frac{1}{\sqrt{2}},0,0,-\frac{1}{\sqrt{2}}); \\
&A_6(1,-\frac{1}{\sqrt{2}},0,0,\frac{1}{\sqrt{2}}); && A_{18}(1,\frac{1}{\sqrt{2}},0,0,-\frac{1}{\sqrt{2}}); \\
&A_7(1,0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0); && A_{19}(1,0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0); \\
&A_8(1,0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0); && A_{20}(1,0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0); \\
&A_9(1,0,\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}); && A_{21}(1,0,-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}});
\end{aligned} \tag{3.2}$$

$$A_{10}(1,0,-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}); \qquad A_{22}(1,0,\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}});$$
$$A_{11}(1,0,0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}); \qquad A_{23}(1,0,0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}});$$
$$A_{12}(1,0,0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}); \qquad A_{24}(1,0,0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}});$$

where the points (vertices) are described in a Cartesian projective coordinate system $E_0, E_1, \dots E_4$ given in Section 1.

The 24-cell is the unique regular four-dimensional polytope having 3-cube vertex figure.

3.1. The structure of the hyperbolic 24-cell honeycomb $\mathcal{P}^{24}$

P^{24} is a tile of the 4-dimensional regular honeycomb $\mathcal{P}^{24}$ with C-Sch symbol $\{3,4,3,4\}$. It has 24 octahedral facets, 96 triangular faces, 96 edges and 24 vertices with 3-cube vertex figures. A hyperbolic 24 cell contains $24 \cdot 48 = 1152$ characteristic simplices $\mathcal{F}^{24}$ and the volume of such a Coxeter simplex with C-Sch symbol $\{3,4,3,4\}$ is $Vol(\mathcal{F}^{24}) = \frac{\pi^2}{864}$ (see [8]), therefore the volume of the hyperbolic 24-cell is $Vol(\mathcal{P}^{24}) = \frac{4}{3}\pi^2$.

The vertices of P^{24} are denoted by A_i ($i \in \{1,2,\dots,24\}$) and they coordinates are given in (3.2).

We introduce the notion of the k-neighbouring points ($k \in \{1,2,3,4\}$) related to the vertices of P^{24}:

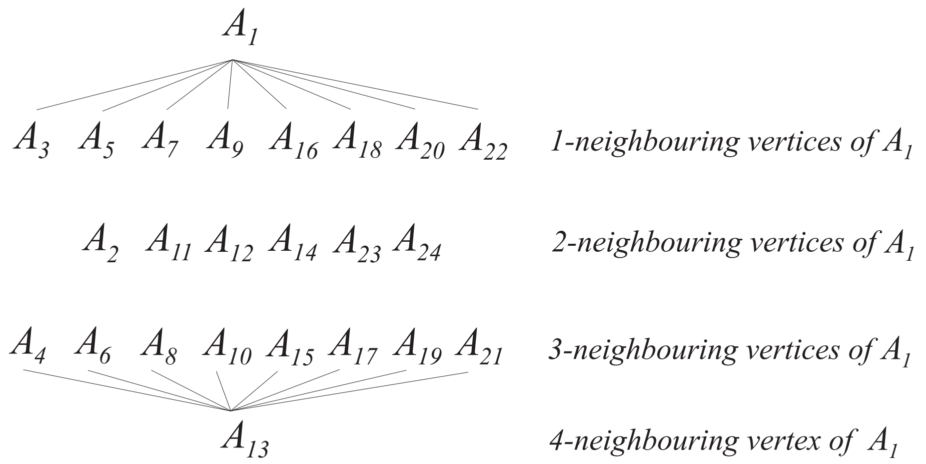


Figure 2: The "neighborhood structure" of P^{24}

Definition 3.1. 1. *The* 1*-neighbouring vertices of A_i are those vertices A_j where A_iA_j is an edge of P^{24}.*

2. *The* 2*-neighbouring vertices of A_i are the vertices A_j where A_iA_j is a diagonal of an octahedral facet of P^{24}.*
3. *The* 4*-neighbouring vertex of A_i is its opposite vertex A_j in P^{24}.*
4. *The* 3*-neighbouring vertices of A_i are those vertices A_j of P^{24} that are not k-neighbouring* ($k = 1,2,4$) *of A_i.*

Fig. 2 shows the k-neighbouring vertices ($k = 1,2,3,4$) of A_1.

Definition 3.2. *Two horoballs B_i and B_j centred in the vertices of P^{24} are k-neighbouring* ($k \in \{1,2,3,4\}$) *if their centres A_i and A_j are k-neighbouring with respect to P^{24}.*

We choose a characteristic simplex (orthoscheme) (see (3.1) and Fig. 3) of P^{24} with vertices $T_0 = A_1\Big(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\Big)$, T_1, T_2, T_3 and $T_4 = O$ and $T_4(1,0,0,0,0)$ is the centre of P^{24} (coinciding with the centre of the model), $T_3\Big(1, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\Big)$ is the centre of the facet-octahedron $A_1A_3A_5A_7A_9A_{11}$. The centre of its regular triangle face $A_1A_3A_7$ is denoted by $T_2\Big(1, \frac{2}{3\sqrt{2}}, \frac{2}{3\sqrt{2}}, \frac{2}{3\sqrt{2}}, 0\Big)$ and $T_1\Big(1, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, 0\Big)$ is the midpoint of the edge A_1A_3 of this face. Moreover, we denote by $T\left(1, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ the centre of the edge A_3A_7. This point coincides with the orthogonal projection of A_1 onto its adjacent octahedral facet $A_3A_4A_7A_8A_{11}A_{24}$ (see Fig. 3).

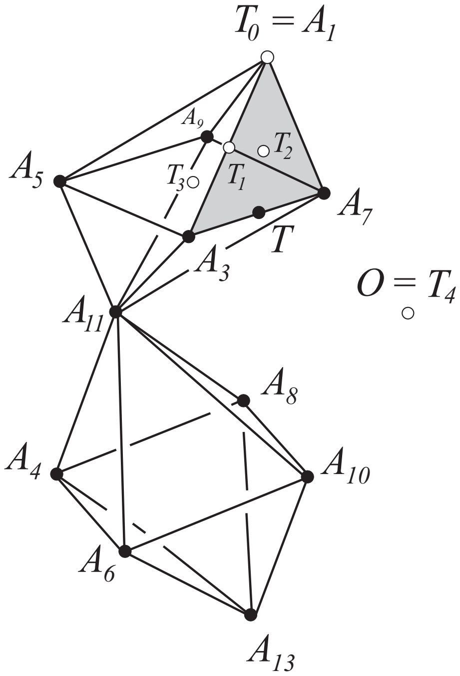


Figure 3: A part of P^{24}

4. Horoball Packings and Polyhedral Density Function

As in the previous section let P^{24} be a tile of the 4-dimensional regular honeycomb $\mathcal{P}_{24}$ with Schläfli symbol $\{3,4,3,4\}$. We study the horoball packings $\mathcal{B}$ with horoballs centred at the infinite vertices of $\mathcal{P}_{24}$. The horoball centred in the vertex A_i is denoted by B_i. The density $\delta(\mathcal{B})$ of the horoball packing $\mathcal{B}$ to the above Coxeter tiling can be defined as the extension of the local density related to the polytope P^{24}. It is well known that for periodic horoball packings the local density can be extended entirely to hyperbolic space $\mathbb{H}^4$.

Definition 4.1. *We consider polytope P^{24} with vertices A_i $(i = 1, \dots, 24)$ in 4-dimensional hyperbolic space $\overline{\mathbb{H}}^4$. Centres of horoballs lie in vertices of P^{24}. We allow horoballs of different types at the various vertices and require that they form a packing. Moreover, we assume that*

$$card\Big[B_i \cap int\Big\{\cup_{j=1}^{18} \mathcal{O}_{i_j}\Big\}\Big] = 0,$$

where the hyperplanes $\mathcal{O}_{i_j}$ $(j = 1, \dots, 18)$ do not contain the vertex A_i. The generalized polyhedral density function for the above polytope and horoballs is defined as

$$\delta(\mathcal{B}) = \frac{\sum_{i=0}^{24} Vol(B_i \cap P^{24})}{Vol(P^{24})}.$$

The aim of this section is to determine the optimal packing arrangements $\mathcal{B}_{opt}$ and their densities for the regular honeycomb $\mathcal{P}^{24}$ in $\overline{\mathbb{H}}^4$. We vary the types of horoballs so that they do not overlap. The packing density is obtained by the above definition.

We will use the consequences of the following Lemma (see Fig. 4, [29]):

Lemma 4.2. *Let B_1 and B_2 denote two horoballs with ideal centers C_1 and C_2, respectively, in the n-dimensional hyperbolic space ($n \geq 2$). Take τ_1 and τ_2 to be two congruent n-dimensional convex pyramid-like regions, with vertices C_1 and C_2. Assume that these horoballs $B_1(x)$ and $B_2(x)$ are tangent at point $I(x) \in C_1C_2$ and C_1C_2 is a common edge of τ_1 and τ_2. We define the point of contact $I(0)$ (the so-called „midpoint") such that the following equality holds for the volumes of horoball sectors:*

$$V(0) := 2vol(B_1(0) \cap \tau_1) = 2vol(B_2(0) \cap \tau_2).$$

If x denotes the hyperbolic distance between $I(0)$ and $I(x)$, then the function

$$V(x) := vol(B_1(x) \cap \tau_1) + vol(B_2(x) \cap \tau_2) = \frac{V(0)}{2}(e^{(n-1)x} + e^{-(n-1)x})$$

strictly increases as $x \to \pm\infty$.

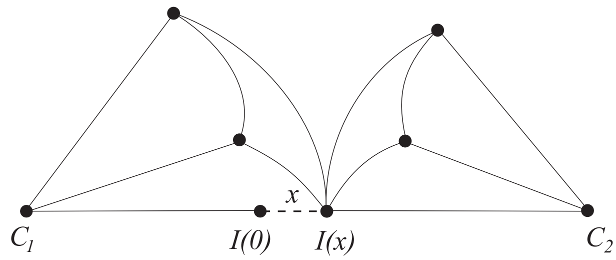


Figure 4: Volume change of touching horoballs

We consider the following four basic horoball configurations $\mathcal{B}_i$, $(i = 0, 1, 2, 3, 4)$:

1. All the 24 horoballs are of the same type and the adjacent horoballs touch each other at the „midpoints" of the corresponding edge. This horoball arrangement is denoted by $\mathcal{B}_0$.
2. We allow *horoballs of different types* so that the opposite horoballs, e.g. B_1 and B_{13} touch their common 2-neighbouring horoballs B_i ($i = 2, 11, 12, 14, 23, 24$) (see Fig. 2) at the centres of the corresponding octahedral facets. E.g. the horoball B_1 touches the horoball B_{11} at the facet center T_3 and B_{13} tangent with B_{11} at the centre of octahedral facet $A_4A_6A_8A_{10}A_{11}A_{13}$ (see Fig. 3). The other "smaller" horoballs are of the same type, and touch their 1-neighbouring "larger" horoballs, e.g. the "larger" horoballs B_1 and B_{11} touch the "smaller" horoballs B_3, B_5, B_7, B_9. At this horoball arrangement let the point $A_1A_3 \cap B_1^s$ be denoted by $C = I_1$ (see Fig. 5.a, B_i^s is the corresponding horosphere of horoball B_i.) This horoball arrangement is denoted by $\mathcal{B}_1$.
3. We set out from the horoball configuration $\mathcal{B}_1$ and we expand B_1 and B_{13} until they comes into contact with their adjacent facets regarding P^{24} while keeping their 1 and 2-neighbouring horoballs tangent to them. At this configuration, denoted by $\mathcal{B}_2$, the horoballs are included in 3 classes related to P^{24}. The horoballs B_1 and B_{13} are of the same type and they touch their corresponding 1-neighbouring horoballs that form the second class. The remaining 8 horoballs are also of the same type and are included in the 3^{rd} type.
   For example the horoball B_1 touches its neighbouring facet at the point T (see Fig. 3, and Fig. 5.b) and touches its 1-neighbouring horoballs e.g. B_3, B_5, B_7, B_9 and its 2-neighbouring horoballs e.g. B_{11}. At this horoball arrangement let the point $A_1A_{11} \cap B_1^s$ be denoted by $E = I_3$ (see Fig. 5.b).
4. We set out also from the horoball configuration $\mathcal{B}_1$ and we blow up the horoball B_1 until it comes into contact with their adjacent facets, while keeping their 1− and 2-neighbouring horoballs tangent to them. Moreover, we blow up the 3-neighbouring horoballs of B_1, while their 1-neighbouring horoballs touch them. At this configuration e.g. the horoball B_1 touches its neighbouring facet $A_3A_4A_7A_8A_{11}A_{24}$

at the point T (see Fig. 3, and Fig. 5.b), and touch its 1-neighbouring horoballs, e.g. B_3, B_5, B_7, B_9 and its 2-neighbouring horoballs, so e.g. B_{11}. Furthermore, the "expanded" horoballs, e.g. B_4, B_6, B_8, B_{10} touch the "shrunk" horoballs B_{11} and B_{13}.
This horoball arrangement is denoted by $\mathcal{B}_3$.

5. Now we start with the configuration $\mathcal{B}_0$ and we choose three arbitrary, mutually 3-neighbouring horoballs and expand them until they contact with each other, while keeping their 1-neighbouring horoballs tangent to them. We note here that this horoball configuration can be realized in $\mathbf{H}^4$ (see the subsection 4.2.4). At this configuration, denoted by $\mathcal{B}_4$, the horoballs are included in 2 classes related to P^{24}, e.g. the horoballs B_1, B_{10}, B_{17} are of the same type touching each other and their "smaller" 1-neighbouring horoballs that are also of the same type.

4.1. *Optimal horoball packings with all horoballs of the same type*

In this Section we consider the packings of horoballs where $Vol(B_i \cap P^{24}) = Vol(B_j \cap P^{24})$ for all $i, j \in \{1, 2, \dots, 24\}$, thus the horoballs B_i are of the same type regarding P^{24}.

It is clear that in this case the maximal density can be achieved if the neighbouring horoballs touch each other at the centres of the edges of P^{24} and the density of this densest packing $\mathcal{B}_0$ is equal to the maximal density of the horoball packings related to the Coxeter simplex tiling $\{3, 4, 3, 4\}$. For example in this case two horoballs B_1 and B_3 touch at the "midpoint" T_1 of edge A_1A_3 as projection of the polyhedron centre on it (see Fig. 3). These ball packings have already been investigated by the author in [25]:

$$\begin{gathered} V_0 := Vol(B_i \cap \mathcal{F}_{24}) = \frac{1}{216}\sqrt{2}\sinh\Big(\frac{1}{2}\operatorname{arcosh}\Big(\frac{11}{8}\Big)\Big) \approx 0.00694, \\ Vol(\mathcal{F}_{24}) = \frac{\pi^2}{864}, \ \ \delta(\mathcal{B}_0) = \frac{Vol(B_i \cap \mathcal{F}_{24})}{Vol(\mathcal{F}_{24})} \approx 0.60793. \end{gathered} \tag{4.1}$$

4.2. *Optimal horoball packings with horoballs of different types*

The type of a horoball is allowed to expand until either it touches another horoball or another adjacent facet of the honeycomb. These conditions are satisfactory to ensure that all horoballs form a well defined packing in $\mathbb{H}^4$.

4.2.1. *Horoball packings $\mathcal{B}_0^1$ and their densities between horoball arrangements $\mathcal{B}_0$ and $\mathcal{B}_1$*

We set out from the horoball configuration $\mathcal{B}_0$ (see the previous Section) and consider two 1-neighbouring horoballs e.g. B_1 and B_3 from it. Let $I_0 = I(0) = T_1$ be their point of tangency on A_1A_3 (see Fig. 3 and Fig. 5.a). Moreover, consider the point $I(x)$ on A_1A_3 where the modified horoballs $B_i(x)$, $(i = 1, 3)$ are tangent to each other and x is the hyperbolic distance between $I(0)$ and $I(x)$ (this x can also be negative if $I(x)$ is on the segment T_1A_1).

We blow up horoballs $B_1(0)$ and $B_{11}(0)$ (and also the horoballs B_2, B_{12}, B_{14}, B_{23}, B_{24} and B_{13} to achieve horoball configuration $\mathcal{B}_1$) until they touch each other at centre T_3 of the octahedral facet $A_1A_3A_5A_7A_9A_{11}$. At this situation (see Fig. 3) the horoball is denoted, by $B_1(\rho_1)$ where ρ_1 is the hyperbolic distance between I_0 and I_1 (see Fig. 5.a).

The foot-point of the perpendicular from T_3 onto the straight line A_1A_3 is $I_0 = T_1$ which is the common point of the horoballs $B_1(0) \in \mathcal{B}_0$ and $B_3(0) \in \mathcal{B}_0$ centred in A_1 and A_3, respectively. The hyperbolic distance $s_1 = T_1T_3$ between points $T_1[\mathbf{t}_1]$ and $T_3[\mathbf{t}_3]$ can be computed by formula (2.4) (see Fig. 5.a): The parallel distance of the angle $\phi_1 = T_1T_3A_1\angle$ is s_1 therefore we obtain by the classical formula of J. Bolyai and by formula (2.4) the following equation (see Fig. 5.a).

$$\frac{1}{\sin(\phi_1)} = \cosh s_1 = \sqrt{2}. \tag{4.2}$$

We consider two horocycles $\mathcal{H}_0$ and $\mathcal{H}_1$ through the points I_0 and I_1 with center A_1 in the plane $A_1A_3T_3$ and the point $\mathcal{H}_1 \cap A_1T_3$ is denoted by M. The horocyclic distances between points I_0, M and I_1, T_3 are

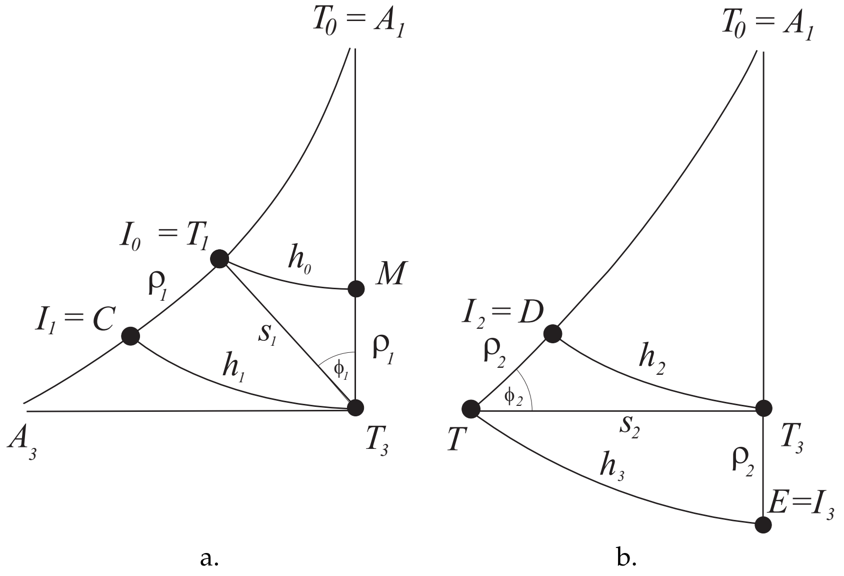


Figure 5: Computations of hyperbolic distances $\rho_1 = I_0 I_1$ and $\rho_2 = I_2 T = I_3 T_3$

denoted by h_0 and h_1. By means of formula of J. Bolyai and of (4.2), we have

$$\frac{h_1}{h_0} = e^{\rho_1} = \frac{1}{\sin(\phi_1)} \Rightarrow \rho_1 = \log(\sqrt{2}) \approx 0.34657. \tag{4.3}$$

We extend the above modifications and notations for all horoballs of packings between horoball arrangements $\mathcal{B}_0$ and $\mathcal{B}_1$ i.e. the horoballs are denoted by $B_i(x)$ ($i \in [0, \rho_1]$). If $x = 0$ then we get the horoball packing $\mathcal{B}_0$ and if $x = \rho_1$ then the $\mathcal{B}_1$ one.

Using the former computations and Lemma 4.2, we obtain the next

Lemma 4.3. *The density of packings $\mathcal{B}_0^1$ (see Fig. 6.a) between the main horoball arrangements $\mathcal{B}_0$ and $\mathcal{B}_1$ can be computed by formula*

$$\delta(\mathcal{B}_0^1(x)) = \frac{\sum_{i=0}^{24} Vol(B_i(x) \cap P^{24})}{Vol(P^{24})} = \frac{384 \cdot V_0 \, (e^{3x} + 2 \cdot e^{-3x})}{\frac{4}{3}\pi^2}, \quad x \in [0, \rho_1]$$

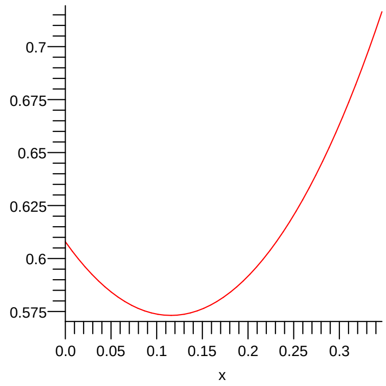
and the maximum of function $\delta(\mathcal{B}_0^1(x))$ (see Fig. 6.a) is realized at $x = \rho_1 \approx 0.34657$ where the horoball packing density is $\delta(\mathcal{B}_0^1(\rho_1)) \approx 0.71645$.

Remark 4.4. *We note, here that the above optimal density $\delta(\mathcal{B}_0^1(\rho_1)) \approx 0.71645$ is equal to the density of the known densest horoball packings in $\mathbb{H}^4$.*

4.2.2. Horoball packings $\mathcal{B}_1^2$ and their densities between horoball arrangements $\mathcal{B}_1$ and $\mathcal{B}_2$

We start our investigation with horoball configuration $\mathcal{B}_1$. Here, e.g. horoballs B_1 and B_3 touch each other at point I_1 (see Fig. 5.a) and B_1 touches B_{11} at point T_3 (see Fig. 3 and Fig. 5.b) etc. Furthermore, in $\mathcal{B}_1$ the common point of horosphere B_1^s with the line segment $A_1 T$ is denoted by $I_2 = I^*(0) = D$ (see Fig. 5.b). We consider the point $I^*(x)$ on $A_1 T$ where a modified horosphere $B_1^s(x)$ intersects the segment $A_1 T$ and x is the hyperbolic distance between $I^*(0)$ and $I^*(x)$ (x can also be negative if $I^*(x)$ is on the segment $A_1 I^*(0)$). According to the above notions, we introduce notations $B_{13}(0)$ and $B_{13}(x)$.

We blow up the horoballs $B_1(0)$ and $B_{13}(0)$ while keeping their 1-neighbouring horoballs tangent to them until they touch their adjacent facets of P^{24} e.g. up to the horoball $B_1(x)$ touches the octahedron facet



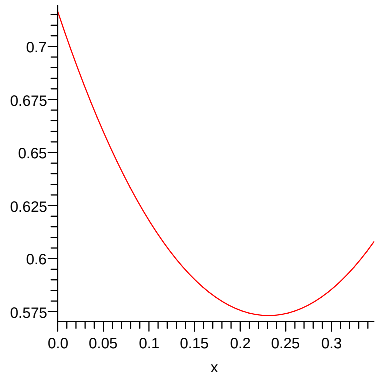


Figure 6: The graphs of functions $\delta(\mathcal{B}_0^1(x))$ and $\delta(\mathcal{B}_1^2(x))$ where $x \in [0, \rho_1]$.

$A_3A_4A_7A_8A_{11}A_{24}$. At this arrangement (Fig. 5.b) the horoball centred in A_1, is denoted by $B_1(\rho_2)$, where ρ_2 is the hyperbolic distance between $I^*(0)$ and T.

The foot-point of the perpendicular from T onto line A_1A_{11} is T_3. The hyperbolic distance $s_2 = TT_3$ between $T[\mathbf{t}]$ and $T_3[\mathbf{t}_1]$ can be computed by formula (2.4) (see Fig. 5.b): The parallel distance of the angle $\phi_2 = A_1TT_3\angle$ is s_2, therefore we obtain by the classical formula of J. Bolyai and by formula (2.4) the following equation (see Fig. 5.b):

$$\frac{1}{\sin(\phi_2)} = \cosh s_2 = \sqrt{2}. \tag{4.4}$$

We consider two horocycles $\mathcal{H}_2$ and $\mathcal{H}_3$ through points I_2 and T, with centre A_1 in the plane A_1TT_3, and the point $\mathcal{H}_3 \cap A_1T_3$ is denoted by $E = I_3$. The horocyclic distances between points I_2, T_3 and T, E are denoted by h_2 and h_3. Similarly to (4.3) we obtain that $\rho_2 = \log(\sqrt{2}) \approx 0.34657$.

We extend the above modifications and notations for all horoballs between arrangements $\mathcal{B}_1$ and $\mathcal{B}_2$, i.e. the horoballs are denoted by $B_i(x)$ ($i \in [0, \rho_2]$). If $x = 0$ then we get the horoball packing $\mathcal{B}_1$ and if $x = \rho_2$ then we get $\mathcal{B}_2$.

By the former computations and Lemma 4.2 we obtain the next

Lemma 4.5. *The density of packings $\mathcal{B}_1^2$ (see Fig. 6.b) between the main horoball arrangements $\mathcal{B}_1$ and $\mathcal{B}_2$ can be computed by the formula*

$$\delta(\mathcal{B}_1^2(x)) = \frac{\sum_{i=0}^{24} Vol(B_i(x) \cap P^{24})}{Vol(P^{24})} =$$
$$= \frac{48 \cdot V_0 \, (2\, e^{3(\rho_1+x)} + 6 \cdot e^{-3(-\rho_1+x)} + 16 \cdot e^{-3(\rho_1+x)})}{\frac{4}{3}\pi^2}, \quad x \in [0, \rho_2]$$

and the maximum of function $\delta(\mathcal{B}_1^2(x))$ (see Fig. 6.b) is realized at $x = 0$ i.e. at horoball packing $\mathcal{B}_1$ (see Lemma 4.3).

Remark 4.6. *The density $\delta(\mathcal{B}_1^2(\rho_2))$ is equal to the maximal packing density with horoballs of the same type: $\delta(\mathcal{B}_1^2(\rho_2)) = \delta(\mathcal{B}_0) \approx 0.60793$.*

4.2.3. Horoball packings $\mathcal{B}_1^3$ and their densities between horoball arrangements $\mathcal{B}_1$ and $\mathcal{B}_3$

Similarly to the above subsection, we set out from horoball configuration $\mathcal{B}_1$, and we will use the notations of subsection 4.2.2. We expand the horoball $B_1(0)$ until it touches adjacent facets of P^{24}, while keeping its 1− and 2-neighbouring horoballs tangent to them. Moreover, we blow up the 3-neighbouring horoballs of $B_1(0)$ while their 1-neighbouring horoballs touch them. At this procedure these horoballs are denoted by $B_1(x)$. If we achieved the endpoint of this extension, then e.g. the horoball $B_1(\rho_2)$ touches its neighbouring facet $A_3A_4A_7A_8A_{11}A_{24}$ at the point T (see Fig. 3, and Fig. 5) and it touches its 1-neighbouring horoballs, e.g. B_3, B_5, B_7, B_9 and its 2-neighbouring horoballs, e.g. B_{11}. Furthermore, the "expanded" horoballs, e.g. B_4, B_6, B_8, B_{10} touch the "shrunk" horoballs B_{11} and B_{13}.

We extend the above modifications and notations for all horoballs between horoball arrangements $\mathcal{B}_1$ and $\mathcal{B}_3$ i.e. the horoballs are denoted by $B_i(x)$ ($i \in [0, \rho_2]$). If $x = 0$ then we get the $\mathcal{B}_1$ horoball packing and if $x = \rho_2$ then the $\mathcal{B}_3$ one. Finally, we obtain the next

Lemma 4.7. *The density of packings $\mathcal{B}_1^3$ between the main horoball arrangements $\mathcal{B}_1$ and $\mathcal{B}_3$ can be computed by formula*

$$\delta(\mathcal{B}_1^3(x)) = \frac{\sum_{i=0}^{24} Vol(B_i(x) \cap P^{24})}{Vol(P^{24})} =$$
$$= \frac{48 \cdot V_0 \left(e^{3(\rho_1+x)} + 7 \cdot e^{-3(-\rho_1+x)} + 8 \cdot e^{-3(\rho_1+x)} + 8 \cdot e^{-3(\rho_1-x)}\right)}{\frac{4}{3}\pi^2}, \ x \in [0, \rho_2]$$

and the maximum of function $\delta(\mathcal{B}_1^3(x))$ are realized at $x = 0$, i.e. at ball packing $\mathcal{B}_1$ (see Lemma 4.3).

Remark 4.8. *Function $\delta(\mathcal{B}_1^3(x))$ is the same as $\delta(\mathcal{B}_1^2(x))$ ($x \in [0, \rho_2]$) (see Fig. 6.b).*

4.2.4. Horoball packings $\mathcal{B}_0^4$ and their densities between horoball arrangements $\mathcal{B}_0$ and $\mathcal{B}_4$

Here we consider horoball configuration $\mathcal{B}_0$ and we arbitrarily choose three mutually 3-neighbouring horoballs e.g. B_1, B_{10} and B_{17}. Let $I_6 = I_*(0)$ be the point of intersection of horosphere $B_1^s(0)$ with segment TA_1. Moreover, consider the point $I_*(x)$ on the segment I_6T where the expanded horosphere $B_1^s(x)$ intersects segment I_6T and x is the hyperbolic distance between $I_*(0)$ and $I_*(x)$ (see Fig. 7.a). We have seen in former subsections that the hyperbolic distance between I_0 and T is $2\rho_1 = 2\rho_2$ (see Fig. 6.a and Fig. 6.b). We consider a horocycles $\mathcal{H}_5$ through point T with center A_1 in the plane $A_1A_{10}T$ and the point $\mathcal{H}_5 \cap A_1A_{10}$ is denoted by $K = I_5$.

The foot-point of the perpendicular from T onto the straight line A_1A_{10} is called Q whose coordinates are $Q\left(1, \frac{5}{7\sqrt{2}}, \frac{3}{7\sqrt{2}}, 0, \frac{2}{7\sqrt{2}}\right)$.

We obtain, by the method described in subsections 4.2.1 and 4.2.2 that the hyperbolic distance ρ_3 of points Q and K is $\rho_3 = \log \frac{10}{3}$.

The midpoint of segment A_1A_{10} is denoted by H (see Fig. 7.a) (in our model this is the Euclidean midpoint of segment A_1A_{10}, as well) whose distance ρ_4 to Q can be computed by formula (2.4): $\rho_4 = \operatorname{arccosh}\left(\frac{7\sqrt{2}}{4\sqrt{5}}\right)$. H lies on the line segment QK because $0.60199 \approx \rho_3 > \rho_4 \approx 0.45815$.

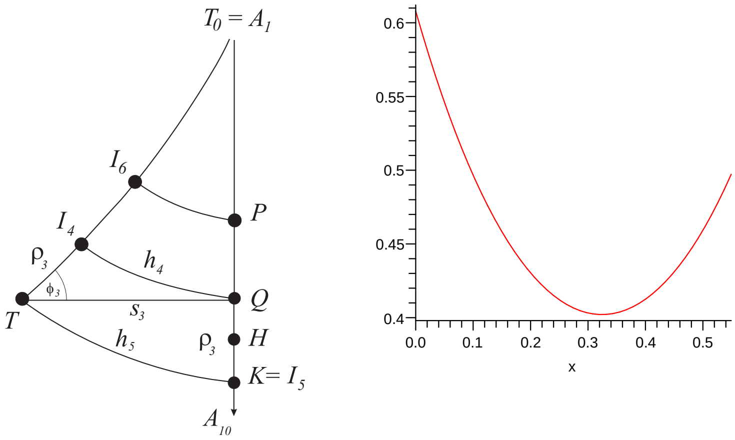
Finally, by the former computations and Lemma 4.2 we obtain the next

Lemma 4.9. *The density of packings $\mathcal{B}_0^4$ (see Fig. 6.b) between the main horoball arrangements $\mathcal{B}_0$ and $\mathcal{B}_4$ can be computed by formula*

$$\delta(\mathcal{B}_0^4(x)) = \frac{\sum_{i=0}^{24} Vol(B_i(x) \cap P^{24})}{Vol(P^{24})} =$$
$$= \frac{48 \cdot V_0 \left(3 \cdot e^{3x} + 21 \cdot e^{-3x}\right)}{\frac{4}{3}\pi^2}, \ \ x \in [0, 2\rho_1 + \rho_4 - \rho_3 \approx 0.54931]$$

and the maximum of function $\delta(\mathcal{B}_0^4(x))$ (see Fig. 7.b) is realized at $x = 0$ where the horoball packing density is $\delta(\mathcal{B}_0^4(0)) \approx 0.60793$.

Remark 4.10. *The density $\delta(\mathcal{B}_0^4(0))$ is equal to the maximal packing density with horoballs of the same type: $\delta(\mathcal{B}_0^4(\rho_2)) = \delta(\mathcal{B}_0) \approx 0.60793$.*



a. b.

Figure 7: a. The computation of hyperbolic distance $\rho_3 = I_4T$ and $\rho_4 = QH$. b. The graph of function $\delta(\mathcal{B}_0^4(x))$ $x \in [0, 2\rho_1 + \rho_3 - \rho_4]$.

4.3. *Optimal horoball packings to hyperbolic* 24*-cell honeycomb*

The main result of this paper is summarized in the following

Theorem 4.11. *The horoball arrangement $\mathcal{B}_1$ (see 4.2.1) provides the maximal horoball packing density related to the hyperbolic tiling $\mathcal{P}^{24}$ with Coxeter-Schläfli symbol $\{3,4,3,4\}$. Its density is $\delta_{opt}(\mathcal{B}) \approx 0.71645$, and horoballs of different types are allowed at asymptotic vertices of the tiling.*

Remark 4.12. *The optimal horoball packing determined and described in this paper is a new horoball packing configuration which provide the known maximal density of hyperbolic space $\mathbb{H}^4$.*

Proof. It is well known that a packing is optimal, then it is locally stable i.e. each ball is fixed by the other ones so that no ball of packing can be moved alone without overlapping another ball of the given ball packing.

The packings of horoballs can be easily classified by the type of "maximally large" horoball regarding the horoball packing to $\mathcal{P}_{24}$. If we fix the "maximally large" horoball related to the above tiling then all possible horoball packing can be modified to achieve one of the above horoball configurations $\mathcal{B}_i^j(x)$ $(i, j \in \{0,1,2,3,4\},\ i < j)$ without decrease of the packing density.

A horoball $B_i(x)$ is "maximally large" if $Vol(B_i(x) \cap P^{24})$ $(i \in 1 \dots 24)$ is maximal. Here the maximal volume is denoted by $Vol(B_i^{max})$.

1. If $\frac{1}{48} Vol(B_i^{max}) \leq V_0$ then the maximal density can be computed by Sect. 4.1 where the maximal density is $\delta(\mathcal{B}_0) \approx 0.60793$.
2. If $V_0 < \frac{1}{48} \cdot Vol(B_i^{max}) \leq V_0 \cdot e^{3\rho_1}$ then the optimal density can be computed by Sections 4.2.1, here the optimal density is $\delta(\mathcal{B}_0^1(\rho_1)) \approx 0.71645$.
3. If $V_0 \cdot e^{3\rho_1} < \frac{1}{48} \cdot Vol(B_i^{max}) \leq V_0 \cdot e^{6\rho_1}$ then the densities can be computed by Sections 4.2.2, 4.2.3 and 4.2.4 where the maximal density is $\delta(\mathcal{B}_0) \approx 0.60793$.

The volume of the "largest horoball" $Vol(B_i^{max}) \leq V_0 \cdot e^{6\rho_1}$, therefore we have proved the above Theorem. □

The above discussions also show that the problem of determining the densest horoball packing and covering in d-dimensional hyperbolic space with horoballs of different types has not been settled yet. Similarly to these, the problems of the densest hyperball packings and coverings are open, as well.

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