Filomat 32:1 (2018), 101–116
https://doi.org/10.2298/FIL1801101T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Tauberian Theorems for the Statistical Convergence and the Statistical $(C, 1, 1)$ Summability

Ümit Totur[a], İbrahim Çanak[b]

[a] *Adnan Menderes University, Department of Mathematics, Aydin, Turkey*
[b] *Ege University, Department of Mathematics, İzmir, Turkey*

Abstract. Every P-convergent double sequence is statistically convergent and every bounded statistically convergent sequence is statistical $(C, 1, 1)$ summable. The converse of these implications are not always true. Theorems on which conditioned converses are searched are known as Tauberian theorems. Inspired by the convergence to zero of the difference sequence between a sequence and its arithmetic means in the single sequence case, we obtain Tauberian theorems for the statistical convergence and statistical $(C, 1, 1)$ summability method by imposing some conditions on the difference sequence between a double sequence and its different arithmetic means.

1. Preliminary Results for Single Sequences

Let $u = (u_n)$ be a single sequence of real numbers. For the sequence of the backward differences of $u = (u_n)$, we use the notation

$$\Delta u_n = u_n - u_{n-1}, \ (u_{-1} = 0)$$

for any nonnegative integer n.

The $(C, 1)$ mean of a sequence $u = (u_n)$ is defined by

$$\sigma_n^{(1)}(u) := \frac{1}{n+1} \sum_{k=0}^{n} u_k.$$

A sequence $u = (u_n)$ is said to be $(C, 1)$ summable to a finite number s if $\lim_{n\to\infty} \sigma_n^{(1)}(u) = s$ (see [13]).

For a sequence $u = (u_n)$, the identity

$$u_n - \sigma_n^{(1)}(u) = V_n(\Delta u),$$

where $V_n(\Delta u) = \dfrac{1}{n+1} \displaystyle\sum_{k=1}^{n} k\Delta u_k$, is called the Kronecker identity.

2010 *Mathematics Subject Classification*. Primary 40E05; Secondary 40G05

Keywords. Tauberian theorems, statistical convergence, statistical $(C, 1, 1)$ summability, slow decrease, statistically slow decrease, one-sided conditions

Received: 11 February 2017; Revised: 14 March 2017; Accepted: 18 April 2017

Communicated by Dragan S. Djordjević

Email addresses: utotur@adu.edu.tr (Ümit Totur), ibrahim.canak@ege.edu.tr (İbrahim Çanak)

A sequence $u = (u_n)$ is called slowly decreasing [17, 21] if

$$\lim_{\lambda \to 1^+} \liminf_{n\to\infty} \min_{n+1\le k\le[\lambda n]} (u_k - u_n) \ge 0.$$

Throughout this paper, $[\lambda n]$ denotes the integer part of the product λn.

Notice that slow decrease of $u = (u_n)$ implies one-sided boundedness of $V(\Delta u) = (V_n(\Delta u))$ (see [2] for more details). If both (u_n) and $(-u_n)$ are slowly decreasing, then $u = (u_n)$ is slowly oscillating. A sequence $u = (u_n)$ is said to be slowly oscillating [24] if

$$\lim_{\lambda \to 1^+} \limsup_{n\to\infty} \max_{n+1\le k\le[\lambda n]} |u_k - u_n| = 0.$$

Dik [9] proved that a sequence $u = (u_n)$ is slowly oscillating if and only if $V(\Delta u) = (V_n(\Delta u))$ is slowly oscillating and bounded. For the summability theory, spaces of single and double sequences and related topics with applications, we refer the interested reader to [1].

A convergent sequence is summable by a regular summability method to the same value. The converse case is not true in general. Historically, the study of the converse situation began with a study of Tauber in 1897. Tauber [25] showed that an Abel summable sequence $u = (u_n)$ of real numbers is convergent under the condition $n\Delta u_n = o(1)$ or $V_n(\Delta u) = o(1)$, where $V(\Delta u) = (V_n(\Delta u))$ is the sequence of arithmetic means of $(n\Delta u_n)$. These type of theorems in the literature are called Tauberian theorems and regarding conditions are called Tauberian conditions.

The condition $n\Delta u_n = o(1)$ which is given by Tauber [25] is weakened to the conditions of two-sided boundedness and one-sided boundedness of the sequence $(n\Delta u_n)$ by Littlewood and Hardy-Littlewood, respectively (see more details [13]). Schmidt [21] generalized these Tauberian conditions to the slow oscillation and slow decrease of (u_n). Here, we note that Çanak and Totur [7] have recently given the alternative proof of Schmidt's Tauberian theorem.

Dik [9] replaced the Tauberian condition that $u = (u_n)$ is slowly oscillating in Schmidt's theorem by a weaker condition that $V(\Delta u) = (V_n(\Delta u))$ is slowly oscillating. Later, Çanak et al. [8] proved that slow decrease of $V(\Delta u) = (V_n(\Delta u))$ is a Tauberian condition for the Abel summability method. Hardy [13] showed that Abel and $(C, 1)$ summability methods are equivalent for bounded sequences.

On the other hand, the concept of the statistical convergence which is a generalization of the concept of ordinary convergence was introduced by Zygmund [28], Fast [11] and Schoenberg [22] independently.

A real sequence $u = (u_n)$ is said to be statistically convergent to s provided that for arbitrary $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n+1} |\{k \le n : |u_k - s| \ge \epsilon\}| = 0$$

where the notation $|\cdot|$ denotes the cardinality of the enclosed set.

Every convergent sequence is statistically convergent. Moreover, the sequence of arithmetic means of every bounded-statistically convergent sequence is also statistically convergent. Since the converse is not always true, several mathematicians have approached to statistical convergence as a summability method, and have proved relevant Tauberian theorems. For instance, Fridy and Khan [12] obtained convergence of a sequence $u = (u_n)$ out of statistical convergence under the classical Tauberian conditions $n\Delta u_n = O(1)$ and $n\Delta u_n \ge -C$ for some $C > 0$. Moricz [17] showed that slow oscillation and slow decrease of a sequence are Tauberian conditions for the statistical convergence. Totur and Çanak [26] obtained the following theorems by imposing a Tauberian condition on the sequence of the differences between a bounded statistically $(C, 1)$ convergent sequence and its Cesàro transform of order one.

Theorem 1.1. *([26]) Let $u = (u_n)$ be bounded and its Cesàro transform of order one be statistical summable to s. If $V(\Delta u) = (V_n(\Delta u))$ is slowly decreasing, then $u = (u_n)$ is convergent to s.*

Theorem 1.2. *([26]) Let $u = (u_n)$ be bounded and its Cesàro transform of order one be statistical summable to s. If $n\Delta V_n(\Delta u) \ge -C$ for some $C > 0$, then $u = (u_n)$ is convergent to s.*

2. The Statistical $(C, 1, 1)$ Summability and Classical Tauberian Theorems

A double sequence $u = (u_{mn})$ is called convergent in the Pringsheim's sense (in short, P-convergent) to s, denoted by $P - \lim_{m,n\to\infty} u_{mn} = s$ if for a given $\varepsilon > 0$ there exists a positive integer N_0 such that $|u_{mn} - s| < \varepsilon$ for all nonnegative integers m, n; (see [20] and [1]).

A double sequence $u = (u_{mn})$ is said to be bounded if there exists a real number $C > 0$ such that $|u_{mn}| \leq C$ for all nonnegative integers m and n. Note that a P-convergent double sequence need not be bounded (see [27] for details).

A double sequence $u = (u_{mn})$ is said to be one-sided bounded if there exists a real number $C > 0$ such that $u_{mn} \geq -C$ for all nonnegative integers m and n.

The $(C, 1, 1)$ means of $u = (u_{mn})$ are defined by

$$\sigma^{(11)}_{mn}(u) = \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} u_{jk}$$

for all nonnegative integers m and n. A sequence $u = (u_{mn})$ is said to be $(C, 1, 1)$ summable to a finite number s if $P - \lim_{m,n\to\infty} \sigma^{(11)}_{mn}(u) = s$.

The $(C, 1, 0)$ and $(C, 0, 1)$ means of $u = (u_{mn})$ are defined respectively by

$$\sigma^{(10)}_{mn}(u) = \frac{1}{m+1} \sum_{j=0}^{m} u_{jn} \quad \text{and} \quad \sigma^{(01)}_{mn}(u) = \frac{1}{n+1} \sum_{k=0}^{n} u_{mk}$$

for all nonnegative integers m and n. A sequence $u = (u_{mn})$ is said to be $(C, 1, 0)$ (or $(C, 0, 1)$) summable to a finite number s if $\lim_{m,n\to\infty} \sigma^{(10)}_{mn}(u) = s$ (or $\lim_{m,n\to\infty} \sigma^{(01)}_{mn}(u) = s$).

For a double sequence $u = (u_{mn})$, we define $\Delta_{01} u_{mn} = u_{mn} - u_{m,n-1}$, $\Delta_{10} u_{mn} = u_{mn} - u_{m-1,n}$, and $\Delta_{11} u_{mn} = \Delta_{10}\Delta_{01} u_{mn} = \Delta_{10}(\Delta_{01} u_{mn}) = \Delta_{01}(\Delta_{10} u_{mn})$ for all nonnegative integers m and n.

The Kronecker identity given for single sequences takes the following form for double sequences (see [14]):
For all nonnegative integers m and n,

$$u_{mn} - \sigma^{(10)}_{mn}(u) - \sigma^{(01)}_{mn}(u) + \sigma^{(11)}_{mn}(u) = V^{(11)}_{mn}(\Delta u), \tag{1}$$

where $V^{(11)}_{mn}(\Delta u) = \frac{1}{(m+1)(n+1)} \sum_{j=1}^{m} \sum_{k=1}^{n} jk\Delta_{11} u_{jk}$.

We write the following identities similar to the Kronecker identity for single sequences:

$$u_{mn} - \sigma^{(10)}_{mn}(u) = V^{(10)}_{mn}(\Delta u), \tag{2}$$

where $V^{(10)}_{mn}(\Delta u) = \frac{1}{m+1} \sum_{j=1}^{m} j\Delta_{10} u_{jn}$, and

$$u_{mn} - \sigma^{(01)}_{mn}(u) = V^{(01)}_{mn}(\Delta u), \tag{3}$$

where $V^{(01)}_{mn}(\Delta u) = \frac{1}{n+1} \sum_{k=1}^{n} k\Delta_{01} u_{mk}$. The identity (1) can be rewritten as

$$u_{mn} - \sigma^{(11)}_{mn}(u) = V^{(10)}_{mn}(\Delta u) + V^{(01)}_{mn}(\Delta u) - V^{(11)}_{mn}(\Delta u) \tag{4}$$

by (2) and (3). Moreover, the following identities show the relationships between $\sigma^{(10)}_{mn}(u)$ and $V^{(10)}_{mn}(\Delta u)$, and $\sigma^{(01)}_{mn}(u)$ and $V^{(01)}_{mn}(\Delta u)$, respectively:

$$m\Delta_{10}\sigma^{(10)}_{mn}(u) = V^{(10)}_{mn}(\Delta u), \tag{5}$$

$$n\Delta_{01}\sigma^{(01)}_{mn}(u) = V^{(01)}_{mn}(\Delta u). \tag{6}$$

Indeed, we can obtain the identity (5) from the calculations

$$\begin{aligned} m\Delta_{10}\sigma_{mn}^{(10)}(u) &= m\left(\frac{1}{m+1}\sum_{i=0}^{m} u_{in} - \frac{1}{m}\sum_{i=0}^{m-1} u_{in}\right) \\ &= \frac{1}{m+1}\left(m\sum_{i=0}^{m} u_{in} - m\sum_{i=0}^{m-1} u_{in} - \sum_{i=0}^{m-1} u_{in}\right) \\ &= \frac{1}{m+1}\left(mu_{mn} - \sum_{i=0}^{m-1} u_{in}\right) \\ &= \frac{1}{m+1}\sum_{j=0}^{m} j\Delta_{10}u_{jn}. \end{aligned}$$

The identity (6) can be similarly obtained.

The concepts of slow decreasing in different senses for double sequences are defined below.

Definition 2.1. *A double sequence $u = (u_{mn})$ is said to be slowly decreasing in sense $(1,1)$ if for $\lambda > 1$*

$$\lim_{\lambda\to 1^+} \liminf_{m,n\to\infty} \min_{\substack{m+1\le j\le[\lambda m]\\ n+1\le k\le[\lambda n]}} \left(u_{jk} - u_{jn} - u_{mk} + u_{mn}\right) \ge 0.$$

A double sequence $u = (u_{mn})$ is said to be slowly decreasing in sense $(1,0)$ if for $\lambda > 1$

$$\lim_{\lambda\to 1^+} \liminf_{m,n\to\infty} \min_{m+1\le j\le[\lambda m]} \left(u_{jn} - u_{mn}\right) \ge 0. \tag{7}$$

A double sequence $u = (u_{mn})$ is said to be slowly decreasing in sense $(0,1)$ if for $\lambda > 1$

$$\lim_{\lambda\to 1^+} \limsup_{m,n\to\infty} \min_{n+1\le k\le[\lambda n]} (u_{mk} - u_{mn}) \ge 0.$$

Moreover, we say that a double sequence $u = (u_{mn})$ is said to be slowly decreasing in the strong sense $(1,0)$ if (7) is satisfied with

$$\min_{\substack{m+1\le j\le[\lambda m]\\ n+1\le k\le[\lambda n]}} (u_{jk} - u_{mk}) \quad \textit{instead of} \quad \min_{m+1\le j\le[\lambda m]} (u_{jn} - u_{mn}). \tag{8}$$

The slow decrease of a double sequence $u = (u_{mn})$ in the strong sense $(0,1)$ can be similarly defined.

A double sequence $u = (u_{mn})$ is said to be slowly oscillating in sense $(1,1)$ if for $\lambda > 1$

$$\lim_{\lambda\to 1^+} \limsup_{m,n\to\infty} \max_{\substack{m+1\le j\le[\lambda m]\\ n+1\le k\le[\lambda n]}} \left|u_{jk} - u_{jn} - u_{mk} + u_{mn}\right| = 0,$$

slowly oscillating in sense $(1,0)$ if for $\lambda > 1$

$$\lim_{\lambda\to 1^+} \limsup_{m,n\to\infty} \max_{m+1\le j\le[\lambda m]} \left|u_{jn} - u_{mn}\right| = 0,$$

and slowly oscillating in sense $(0,1)$ if for $\lambda > 1$

$$\lim_{\lambda\to 1^+} \limsup_{m,n\to\infty} \max_{n+1\le k\le[\lambda n]} |u_{mk} - u_{mn}| = 0.$$

Similar to Definition 2.1, the concept of slowly oscillating sequence in strong senses can be defined.

For related topics on slowly oscillating sequences and double sequences, we refer the reader to [3–5, 18, 19].

Notice that every P-convergent sequence is slowly oscillating in senses $(1,1)$, $(1,0)$, and $(0,1)$, and every slowly oscillating sequence is slowly decreasing in the same sense. However, the converses may not be true.

Example 2.2. *The double sequence* $(\log m \log n)$ *is slowly oscillating in sense* $(1,1)$ *but not P-convergent, and the double sequence* $\left(\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{1}{\sqrt{jk}}\right)$ *is slowly decreasing in sense* $(1,1)$ *but not slowly oscillating in sense* $(1,1)$.

In the space of bounded double sequences, the $(C,1,1)$ summability method is regular. That is, a bounded P-convergent double sequence is $(C,1,1)$ summable to the same value. The converse of this implication is not true in general. An example that the converse implication does not hold is given by Totur [27].

The first study regarding classical Tauberian theorems for the $(C,1,1)$ summability method is obtained by Knopp [14]. Knopp [14] showed that Abel summability and $(C,1,1)$ summability methods are equivalent in the space of bounded double sequences and discovered a Tauberian theorem for a real double sequence $u=(u_{mn})$ as follows.

Theorem 2.3. *Let* $u=(u_{mn})$ *be a bounded double sequence. If* $u=(u_{mn})$ *is* $(C,1,1)$ *summable to s and the conditions*

$$V_{mn}^{(10)}(\Delta u)=o(1),$$
$$V_{mn}^{(01)}(\Delta u)=o(1)$$

are satisfied, then $u=(u_{mn})$ *is P-convergent to s.*

Móricz [15] obtained two Tauberian theorems for a double sequence which P-convergence follows from $(C,1,1)$ summability under the classical one-sided Tauberian conditions of Landau and slow decreasing conditions in certain senses defined in his paper.

Theorem 2.4. *([15]) Let* $u=(u_{mn})$ *be* $(C,1,1)$ *summable to s. If the conditions*

$$mn\Delta_{11}u_{mn}\geq -C, \tag{9}$$
$$m\Delta_{10}u_{mn}\geq -C, \tag{10}$$
$$n\Delta_{01}u_{mn}\geq -C \tag{11}$$

are satisfied for some $C>0$, *then* $u=(u_{mn})$ *is P-convergent to s.*

Stadtmüller [23] indicated that condition (9) in Theorem 2.4 is superfluous.

Theorem 2.5. *([15]) If* $u=(u_{mn})$ *is* $(C,1,1)$ *summable to s and slowly decreasing in senses* $(1,1)$, $(1,0)$ *and* $(0,1)$, *then* $u=(u_{mn})$ *is P-convergent to s.*

We note here that if we remove the slow decrease condition in sense $(1,1)$ in Theorem 2.5 as Stadtmüller [23] did, one of slow decreasing conditions in sense $(1,0)$ or $(0,1)$ must be strong. The proof can be made by using the same technique in our main theorem. Therefore, we do not give the details of the proof.

In addition, Totur [27] obtained Tauberian theorems of classical type by imposing different type of conditions such as one-sided boundedness and slow oscillation on the sequences $V^{(10)}(\Delta u)=(V_{mn}^{(10)}(\Delta u))$ and $V^{(01)}(\Delta u)=(V_{mn}^{(01)}(\Delta u))$.

A double sequence $u=(u_{mn})$ is said to be statistically convergent to s provided that for arbitrary $\epsilon>0$,

$$\lim_{M,N\to\infty}\frac{1}{(M+1)(N+1)}\left|\{j\leq M, k\leq N : |u_{jk}-s|\geq\epsilon\}\right|=0$$

where the notation $|\cdot|$ denotes the cardinality of the enclosed set.

It is clear that if a double sequence is P-convergent to a real number, then it is also statistical convergent to the same number, but the converse is not necessarily true (see [10] for details).

On the other hand, a double sequence $u=(u_{mn})$ is said to be statistically slowly decreasing in sense $(1,0)$ if for $\lambda>1$ and every $\epsilon>0$

$$\lim_{\lambda\to 1^+}\limsup_{M,N\to\infty}\frac{1}{(M+1)(N+1)}\left|\left\{m\leq M, n\leq N : \min_{m+1\leq j\leq[\lambda m]}(u_{jn}-u_{mn})\leq -\epsilon\right\}\right|=0. \tag{12}$$

The statistically slow decrease of $u = (u_{mn})$ in sense $(0, 1)$ can be defined similarly.

We note that if $u = (u_{mn})$ is slowly decreasing in sense $(1, 0)$, then it is statistically slowly decreasing in sense $(1, 0)$.
In fact, assume that (u_{mn}) is slowly decreasing in sense $(1, 0)$. Given an $\epsilon > 0$. For every large enough m and n, that is, $m, n \geq N_0(\epsilon)$, we can write that

$$\begin{aligned} 0 &\leq \frac{1}{(M+1)(N+1)} \left| \left\{ m \leq M, n \leq N : \min_{m+1 \leq j \leq [\lambda m]} (u_{jn} - u_{mn}) \leq -\epsilon \right\} \right| \\ &\leq \frac{N_0(\epsilon)}{M+1} + \frac{N_0(\epsilon)}{N+1}. \end{aligned} \tag{13}$$

By applying lim sup to the inequality (13) as $M, N \to \infty$, the term on the right-hand side of the inequality (13) tends to 0. Then taking the limit of both sides of the last inequality as $\lambda \to 1^+$, we find that $u = (u_{mn})$ is statistically slowly decreasing in sense $(1, 0)$.

Note that there is a similar relation between slow decrease in sense $(0, 1)$ and statistical slow decrease in sense $(0, 1)$. A sequence $u = (u_{mn})$ is said to be statistically slowly decreasing in the strong sense $(1, 0)$ if (12) is satisfied with (8).

If $u = (u_{mn})$ is bounded and statistically convergent to a number L, then $u = (u_{mn})$ is also $(C, 1, 1)$ summable to the same number, but not conversely ([10]).

Briefly, in the space of bounded sequences, the implications

$$P\text{- convergence} \Rightarrow \text{statistical convergence} \Rightarrow \text{statistical } (C, 1, 1) \text{ summable} \tag{14}$$

are satisfied.

Example 2.6. *(a) The sequence $u = (u_{mn})$ defined by*

$$u_{mn} = \begin{cases} 1, & n^2 = m, \text{for all } n; \\ 0, & \text{otherwise.} \end{cases}$$

is bounded and statistically convergent to 0*, but not P-convergent.*
(b) The sequence $u = (u_{mn})$ defined by

$$u_{mn} = \begin{cases} 1, & m \text{ and } n \text{ are even;} \\ 0, & \text{otherwise.} \end{cases}$$

is bounded but not statistically convergent. However, $u = (u_{mn})$ is statistical $(C, 1, 1)$ *summable to* $1/4$.

Tauberian theorems on which conditioned converses of each implication are searched are given. Edely and Mursaleen proved the following theorem.

Theorem 2.7. *([10]) If $u = (u_{mn})$ is statistical* $(C, 1, 1)$ *summable to s and the conditions* (10) *and* (11) *are satisfied, then $u = (u_{mn})$ is P-convergent to s.*

Instead of using the conditions (10) and (11) as Tauberian conditions, Chen and Chang [6] recovered P-convergence of a double sequence from its statistical $(C, 1, 1)$ summability if it is slowly decreasing in senses $(1, 0)$ and $(0, 1)$, in addition, in the strong sense with respect to one of the types.

Theorem 2.8. *([6]) If $u = (u_{mn})$ is statistical* $(C, 1, 1)$ *summable to s and slowly decreasing in senses* $(1, 0)$ *and* $(0, 1)$*, in addition, in the strong sense with respect to one of the types, then $u = (u_{mn})$ is P-convergent to s.*

Under weaker conditions, Móricz [16] obtained statistical convergence of a double sequence from its statistical $(C, 1, 1)$ summability.

Theorem 2.9. *([15]) Let $u = (u_{mn})$ be statistical* $(C, 1, 1)$ *summable to L. If $u = (u_{mn})$ is statistically slowly decreasing in senses* $(1, 0)$ *and* $(0, 1)$*, in addition, in the strong sense with respect to one of the types, then $u = (u_{mn})$ is statistically convergent to L.*

Remark 2.10. *In conditions of Theorem 2.8 and Theorem 2.9, the term "strongly slowly decreasing" is very important. If we do not use this condition, then we need the slowly decreasing in sense* $(1,1)$ *or statistically slowly decreasing in sense* $(1,1)$ *of a sequence.*

Our aim in this paper is to obtain Tauberian conditions for each implication in the diagram (14) which generalizes and extends the results of Moricz [15] and Chen and Chang [6] for double sequences.

3. Lemmas

In this part of the paper, we present the following Lemmas which will be used in the proof of our main theorems.

In the following Lemma, we give relations among the sequences $V^{(10)}(\Delta u) = (V^{(10)}_{mn}(\Delta u))$, $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$ and $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$.

Lemma 3.1. *The following identities are satisfied:*

$$V^{(10)}_{mn}(\Delta V^{(01)}(\Delta u)) = V^{(11)}_{mn}(\Delta u), \tag{15}$$

and

$$V^{(01)}_{mn}(\Delta V^{(10)}(\Delta u)) = V^{(11)}_{mn}(\Delta u). \tag{16}$$

Proof. By the definition of $V^{(10)}_{mn}(\Delta u)$, we have

$$\begin{aligned} V^{(10)}_{mn}(\Delta V^{(01)}(\Delta u)) &= V^{(01)}_{mn}(\Delta u) - \sigma^{(10)}_{mn}(V^{(01)}(\Delta u)) \\ &= u_{mn} - \sigma^{(01)}_{mn}(u) - \sigma^{(10)}_{mn}(u) + \sigma^{(11)}_{mn}(u) \\ &= V^{(11)}_{mn}(\Delta u). \end{aligned}$$

The proof of (16) is similar to that of the identity (15). Therefore, we omit the details. □

Lemma 3.2. *(i) The one-sided conditions* $m\Delta_{10}u_{mn} \geq -C$ *and* $n\Delta_{01}u_{mn} \geq -C$ *for some* $C > 0$ *imply the slow decrease of* $u = (u_{mn})$ *in senses* $(1,0)$ *and* $(0,1)$ *and in the strong senses* $(1,0)$ *and* $(0,1)$*, respectively.*

(ii) The sequence of $(C,1,0)$ *means of bounded slowly decreasing sequence in sense* $(1,0)$ *is also slowly decreasing in sense of* $(1,0)$*.*

Similar result can be given for a slowly decreasing sequence in sense $(0,1)$.

Proof. (i) Let the condition $m\Delta_{10}u_{mn} \geq -C$ for some $C > 0$ be satisfied. Then,

$$u_{jk} - u_{mk} = \sum_{r=m+1}^{j} \Delta_{10}u_{rk} \geq -C \sum_{r=m+1}^{j} \frac{1}{r} > -C \log \frac{j}{m} \geq -C \log \lambda.$$

Therefore, $u = (u_{mn})$ is slowly decreasing in sense and the strong sense $(1,0)$ by (7).
(ii) For $\lambda > 1$, we write the finite sum $\sum_{j=1}^{m} j\Delta_{10}u_{jn}$ as a series $\sum_{k=0}^{\infty} \sum_{\frac{n}{2^{k+1}} \leq k < \frac{n}{2^k}} j\Delta_{10}u_{jn}$. Hence,

$$\begin{aligned} \sum_{j=1}^{m} j\Delta_{10}u_{jn} &= \sum_{k=0}^{\infty} \sum_{\frac{m}{2^{k+1}} \leq j < \frac{m}{2^k}} j\Delta_{10}u_{jn} \geq \sum_{k=0}^{\infty} \left(\min_{\frac{m}{2^{k+1}} \leq j < \frac{m}{2^k}} \sum_{r=m+1}^{j} \Delta_{10}u_{rn} \right) \\ &\geq -Km \sum_{k=0}^{\infty} \frac{1}{2^k} \geq -2K(m+1). \end{aligned}$$

for some $K > 0$. Therefore, we obtain

$$V_{mn}^{(10)}(\Delta u) = \frac{1}{m+1}\sum_{j=1}^{m} j\Delta_{10}u_{jn} \geq -C$$

for some $C > 0$, and it follows from the identity (5) that $\sigma^{(10)}(u) = (\sigma_{mn}^{(10)}(u))$ is slowly decreasing in the strong sense $(1,0)$. □

Lemma 3.3. *([27]) Let $u = (u_{mn})$ be a double sequence of real numbers. For sufficiently large integers m and n,*

(i) For $\lambda > 1$,

$$\begin{aligned} u_{mn} - \sigma_{mn}^{(11)}(u) &= \frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)}\left(\sigma_{[\lambda m],[\lambda n]}^{(11)}(u) - \sigma_{[\lambda m],n}^{(11)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u) + \sigma_{mn}^{(11)}(u)\right) \\ &+ \frac{[\lambda m]+1}{[\lambda m]-m}\left(\sigma_{[\lambda m],n}^{(11)}(u) - \sigma_{m,n}^{(11)}(u)\right) + \frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{m,[\lambda n]}^{(11)}(u) - \sigma_{m,n}^{(11)}(u)\right) \\ &- \frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}(u_{jk} - u_{mn}). \end{aligned}$$

(ii) For $0 < \lambda < 1$,

$$\begin{aligned} u_{mn} - \sigma_{mn}^{(11)}(u) &= \frac{([\lambda m]+1)([\lambda n]+1)}{(m-[\lambda m])(n-[\lambda n])}\left(\sigma_{mn}^{(11)}(u) - \sigma_{[\lambda m],n}^{(11)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u) + \sigma_{[\lambda m],[\lambda n]}^{(11)}(u)\right) \\ &+ \frac{[\lambda m]+1}{m-[\lambda m]}\left(\sigma_{mn}^{(11)}(u) - \sigma_{[\lambda m],n}^{(11)}(u)\right) + \frac{[\lambda n]+1}{n-[\lambda n]}\left(\sigma_{mn}^{(11)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u)\right) \\ &+ \frac{1}{(m-[\lambda m])(n-[\lambda n])}\sum_{j=[\lambda m]+1}^{m}\sum_{k=[\lambda n]+1}^{n}(u_{mn} - u_{jk}). \end{aligned}$$

Remark 3.4. *In analogy to Lemma 3.3, we have the following identities.*

(i) For $\lambda > 1$,

$$u_{mn} - \sigma_{mn}^{(10)}(u) = \frac{[\lambda m]+1}{[\lambda m]-m}\left(\sigma_{[\lambda m],n}^{(10)}(u) - \sigma_{mn}^{(10)}(u)\right) - \frac{1}{[\lambda m]-m}\sum_{j=m+1}^{[\lambda m]}(u_{jn} - u_{mn}).$$

(ii) For $0 < \lambda < 1$,

$$u_{mn} - \sigma_{mn}^{(10)}(u) = \frac{[\lambda m]+1}{[\lambda m]-m}\left(\sigma_{mn}^{(10)}(u) - \sigma_{[\lambda m],n}^{(10)}(u)\right) + \frac{1}{[\lambda m]-m}\sum_{j=[\lambda m]+1}^{m}(u_{mn} - u_{jn}).$$

The identities can be easily obtained as in the proof of the corresponding lemma for single sequences given by [9]. We do not give details. Moreover, we note that we can represent the difference $u_{mn} - \sigma_{mn}^{(01)}(u)$ in a similar way above.

4. The Main Results

4.1. Tauberian theorems for statistical convergence

Theorem 4.1. *Let $u = (u_{mn})$ be a bounded double sequence. If $u = (u_{mn})$ is statistically convergent to s and there exists a constant N such that*

$$m\Delta_{10}V_{mn}^{(11)}(\Delta u) \geq -C \quad \textit{and} \quad n\Delta_{01}V_{mn}^{(11)}(\Delta u) \geq -C, \qquad (m,n > N) \tag{17}$$

$$m\Delta_{10}V^{(10)}_{mn}(\Delta u) \geq -C \quad \textit{and} \quad n\Delta_{01}V^{(01)}_{mn}(\Delta u) \geq -C, \qquad (m, n > N) \tag{18}$$

are satisfied for some $C > 0$, then $u = (u_{mn})$ is P-convergent to s.

Proof. Since $u = (u_{mn})$ is bounded and statistically convergent to s, then $\sigma^{(10)}(u) = (\sigma^{(10)}_{mn}(u))$, $\sigma^{(01)}(u) = (\sigma^{(01)}_{mn}(u))$ and $\sigma^{(11)}(u) = (\sigma^{(11)}_{mn}(u))$ are statistically convergent to s. By the identity (1), we have $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is statistically convergent to 0. Applying Theorem 2.7 to the sequence $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$, we have

$$V^{(11)}_{mn}(\Delta u) = o(1)$$

by the condition (17).

On the other hand, since $u = (u_{mn})$ is bounded and statistically convergent to s, then $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$ is statistically convergent to 0 by the identity (3). Moreover, the boundedness of $u = (u_{mn})$ implies the boundedness of $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$. Therefore, by the identity

$$V^{(01)}_{mn}(\Delta u) - \sigma^{(10)}_{mn}(V^{(01)}(\Delta u)) = V^{(11)}_{mn}(\Delta u)$$

and (15), we have

$$m\Delta_{10}V^{(01)}_{mn}(\Delta u) - m\Delta_{10}\sigma^{(10)}_{mn}(V^{(01)}(\Delta u)) = m\Delta_{10}V^{(11)}_{mn}(\Delta u). \tag{19}$$

By (5) and (17), we obtain

$$m\Delta_{10}V^{(01)}_{mn}(\Delta u) \geq -C$$

for some $C > 0$. Moreover, applying Theorem 2.7 to the sequence $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$, we get

$$V^{(01)}_{mn}(\Delta u) = o(1)$$

by (18).

Similarly, $V^{(10)}(\Delta u) = (V^{(10)}_{mn}(\Delta u))$ is statistically convergent to 0 by the identity (3). Since $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is bounded, then we have

$$n\Delta_{01}V^{(10)}_{mn}(\Delta u) - n\Delta_{01}\sigma^{(01)}_{mn}(V^{(10)}(\Delta u)) = n\Delta_{01}V^{(11)}_{mn}(\Delta u)$$

by the identity

$$V^{(10)}_{mn}(\Delta u) - \sigma^{(01)}_{mn}(V^{(10)}(\Delta u)) = V^{(11)}_{mn}(\Delta u),$$

(6) and (17). Therefore, it follows

$$n\Delta_{01}V^{(10)}_{mn}(\Delta u) \geq -C \tag{20}$$

for some $C > 0$. Applying Theorem 2.7 to the sequence $V^{(10)}(\Delta u) = (V^{(10)}_{mn}(\Delta u))$, we obtain

$$V^{(10)}_{mn}(\Delta u) = o(1).$$

The proof is completed by the identity (4). □

Corollary 4.2. *Let $u = (u_{mn})$ be a bounded double sequence. If $u = (u_{mn})$ is statistically convergent to s and*

$$m\Delta_{10}V^{(11)}_{mn}(\Delta u) = O(1) \quad \textit{and} \quad n\Delta_{01}V^{(11)}_{mn}(\Delta u) = O(1),$$

$$m\Delta_{10}V^{(10)}_{mn}(\Delta u) = O(1) \quad \textit{and} \quad n\Delta_{01}V^{(01)}_{mn}(\Delta u) = O(1)$$

are satisfied, then $u = (u_{mn})$ is P-convergent to s.

Theorem 4.3. *Let $u = (u_{mn})$ be a bounded double sequence. If $u = (u_{mn})$ is statistically convergent to s, and $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is slowly decreasing in the strong senses $(1,0)$ and $(0,1)$, $V^{(10)}(\Delta u) = (V^{(10)}_{mn}(\Delta u))$ is slowly decreasing in sense $(1,0)$ and $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$ is slowly decreasing in sense $(0,1)$, then $u = (u_{mn})$ is P-convergent to s.*

Proof. Since $u = (u_{mn})$ is bounded and statistically convergent to s, then $\sigma^{(10)}(u) = (\sigma^{(10)}_{mn}(u))$, $\sigma^{(01)}(u) = (\sigma^{(01)}_{mn}(u))$ and $\sigma^{(11)}(u) = (\sigma^{(11)}_{mn}(u))$ are statistically convergent to s. By the identity (1), we have $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is statistically convergent to 0. Applying Theorem 2.8 to the sequence $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$, we have

$$V^{(11)}_{mn}(\Delta u) = o(1).$$

On the other hand, the sequence $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$ is statistically convergent to 0 by the identity (3). Since $u = (u_{mn})$ is bounded, then $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is bounded. Therefore, it follows from Lemma 3.1 and the identity (5) that

$$m\Delta_{10}\sigma^{(10)}_{mn}(V^{(01)}(\Delta u)) = V^{(11)}_{mn}(\Delta u).$$

Then, we obtain that the sequence $\sigma^{(10)}(V^{(01)}(\Delta u)) = (\sigma^{(10)}_{mn}(V^{(01)}(\Delta u)))$ is slowly decreasing in the strong sense $(1,0)$. Hence, using the identity

$$V^{(01)}_{mn}(\Delta u) - \sigma^{(10)}_{mn}(V^{(01)}(\Delta u)) = V^{(11)}_{mn}(\Delta u),$$

we have that the sequence $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$ is slowly decreasing in the strong sense $(1,0)$.

Applying Theorem 2.8 to the sequence $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$, we have

$$V^{(01)}_{mn}(\Delta u) = o(1)$$

by the hypothesis. Similarly, $V^{(10)}(\Delta u) = (V^{(10)}_{mn}(\Delta u))$ is statistically convergent to 0 by the identity (2). It follows from the identity (6) and Lemma 3.1 that the identity

$$n\Delta_n\sigma^{(01)}_{mn}(V^{(10)}(\Delta u)) = V^{(11)}_{mn}(\Delta u)$$

is satisfied. Then, $\sigma^{(01)}(V^{(10)}(\Delta u)) = (\sigma^{(01)}_{mn}(V^{(10)}(\Delta u)))$ is slowly decreasing in the strong sense $(0,1)$. Therefore, using the identity

$$V^{(10)}_{mn}(\Delta u) - \sigma^{(01)}_{mn}(V^{(10)}(\Delta u)) = V^{(11)}_{mn}(\Delta u),$$

we obtain that the sequence $V^{(10)}(\Delta u) = (V^{(10)}_{mn}(\Delta u))$ is slowly decreasing in the strong sense $(0,1)$. Applying Theorem 2.8 to the sequence $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$, we have

$$V^{(01)}_{mn}(\Delta u) = o(1)$$

by the hypothesis. The proof is completed by identity (4). □

Corollary 4.4. *Let $u = (u_{mn})$ be a bounded double sequence. If $u = (u_{mn})$ is statistically convergent to s, $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is slowly oscillating in the strong senses $(1,0)$ and $(0,1)$, $V^{(10)}(\Delta u) = (V^{(10)}_{mn}(\Delta u))$ is slowly oscillating in sense $(1,0)$ and $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$ is slowly oscillating in sense $(0,1)$, then $u = (u_{mn})$ is P-convergent to s.*

4.2. Tauberian theorems for the statistical $(C,1,1)$ summability

Theorem 4.5. *Let $u = (u_{mn})$ be a bounded double sequence. If $u = (u_{mn})$ is statistical $(C,1,1)$ summable to s and the conditions (17) and (18) are satisfied, then $u = (u_{mn})$ is P-convergent to s.*

Proof. Since $u = (u_{mn})$ is bounded and statistical $(C, 1, 1)$ summable to s, then $\sigma^{(10)}(u) = (\sigma^{(10)}_{mn}(u))$, $\sigma^{(01)}(u) = (\sigma^{(01)}_{mn}(u))$ and $\sigma^{(11)}(u) = (\sigma^{(11)}_{mn}(u))$ are statistical $(C, 1, 1)$ summable to s. By the identity (1), we have $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is statistical $(C, 1, 1)$ summable to 0. Namely, $\sigma^{(11)}(V^{(11)}(\Delta u)) = (\sigma^{(11)}_{mn}(V^{(11)}(\Delta u)))$ is statistically convergent to 0.

The inequalities (17) imply

$$m\Delta_{10}\sigma^{(11)}_{mn}(V^{(11)}(\Delta u)) \geq -C \quad \text{and} \quad n\Delta_{01}\sigma^{(11)}_{mn}(V^{(11)}(\Delta u)) \geq -C$$

for some $C > 0$. Applying Theorem 2.7 to the sequence $\sigma^{(11)}(V^{(11)}(\Delta u)) = (\sigma^{(11)}_{mn}(V^{(11)}(\Delta u)))$, we have

$$\sigma^{(11)}_{mn}(V^{(11)}(\Delta u)) = o(1).$$

Hence, we obtain that $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is $(C, 1, 1)$ summable to 0. Applying Theorem 2.4 to the sequence $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ this time, we have

$$V^{(11)}_{mn}(\Delta u) = o(1)$$

by the identity (17).

On the other hand, $\sigma^{(11)}(V^{(01)}(\Delta u)) = (\sigma^{(11)}_{mn}(V^{(01)}(\Delta u)))$ is statistically convergent 0 by the identity (3). Since the boundedness of $u = (u_{mn})$ implies the boundedness of $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$, we have

$$m\Delta_{10}V^{(01)}_{mn}(\Delta u) \geq -C$$

by the identity (19) and then

$$m\Delta_{10}\sigma^{(11)}_{mn}(V^{(01)}(\Delta u)) \geq -C$$

for some $C > 0$. Moreover, by (18), we obtain

$$n\Delta_{01}\sigma^{(11)}_{mn}(V^{(01)}(\Delta u)) \geq -C$$

for some $C > 0$. Applying Theorem 2.7 to the sequence $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$, we have

$$\sigma^{(11)}_{mn}(V^{(01)}(\Delta u)) = o(1).$$

Hence, we obtain that $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$ is $(C, 1, 1)$ summable to 0. Again, applying Theorem 2.4 to the sequence $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$, we have

$$V^{(01)}_{mn}(\Delta u) = o(1)$$

by the identity (17).

The proof of $V^{(10)}_{mn}(\Delta u) = o(1)$ can be established by following up the similar technique. Finally, the proof is completed by the identity (4).

□

Theorem 4.6. *Let $u = (u_{mn})$ be a bounded double sequence. If $u = (u_{mn})$ is statistical $(C, 1, 1)$ summable to s, $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is slowly decreasing in the strong senses $(1, 0)$ and $(0, 1)$, $V^{(10)}(\Delta u) = (V^{(10)}_{mn}(\Delta u))$ is slowly decreasing in sense $(1, 0)$ and $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$ is slowly decreasing in sense $(0, 1)$, then $u = (u_{mn})$ is P-convergent to s.*

Proof. Since $u = (u_{mn})$ is bounded and statistical $(C, 1, 1)$ summable to s, then $\sigma^{(11)}(V^{(11)}(\Delta u)) = (\sigma^{(11)}_{mn}(V^{(11)}(\Delta u)))$ is statistically convergent to 0 by the identity (1). Since $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is slowly decreasing in the strong senses $(1, 0)$ and $(0, 1)$, we obtain $V^{(10)}_{mn}(\Delta V^{(11)}(\Delta u)) \geq -C$ and $V^{(01)}_{mn}(\Delta V^{(11)}(\Delta u)) \geq -C$ for some constant $C > 0$ by Lemma 3.2 (ii), respectively. Therefore, we obtain

$$\sigma^{(01)}_{mn}(V^{(10)}(\Delta V^{(11)}(\Delta u))) \geq -C$$
$$\sigma^{(10)}_{mn}(V^{(01)}(\Delta V^{(11)}(\Delta u))) \geq -C$$

for some constant $C > 0$. It follows from the identities

$$\sigma^{(01)}_{mn}(V^{(10)}(\Delta V^{(11)}(\Delta u))) = m\Delta_{10}\sigma^{(11)}_{mn}(V^{(11)}(\Delta))$$
$$\sigma^{(10)}_{mn}(V^{(01)}(\Delta V^{(11)}(\Delta u))) = n\Delta_{01}\sigma^{(11)}_{mn}(V^{(11)}(\Delta))$$

that $\sigma^{(11)}(V^{(11)}(\Delta u)) = (\sigma^{(11)}_{mn}(V^{(11)}(\Delta u)))$ is slowly decreasing in the strong senses $(1, 0)$ and $(0, 1)$ by Lemma 3.2 (i).

Applying Theorem 2.8 to the sequence $\sigma^{(11)}(V^{(11)}(\Delta u)) = (\sigma^{(11)}_{mn}(V^{(11)}(\Delta u)))$, we have

$$\sigma^{(11)}_{mn}(V^{(11)}(\Delta u)) = o(1).$$

Hence, we obtain that $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is $(C, 1, 1)$ summable to 0. Therefore, applying Theorem 2.5 to the sequence $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$, we obtain

$$V^{(11)}_{mn}(\Delta u) = o(1)$$

by the hypothesis. From the identity (3), $\sigma^{(11)}(V^{(01)}(\Delta u)) = (\sigma^{(11)}_{mn}(V^{(01)}(\Delta u)))$ is statistically convergent. Since $u = (u_{mn})$ is bounded, $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is bounded. It follows from the identity (5) and Lemma 3.1 that

$$m\Delta_{10}\sigma^{(10)}_{mn}(V^{(01)}(\Delta u)) = V^{(11)}_{mn}(\Delta u).$$

Hence, $\sigma^{(10)}(V^{(01)}(\Delta u)) = (\sigma^{(10)}_{mn}(V^{(01)}(\Delta u)))$ is slowly decreasing in the strong sense $(1, 0)$. Therefore, using the identity

$$V^{(01)}_{mn}(\Delta u) - \sigma^{10)}_{mn}(V^{(01)}(\Delta u)) = V^{(11)}_{mn}(\Delta u),$$

we obtain that the sequence $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$ is slowly decreasing in the strong sense $(1, 0)$. Hence, it follows that $\sigma^{10)}(V^{(01)}(\Delta u)) = (\sigma^{10)}_{mn}(V^{(01)}(\Delta u)))$ is slowly decreasing in the strong sense $(1, 0)$.

Similarly, applying Theorem 2.7 to the sequence $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$, we get

$$\sigma^{(11)}_{mn}(V^{(01)}(\Delta u)) = o(1).$$

Namely, we obtain that $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$ is $(C, 1, 1)$ summable to 0. Applying Theorem 2.5 to the sequence $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$, then we have

$$V^{(01)}_{mn}(\Delta u) = o(1)$$

by the identity (17).

The proof of $V^{(10)}_{mn}(\Delta u) = o(1)$ can be established following similar technique. Finally, the proof is completed by the identity (1). □

Corollary 4.7. *Let $u = (u_{mn})$ be a bounded double sequence. If $u = (u_{mn})$ is statistical $(C, 1, 1)$ summable to s, $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is slowly oscillating in the strong senses $(1, 0)$ and $(0, 1)$, $V^{(10)}(\Delta u) = (V^{(10)}_{mn}(\Delta u))$ is slowly oscillating in sense $(1, 0)$ and $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$ is slowly oscillating in sense $(0, 1)$, then $u = (u_{mn})$ is P-convergent to s.*

If we replace the conditions of slow decrease by the conditions of statistically slow decrease in Theorem 4.6, we can not obtain convergence of $u = (u_{mn})$, but we recover its statistical convergence.

Theorem 4.8. *Let $u = (u_{mn})$ be a bounded double sequence. If $u = (u_{mn})$ is statistical $(C, 1, 1)$summable to s, and $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is statistically slowly decreasing in senses $(1, 0)$ and $(0, 1)$, in addition, in the strong sense with respect to one of the types, and $V^{(10)}(\Delta u) = (V^{(10)}_{mn}(\Delta u))$ is statistically slowly decreasing in sense $(1, 0)$ and $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$ is statistically slowly decreasing in sense $(0, 1)$, then $u = (u_{mn})$ is statistically convergent to s.*

Proof. Since $u = (u_{mn})$ is bounded and statistical $(C, 1, 1)$ summable to s, then $\sigma^{(11)}(V^{(11)}(\Delta u)) = (\sigma^{(11)}_{mn}(V^{(11)}(\Delta u)))$ is statistically convergent to 0. We should show that $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is statistically convergent to 0.

Let $\lambda > 1$. It follows from Lemma 3.3 (i) that for every $\epsilon > 0$, we have

$$\begin{aligned}
& \Big\{m \le M, n \le N : V^{(11)}_{mn}(\Delta u) - \sigma^{(11)}_{mn}(V^{(11)}(\Delta u)) \ge \epsilon\Big\} \\
\subseteq\ & \Big\{m \le M, n \le N : \frac{([\lambda m] + 1)([\lambda n] + 1)}{([\lambda m] - m)([\lambda n] - n)}(\sigma^{(11)}_{[\lambda m],[\lambda n]}(V^{(11)}(\Delta u)) - \sigma^{(11)}_{m,[\lambda n]}(V^{(11)}(\Delta u)) \\
& \qquad - \sigma^{(11)}_{[\lambda m],n}(V^{(11)}(\Delta u)) + \sigma^{(11)}_{mn}(V^{(11)}(\Delta u))) \\
+\ & \frac{([\lambda m] + 1)}{([\lambda m] - m)}(\sigma^{(11)}_{[\lambda m],n}(V^{(11)}(\Delta u)) - \sigma^{(11)}_{mn}(V^{(11)}(\Delta u))) \\
+\ & \frac{([\lambda n] + 1)}{([\lambda n] - n)}(\sigma^{(11)}_{m,[\lambda n]}(V^{(11)}(\Delta u)) - \sigma^{(11)}_{mn}(V^{(11)}(\Delta u))) \ge \frac{\epsilon}{2}\Big\} \\
\cup\ & \Big\{m \le M, n \le N : \frac{1}{([\lambda m] - m)([\lambda n] - n)} \sum_{i=m+1}^{[\lambda m]} \sum_{j=n+1}^{[\lambda n]} (V^{(11)}_{ij}(\Delta u) - V^{(11)}_{mn}(\Delta u)) \le -\frac{\epsilon}{2}\Big\} \\
:=\ & A_{MN}(\epsilon) \cup B_{MN}(\epsilon)
\end{aligned} \tag{21}$$

For the first term on the right-hand side of the relation (21), since

$$\frac{[\lambda m] + 1}{[\lambda m] - m} \le \frac{2\lambda}{\lambda - 1}, \ \frac{[\lambda n] + 1}{[\lambda n] - n} \le \frac{2\lambda}{\lambda - 1},$$

we have

$$\lim_{M,N\to\infty} \frac{1}{(M + 1)(N + 1)}|A_{MN}(\epsilon)| = 0.$$

On the other hand, for the second term on the right-hand side of the relation (21), since

$$\begin{aligned}
& \frac{1}{([\lambda m] - m)([\lambda n] - n)} \sum_{i=m+1}^{[\lambda m]} \sum_{j=n+1}^{[\lambda n]} (V^{(11)}_{ij}(\Delta u) - V^{(11)}_{mn}(\Delta u)) \\
& \qquad \ge \min_{\substack{m+1\le i\le[\lambda m] \\ n+1\le j\le[\lambda n]}} (V^{(11)}_{ij}(\Delta u) - V^{(11)}_{mj}(\Delta u)) + \min_{n+1\le j\le[\lambda n]} (V^{(11)}_{mj}(\Delta u) - V^{(11)}_{mn}(\Delta u)),
\end{aligned} \tag{22}$$

we obtain for every $\epsilon > 0$

$$\begin{aligned}
& \Big\{m \le M, n \le N : \frac{1}{([\lambda m] - m)([\lambda n] - n)} \sum_{i=m+1}^{[\lambda m]} \sum_{j=n+1}^{[\lambda n]} (V^{(11)}_{ij}(\Delta u) - V^{(11)}_{mn}(\Delta u)) \le -\frac{\epsilon}{2}\Big\} \\
\subseteq\ & \Big\{m \le M, n \le N : \min_{\substack{m+1\le i\le[\lambda m] \\ n+1\le j\le[\lambda n]}} (V^{(11)}_{ij}(\Delta u) - V^{(11)}_{mj}(\Delta u)) \le -\frac{\epsilon}{4}\Big\} \\
\cup\ & \Big\{m \le M, n \le N : \min_{n+1\le j\le[\lambda n]} (V^{(11)}_{mj}(\Delta u) - V^{(11)}_{mn}(\Delta u)) \le -\frac{\epsilon}{4}\Big\} \\
:=\ & B^{(1)}_{MN}(\epsilon) \cup B^{(2)}_{MN}(\epsilon).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \left|\left\{m \le M, n \le N : V^{(11)}_{mn}(\Delta u) - \sigma^{(11)}_{mn}(V^{(11)}(\Delta u)) \ge \epsilon\right\}\right| \\
\le\ &\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} |B^{(1)}_{MNS}(\epsilon)| + \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} |B^{(2)}_{MN}(\epsilon)|.
\end{aligned}$$

Therefore, given any $\delta > 0$, since $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is statistically slowly decreasing in the strong senses $(1,0)$, $(0,1)$, for some $\lambda > 1$ we have

$$\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \left|\left\{m \le M, n \le N : V^{(11)}_{mn}(\Delta u) - \sigma^{(11)}_{mn}(V^{(11)}(\Delta u)) \ge \epsilon\right\}\right| \le \delta.$$

Since $\delta > 0$ is arbitrary, we get for every $\epsilon > 0$

$$\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \left|\left\{m \le M, n \le N : V^{(11)}_{mn}(\Delta u) - \sigma^{(11)}_{mn}(V^{(11)}(\Delta u)) \ge \epsilon\right\}\right| = 0. \tag{23}$$

We omit the proof for $0 < \lambda < 1$ since it is similar to $\lambda > 1$. Therefore, we get

$$\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \left|\left\{m \le M, n \le N : V^{(11)}_{mn}(\Delta u) - \sigma^{(11)}_{mn}(V^{(11)}(\Delta u)) \le -\epsilon\right\}\right| = 0. \tag{24}$$

Combining (23) and (24) yields

$$\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \left|\left\{m \le M, n \le N : |V^{(11)}_{mn}(\Delta u) - \sigma^{(11)}_{mn}(V^{(11)}(\Delta u))| \ge \epsilon\right\}\right| = 0.$$

Therefore, we obtain that $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is statistically convergent to 0.

Now, let's show that $V^{(10)}(\Delta u) = (V^{(10)}_{mn}(\Delta u))$ is statistically convergent to 0. If we replace u_{mn} by $V^{(10)}_{mn}(\Delta u)$ in (20), then we have

$$\begin{aligned}
V^{(10)}_{mn}(\Delta u) - \sigma^{(10)}_{mn}(V^{(10)}(\Delta u)) = &\frac{[\lambda m]+1}{[\lambda m]-m}\left(\sigma^{(10)}_{[\lambda m],n}(V^{(10)}(\Delta u)) - \sigma^{(10)}_{mn}(V^{(10)}(\Delta u))\right) \\
&- \frac{1}{[\lambda m]-m}\sum_{i=m+1}^{[\lambda m]} (V^{(10)}_{in}(\Delta u) - V^{(10)}_{mn}(\Delta u)),
\end{aligned}$$

for $\lambda > 1$. From this, we obtain

$$\begin{aligned}
&\left\{m \le M, n \le N : V^{(10)}_{mn}(\Delta u) - \sigma^{(10)}_{mn}(V^{(10)}(\Delta u)) \ge \varepsilon\right\} \\
&\subset \left\{m \le M, n \le N : \frac{[\lambda m]+1}{[\lambda m]-m}\left(\sigma^{(10)}_{[\lambda m],n}(V^{(10)}(\Delta u)) - \sigma^{(10)}_{mn}(V^{(10)}(\Delta u))\right) \ge \varepsilon/2\right\} \\
&\cup \left\{m \le M, n \le N : \frac{1}{[\lambda m]-m}\sum_{i=m+1}^{[\lambda m]} (V^{(10)}_{in}(\Delta u) - V^{(10)}_{mn}(\Delta u)) \le -\varepsilon/2\right\} \\
&:= A^*_{MN}(\epsilon) \cup B^*_{MN}(\epsilon).
\end{aligned}$$

Since

$$\frac{[\lambda m]+1}{[\lambda m]-m} \le \frac{2\lambda}{\lambda-1},$$

we have

$$\lim_{M,N\to\infty} \frac{1}{(M+1)(N+1)} |A^*_{MN}(\epsilon)| = 0.$$

On the other hand, for $B^*_{MN}(\epsilon)$, since

$$\frac{1}{[\lambda m]-m} \sum_{i=m+1}^{[\lambda m]} (V^{(10)}_{in}(\Delta u) - V^{(10)}_{mn}(\Delta u)) \geq \min_{m+1\leq i\leq [\lambda m]} (V^{(10)}_{in}(\Delta u) - V^{(10)}_{mn}(\Delta u)),$$

we obtain for every $\epsilon > 0$

$$\begin{aligned} &\Big\{m \leq M, n \leq N : \frac{1}{[\lambda m]-m} \sum_{i=m+1}^{[\lambda m]} (V^{(10)}_{in}(\Delta u) - V^{(10)}_{mn}(\Delta u)) \leq -\frac{\epsilon}{2}\Big\} \\ \subseteq \ &\Big\{m \leq M, n \leq N : \min_{m+1\leq i\leq [\lambda m]} (V^{(10)}_{in}(\Delta u) - V^{(10)}_{mn}(\Delta u)) \leq -\frac{\epsilon}{2}\Big\}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Big|\Big\{m \leq M, n \leq N : V^{(10)}_{mn}(\Delta u) - \sigma^{(10)}_{mn}(V^{(10)}(\Delta u)) \geq \epsilon\Big\}\Big| \\ \leq \ &\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} |\Big\{m \leq M, n \leq N : \min_{m+1\leq i\leq [\lambda m]} (V^{(10)}_{in}(\Delta u) - V^{(10)}_{mn}(\Delta u)) \leq -\frac{\epsilon}{2}\Big\}|. \end{aligned}$$

Therefore, given any $\delta > 0$, since $V^{(10)}(\Delta u) = (V^{(10)}_{mn}(\Delta u))$ is statistically slowly decreasing in senses $(1, 0)$ for some $\lambda > 1$ we have

$$\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Big|\Big\{m \leq M, n \leq N : V^{(10)}_{mn}(\Delta u) - \sigma^{(10)}_{mn}(V^{(10)}(\Delta u)) \geq \epsilon\Big\}\Big| \leq \delta.$$

Since $\delta > 0$ is arbitrary, we get for every $\epsilon > 0$

$$\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Big|\Big\{m \leq M, n \leq N : V^{(10)}_{mn}(\Delta u) - \sigma^{(10)}_{mn}(V^{(10)}(\Delta u)) \geq \epsilon\Big\}\Big| = 0. \tag{25}$$

If similar steps are applied for $0 < \lambda < 1$, then we obtain

$$\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Big|\Big\{m \leq M, n \leq N : V^{(10)}_{mn}(\Delta u) - \sigma^{(10)}_{mn}(V^{(10)}(\Delta u)) \leq -\epsilon\Big\}\Big| = 0. \tag{26}$$

Combining (25) and (26) yields

$$\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Big|\Big\{m \leq M, n \leq N : |V^{(10)}_{mn}(\Delta u) - \sigma^{(10)}_{mn}(V^{(10)}(\Delta u))| \geq \epsilon\Big\}\Big| = 0.$$

Therefore, we obtain that $V^{(10)}(\Delta u) = (V^{(10)}_{mn}(\Delta u))$ is statistically convergent to 0. Correlatively, we can show that $V^{(10)}(\Delta u) = (V^{(10)}_{mn}(\Delta u))$ is statistically convergent to 0. The proof is completed by the Kronecker identity. □

In Theorem 4.8 without the hypothesis that the statistically slowly decreasing of $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ in the strong sense, we obtain the following theorem.

Theorem 4.9. *Let $u = (u_{mn})$ be a bounded double sequence. If $u = (u_{mn})$ is statistical $(C, 1, 1)$ summable to s, and $V^{(11)}(\Delta u) = (V^{(11)}_{mn}(\Delta u))$ is statistically slowly decreasing in senses $(1, 1)$, $(1, 0)$ and $(0, 1)$, $V^{(10)}(\Delta u) = (V^{(10)}_{mn}(\Delta u))$ is statistically slowly decreasing in sense $(1, 0)$ and $V^{(01)}(\Delta u) = (V^{(01)}_{mn}(\Delta u))$ is statistically slowly decreasing in sense $(0, 1)$, then $u = (u_{mn})$ is statistically convergent to s.*

Theorem 4.9 can be proved in the similar way used in proving Theorem 4.8, but the inequality (22) should be replaced by the following one:

$$\frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{i=m+1}^{[\lambda m]}\sum_{j=n+1}^{[\lambda n]}(V_{ij}^{(11)}(\Delta u)-V_{mn}^{(11)}(\Delta u))$$
$$\geq \min_{\substack{m+1\leq j\leq[\lambda m]\\ n+1\leq k\leq[\lambda n]}}\left(V_{jk}^{(11)}(\Delta u)-V_{mk}^{(11)}(\Delta u)-V_{jn}^{(11)}(\Delta u)+V_{mn}^{(11)}(\Delta u)\right)$$
$$+\min_{m+1\leq j\leq[\lambda m]}\left(V_{jn}^{(11)}(\Delta u)-V_{mn}^{(11)}(\Delta u)\right)+\min_{n+1\leq k\leq[\lambda n]}\left(V_{mk}^{(11)}(\Delta u)V_{mn}^{(11)}(\Delta u)\right).$$

Acknowledgement

The authors would like to thank the anonymous referees for many constructive comments and helpful suggestions which improved the manuscript.

References

[1] F. Başar, Summability theory and its applications, İstanbul, Bentham Science Publishers, e-books, Monographs, 2012.
[2] J. Boos, Classical and modern methods in summability, New York, Oxford University Press, 2000.
[3] H. Çakallı and R. F. Patterson, Functions preserving slowly oscillating double sequences, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) 62 (2) (2016), 531–536.
[4] H. Çakallı and E. Savaş, Statistical convergence of double sequences in topological groups, J. Comput. Anal. Appl. 12 (2) (2010), 421-426.
[5] D. Djurčić, L. D. R. Kočinac and M. R. Žižović, Double sequences and selections, Abstr. Appl. Anal. 2012 (2012), Article ID 497594, 1–6.
[6] C. Chen and C. Chang, Tauberian conditions under which the original convergence of double sequences follows from the statistical convergence of their weighted means, J. Math. Anal. Appl. 332 (2) (2007), 1242–1248.
[7] İ. Çanak and Ü. Totur, An alternative proof of a Tauberian theorem for Abel summability method, Proyecciones. 35 (3) (2016), 235–244.
[8] İ. Çanak, Ü. Totur and B. P. Allahverdiev, Tauberian conditions with controlled oscillatory behavior, Appl. Math. Lett. 25 (3) (2012), 252–256.
[9] M. Dik, Tauberian theorems for sequences with moderately oscillatory control moduli, Math. Morav. 5 (2001), 57–94.
[10] O. H. Edely and M. Mursaleen, Tauberian theorems for statistically convergent double sequences, Inform. Sci. 176 (7) (2006), 875–886.
[11] H. Fast, Sur la convergence statistique, Colloquium Math. 2 (1951), 241–244.
[12] J. A. Fridy and M. K. Khan, Statistical extensions of some classical Tauberian theorems, Proc. Amer. Math. Soc. 128 (8) (2000), 2347–2355.
[13] G. H. Hardy, Divergent Series, (2nd edition), New York, Chelsea, 1991.
[14] K. Knopp, Limitierungs-Umkehrsätze für Doppelfolgen, Math. Z. 45 (1939), 573–589.
[15] F. Móricz, Tauberian theorems for Cesàro summable double sequences, Studia Math. 110 (1) (1994), 83–96.
[16] F. Móricz, Tauberian theorems for double sequences that are statistically summable $(C, 1, 1)$, J. Math. Anal. Appl. 286 (1) (2003), 340–350.
[17] F. Móricz, Ordinary convergence follows from statistical summability (C,1) in the case of slowly decreasing or oscillating sequences, Colloq. Math. 99 (2) (2004), 207–219.
[18] M. Mursaleen and S. A. Mohiuddine, Banach limit and some new spaces of double sequences, Turkish J. Math. 36 (1) (2012), 121–130.
[19] R. F. Patterson and H. Çakallı, Quasi Cauchy double sequences, Tbilisi Math. J. 8 (2) (2015), 211–219.
[20] A. Pringsheim, Zur Theorie der zweifach unendlichen Zahlenfolgen, Math. Ann. 53 (3) (1900), 289–321.
[21] R. Schmidt, Über divergente Folgen und lineare Mittelbildungen, Math. Z. 22 (1) (1925), 89–152.
[22] I. J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly 66 (1959), 361–375.
[23] U. Stadtmüller, Tauberian theorems for weighted means of double sequences, Anal. Math. 25 (1) (1999), 57–68.
[24] Č. V. Stanojević, Analysis of Divergence: Applications to the Tauberian Theory, Graduate Research Seminar, University of Missouri-Rolla, Winter 1999.
[25] A. Tauber, Ein Satz aus der Theorie der Unendlichen Reihen, Monatsh. f. Math. 7 (1897), 273–277.
[26] Ü. Totur and İ. Çanak, Some Tauberian conditions for statistical convergence, C. R. Acad. Bulgare Sci. 67 (7) (2014), 889–896.
[27] Ü. Totur, Classical Tauberian theorems for $(C, 1, 1)$ summability method, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) 61 (2) (2015), 401–414.
[28] A. Zygmund, Trigonometric series, Cambridge University Press, 1959.