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Further Refinements of Some Inequalities Involving Unitarily Invariant Norm

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Abstract. Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ and $|||\cdot|||$ be an arbitrary unitarily invariant norm. We give a new function $f(t,s)$ that is log-convex in each of its variables such that $f(1/2,1/2) \le f(t,s)$ for any $t,s \in [0,1]$ which generalize the log-convex function defined in [4] and obtain the inequalities as follows:

$$\begin{aligned}|||AXB^*|||^2 &= f(1/2,1/2)\\ &\le f(t,1-t)\\ &\le (t|||A^*AX||| + (1-t)|||XB^*B||| - r(\sqrt{|||A^*AX|||} - \sqrt{|||XB^*B|||})^2)\\ &\quad \times ((1-t)|||A^*AX||| + t|||XB^*B||| - r(\sqrt{|||A^*AX|||} - \sqrt{|||XB^*B|||})^2),\end{aligned}$$

where $t \in [0,1]$ and $r = \min\{t, 1-t\}$. Furthermore, we refine some inequalities as well.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the space of $n \times n$ complex matrices and let $|||\cdot|||$ denote any unitarily invariant norm. Therefore, $|||UAV||| = |||A|||$ for every $A \in \mathbb{M}_n(\mathbb{C})$ and for all unitarily matrices $U, V \in \mathbb{M}_n(\mathbb{C})$. If A is arbitrary, then its singular values are enumerated as $s_1(A) \ge s_2(A) \ge \cdots \ge s_n(A)$. These are the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{\frac{1}{2}}$ arranged in a decreasing order and repeated according to multiplicity. So $\|A\| = s_1(A)$ is the operator norm of A. Note that $s_j(A) = s_j(A^*) = s_j(|A|)$ for $j = 1, 2, \cdots, n$. The Ky Fan norm of a matrix A is defined as $\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A)$, $k = 1, 2, \cdots, n$. The Fan dominance theorem asserts that $\|A\|_{(k)} \le \|B\|_{(k)}$ for $k = 1, 2, \cdots, n$ if and only if $|||A||| \le |||B|||$ for every unitarily invariant norm. A special unitarily invariant norm is the Hilbert-Schmidt norm which is defined by $\|A\|_2 = (\sum_{j=1}^{k} s_j^2(A))^{\frac{1}{2}}$.

The classical Young inequality for two scalars is the weighted arithmetic-geometric mean inequality for two nonnegative real numbers. The inequality states that if $a, b \ge 0$ and $0 \le t \le 1$, then

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$$a^t b^{1-t} \leq ta + (1-t)b$$

with equality if and only if $a = b$.

The matrix version of Young inequality [1] for unitarily invariant norm proved by Ando states that if A, B are positive definite matrices and $0 \leq t \leq 1$, then

$$|||A^t B^{1-t}||| \leq |||tA + (1-t)B|||. \tag{1}$$

Recently, Kittaneh et al. gave a refinement of inequality (1) in [9] as follows:

$$|||A^t X B^{1-t}||| + r(\sqrt{|||AX|||} - \sqrt{|||XB|||})^2 \leq t|||AX||| + (1-t)|||XB|||,$$

where A, B are positive definite matrices, $0 \leq t \leq 1$, $r = \min\{t, 1-t\}$.

The classical Cauchy-Schwarz inequality for scalars $a_i, b_i \geq 0\,(1 \leq i \leq n)$ states that

$$(\textstyle\sum_{i=1}^n a_i b_i)^2 \leq (\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2).$$

with equality if and only if $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ are proportional.

The matrix Cauchy-Schwarz inequality [5] was proved by Bhatia as follows:

$$|||AXB^*|||^2 \leq |||A^*AX|||\,|||XB^*B|||, \tag{2}$$

where $A, B, X \in \mathbb{M}_n(\mathbb{C})$.

In [3], Kittaneh et al. extend ineqaulity (2) as follows:

$$|||AXB^*|||^2 \leq |||(A^*A)^p X (B^*B)^{1-p}|||\,|||(A^*A)^{1-p} X (B^*B)^p|||,$$

In [2], Audenaert give an inequality that interpolates between arithmetic-geometric mean inequality ($t = \frac{1}{2}$) and Cauchy-Schwarz inequality ($t = 0$ or $t = 1$) as follows:

$$|||AB^*|||^2 \leq |||tA^*A + (1-t)B^*B|||\,|||(1-t)A^*A + tB^*B|||, \tag{3}$$

where $A, B \in \mathbb{M}_n(\mathbb{C})$, $0 \leq t \leq 1$.

In the paper [10], Zou gave a generalization of (3) as follows:

$$|||AXB^*|||^2 \leq |||tA^*AX + (1-t)XB^*B|||\,|||(1-t)A^*AX + tXB^*B|||, \tag{4}$$

where $A, B \in \mathbb{M}_n(\mathbb{C})$, $0 \leq t \leq 1$.

In [4], Alakhrass utilized a log-convex function to refine the inequalities presented above. We shall generalize this log-convex function and present some inequalities that further refine some known results.

2. Main Results

Lemma 2.1. [6] Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite. If $0 \le t \le 1$, then

$$|||A^t XB^{1-t}||| \le |||AX|||^t \, |||XB|||^{1-t}.$$

Theorem 2.2. Let $P_1, P_2, X \in \mathbb{M}_n(\mathbb{C})$ such that P_1, P_2 are positive semidefinite. Then the function $f : [0,1] \times [0,1] \longmapsto [0,\infty)$ defined by

$$f(t,s) = |||P_1^t XP_2^s||| \, |||P_1^{1-t} XP_2^{1-s}|||$$

is log-convex in each of its variables.

Proof. We only prove the log-convexity for t, cause the other case can be proved similarly. First we give a more general result. Set $t_1, t_2, s_1, s_2, \alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$.

Compute

$$\begin{aligned}
&|||P_1^{\alpha t_1+\beta t_2} XP_2^{\alpha s_1+\beta s_2}|||\\
&= |||P_1^{\alpha t_1+(1-\alpha)t_2} XP_2^{(1-\beta)s_1+\beta s_2}|||\\
&= |||P_1^{\alpha(t_1-t_2)} P_1^{t_2} XP_2^{s_1} P_2^{\beta(s_2-s_1)}||| \qquad (5)\\
&\le |||P_1^{t_1-t_2} P_1^{t_2} XP_2^{s_1}|||^{\alpha} \, |||P_1^{t_2} XP_2^{s_1} P_2^{s_2-s_1}|||^{\beta} \ (by\ Lemma\ 2.1)\\
&= |||P_1^{t_1} XP_2^{s_1}|||^{\alpha} \, |||P_1^{t_2} XP_2^{s_2}|||^{\beta}.
\end{aligned}$$

Similarly,

$$|||P_1^{1-(\alpha t_1+\beta t_2)} XP_2^{1-(\alpha s_1+\beta s_2)}||| \le |||P_1^{1-t_1} XP_2^{1-s_1}|||^{\alpha} \, |||P_1^{1-t_2} XP_2^{1-s_2}|||^{\beta}. \qquad (6)$$

Apply inequalities (5) and (6) we have

$$\begin{aligned}
&f(\alpha t_1 + \beta t_2, \alpha s_1 + \beta s_2)\\
&= |||P_1^{\alpha t_1+\beta t_2} XP_2^{\alpha s_1+\beta s_2}||| \, |||P_1^{1-(\alpha t_1+\beta t_2)} XP_2^{1-(\alpha s_1+\beta s_2)}|||.\\
&\le (|||P_1^{t_1} XP_2^{s_1}||| \, |||P_1^{1-t_1} XP_2^{1-s_1}|||)^{\alpha} (|||P_1^{t_2} XP_2^{s_2}||| \, |||P_1^{1-t_2} XP_2^{1-s_2}|||)^{\beta}\\
&= f(t_1, s_1)^{\alpha} f(t_2, s_2)^{\beta}.
\end{aligned}$$

Putting $s_1 = s_2 = s$ in the preceding inequality we thus obtain

$$f(\alpha t_1 + \beta t_2, s) \le f(t_1, s)^{\alpha} f(t_2, s)^{\beta}.$$

This completes the proof. □

Corollary 2.3. Let $P_1, P_2, X \in \mathbb{M}_n(\mathbb{C})$ such that P_1 and P_2 are positive semidefinite. Then the function

$$t \mapsto |||P_1^t XP_2^{1-t}||| \, |||P_1^{1-t} XP_2^t|||$$

is log-convex on the inverval $[0,1]$.

Proof. By putting $s = 1 - t$ in Theorem 2.2, we obtain the result. □

Lemma 2.4. [8] The generalization of Fan Dominance Theorem states as follows:

$$\|A\|_{(k)}^2 \le \|B\|_{(k)} \|C\|_{(k)} \ for\ k = 1, 2, \cdots, n\ if\ and\ only\ if\ |||A|||^2 \le |||B|||\, |||C|||$$

for every $A, B, C \in \mathbb{M}_n(\mathbb{C})$.

The following theorem was proved in [7] using Fan Dominance Theorem, which should be corrected by using Lemma 2.4.

Theorem 2.5. Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ and $t, s \in [0, 1]$. Then

$$|||AXB^*|||^2 \le |||(A^*A)^t X (B^*B)^s|||\, |||(A^*A)^{1-t} X (B^*B)^{1-s}|||$$

for every unitarily invariant norm.

Corollary 2.6. Let $P_1, P_2, X \in \mathbb{M}_n(\mathbb{C})$ such that P_1, P_2 are positive semidefinite and $f(t, s)$ is defined as in Theorem 2.2. Then we get

$$f(1/2, 1/2) \le f(t, s)$$

for every $t, s \in [0, 1]$.

Proof. Put $A = P_1^{\frac{1}{2}}$ and $B = P_2^{\frac{1}{2}}$ in Theorem 2.5, we obtain

$$f(1/2, 1/2) = |||P_1^{\frac{1}{2}} X P_2^{\frac{1}{2}}|||^2 \le |||P_1^t X P_2^s|||\, |||P_1^{1-t} X P_2^{1-s}||| = f(t, s). \quad \square$$

The following result is a direct consequence of Theorem 2.5 by Young inequality.

Corollary 2.7. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ and $t, s, \alpha \in [0, 1]$. Then

$$|||AB^*|||^2 \le |||\alpha(A^*A)^{\frac{t}{\alpha}} + (1-\alpha)(B^*B)^{\frac{s}{1-\alpha}}|||\, |||(1-\alpha)(A^*A)^{\frac{1-t}{1-\alpha}} + \alpha(B^*B)^{\frac{1-s}{\alpha}}|||.$$

Corollary 2.7 is a refinement of Theorem 2.2 in [4] by putting $\alpha = t$ and $s = 1 - t$.

Proposition 2.8. Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that X is positive semidefinite with $|A|, |B| \le X \le mI$ and $t, s, \alpha \in [0, 1]$ with $\max\left\{\frac{3(1-s)}{2}, \frac{t}{2}\right\} \le \alpha \le \min\left\{\frac{3t-1}{2}, \frac{2-s}{2}\right\}$. Then

$$|||AXB^*|||^2 \le |||X|||\, |||(1-\alpha)X^{\frac{3(1-t)}{1-\alpha}} + \alpha X^{\frac{3(1-s)}{\alpha}}|||\, |||\alpha m^{\frac{t-\alpha}{\alpha}} (A^*A)^{\frac{t}{\alpha}} + (1-\alpha) m^{\frac{s+\alpha-1}{1-\alpha}} (B^*B)^{\frac{s}{1-\alpha}}|||.$$

Proof. . We have

$$
\begin{aligned}
|||AXB^*|||^2 &= |||AX^{\frac{1}{2}}X^{\frac{1}{2}}B^*|||^2 \\
&\le |||\alpha|AX^{\frac{1}{2}}|^{\frac{2t}{\alpha}} + (1-\alpha)|BX^{\frac{1}{2}}|^{\frac{2s}{1-\alpha}}|||\,|||(1-\alpha)|AX^{\frac{1}{2}}|^{\frac{2(1-t)}{1-\alpha}} + \alpha|BX^{\frac{1}{2}}|^{\frac{2(1-s)}{\alpha}}||| \\
&= |||\alpha(X^{\frac{1}{2}}|A|^2X^{\frac{1}{2}})^{\frac{t}{\alpha}} + (1-\alpha)(X^{\frac{1}{2}}|B|^2X^{\frac{1}{2}})^{\frac{s}{1-\alpha}}|||\,|||(1-\alpha)(X^{\frac{1}{2}}|A|^2X^{\frac{1}{2}})^{\frac{1-t}{1-\alpha}} + \alpha(X^{\frac{1}{2}}|B|^2X^{\frac{1}{2}})^{\frac{1-s}{\alpha}}||| \\
&= |||\alpha m^{\frac{t}{\alpha}}((X/m)^{\frac{1}{2}}|A|^2(X/m)^{\frac{1}{2}})^{\frac{t}{\alpha}} + (1-\alpha)m^{\frac{s}{1-\alpha}}((X/m)^{\frac{1}{2}}|B|^2(X/m)^{\frac{1}{2}})^{\frac{s}{1-\alpha}}||| \\
&\times |||(1-\alpha)(X^{\frac{1}{2}}|A|^2X^{\frac{1}{2}})^{\frac{1-t}{1-\alpha}} + \alpha(X^{\frac{1}{2}}|B|^2X^{\frac{1}{2}})^{\frac{1-s}{\alpha}}||| \\
&\le |||X^{\frac{1}{2}}(\alpha m^{\frac{t-\alpha}{\alpha}}|A|^{\frac{2t}{\alpha}} + (1-\alpha)m^{\frac{s+\alpha-1}{1-\alpha}}|B|^{\frac{2s}{1-\alpha}})X^{\frac{1}{2}}|||\,|||(1-\alpha)(X^{\frac{1}{2}}|A|^2X^{\frac{1}{2}})^{\frac{1-t}{1-\alpha}} + \alpha(X^{\frac{1}{2}}|B|^2X^{\frac{1}{2}})^{\frac{1-s}{\alpha}}||| \\
&\quad \text{(by operator convexity of } x^{\frac{t}{\alpha}} \text{ and } x^{\frac{s}{1-\alpha}}) \\
&\le |||X|||\,|||\alpha m^{\frac{t-\alpha}{\alpha}}(A^*A)^{\frac{t}{\alpha}} + (1-\alpha)m^{\frac{s+\alpha-1}{1-\alpha}}(B^*B)^{\frac{s}{1-\alpha}}|||\,|||(1-\alpha)(X^{\frac{1}{2}}|A|^2X^{\frac{1}{2}})^{\frac{1-t}{1-\alpha}} + \alpha(X^{\frac{1}{2}}|B|^2X^{\frac{1}{2}})^{\frac{1-s}{\alpha}}||| \\
&\le |||X|||\,|||(1-\alpha)X^{\frac{3(1-t)}{1-\alpha}} + \alpha X^{\frac{3(1-s)}{\alpha}}|||\,|||\alpha m^{\frac{t-\alpha}{\alpha}}(A^*A)^{\frac{t}{\alpha}} + (1-\alpha)m^{\frac{s+\alpha-1}{1-\alpha}}(B^*B)^{\frac{s}{1-\alpha}}|||. \quad \text{(by [12, p 12])}
\end{aligned}
$$

This completes the proof. □

Proposition 2.9. Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that X is positive semidefinite with $|A|, |B| \le X$ and $t, s, \alpha \in [0,1]$ with $\max\left\{\frac{3(1-s)}{2}, \frac{t}{2}\right\} \le \alpha \le \min\left\{\frac{3t-1}{2}, \frac{2-s}{2}\right\}$. Then

$$
|||AXB^*|||^2 \le |||X|||\,|||(1-\alpha)X^{\frac{3(1-t)}{1-\alpha}} + \alpha X^{\frac{3(1-s)}{\alpha}}|||\,|||\alpha(A^*A)^{\frac{3t-\alpha}{2\alpha}} + (1-\alpha)(B^*B)^{\frac{3s+\alpha-1}{2(1-\alpha)}}|||.
$$

Proof. We have

$$
\begin{aligned}
|||AXB^*|||^2 &= |||AX^{\frac{1}{2}}X^{\frac{1}{2}}B^*|||^2 \\
&\le |||\alpha|AX^{\frac{1}{2}}|^{\frac{2t}{\alpha}} + (1-\alpha)|BX^{\frac{1}{2}}|^{\frac{2s}{1-\alpha}}|||\,|||(1-\alpha)|AX^{\frac{1}{2}}|^{\frac{2(1-t)}{1-\alpha}} + \alpha|BX^{\frac{1}{2}}|^{\frac{2(1-s)}{\alpha}}||| \\
&= |||\alpha(X^{\frac{1}{2}}|A|^2X^{\frac{1}{2}})^{\frac{t}{\alpha}} + (1-\alpha)(X^{\frac{1}{2}}|B|^2X^{\frac{1}{2}})^{\frac{s}{1-\alpha}}|||\,|||(1-\alpha)(X^{\frac{1}{2}}|A|^2X^{\frac{1}{2}})^{\frac{1-t}{1-\alpha}} + \alpha(X^{\frac{1}{2}}|B|^2X^{\frac{1}{2}})^{\frac{1-s}{\alpha}}||| \\
&= |||\alpha X^{\frac{1}{2}}|A|(|A|X|A|)^{\frac{t}{\alpha}-1}|A|X^{\frac{1}{2}} + (1-\alpha)X^{\frac{1}{2}}|B|(|B|X|B|)^{\frac{s}{1-\alpha}-1}|B|X^{\frac{1}{2}}||| \\
&\times |||(1-\alpha)(X^{\frac{1}{2}}|A|^2X^{\frac{1}{2}})^{\frac{1-t}{1-\alpha}} + \alpha(X^{\frac{1}{2}}|B|^2X^{\frac{1}{2}})^{\frac{1-s}{\alpha}}||| \\
&\le |||X^{\frac{1}{2}}(\alpha|A|^{\frac{3t-\alpha}{\alpha}} + (1-\alpha)|B|^{\frac{3s+\alpha-1}{1-\alpha}})X^{\frac{1}{2}}|||\,|||(1-\alpha)(X^{\frac{1}{2}}|A|^2X^{\frac{1}{2}})^{\frac{1-t}{1-\alpha}} + \alpha(X^{\frac{1}{2}}|B|^2X^{\frac{1}{2}})^{\frac{1-s}{\alpha}}||| \\
&\quad \text{(by [12, p 12])} \\
&\le |||X|||\,|||\alpha(A^*A)^{\frac{3t-\alpha}{2\alpha}} + (1-\alpha)(B^*B)^{\frac{3s+\alpha-1}{2(1-\alpha)}}|||\,|||(1-\alpha)(X^{\frac{1}{2}}|A|^2X^{\frac{1}{2}})^{\frac{1-t}{1-\alpha}} + \alpha(X^{\frac{1}{2}}|B|^2X^{\frac{1}{2}})^{\frac{1-s}{\alpha}}||| \\
&\le |||X|||\,|||(1-\alpha)X^{\frac{3(1-t)}{1-\alpha}} + \alpha X^{\frac{3(1-s)}{\alpha}}|||\,|||\alpha(A^*A)^{\frac{3t-\alpha}{2\alpha}} + (1-\alpha)(B^*B)^{\frac{3s+\alpha-1}{2(1-\alpha)}}|||. \quad \text{(by [12, p 12])}
\end{aligned}
$$

This completes the proof. □

Lemma 2.10. [9] Let $P_1, P_2, X \in \mathbb{M}_n(\mathbb{C})$ such that P_1 and P_2 are positive semidefinite. If $t \in [0,1]$, then

$$
|||{P_1}^tX{P_2}^{1-t}||| + r(\sqrt{|||P_1X|||} - \sqrt{|||XP_2|||})^2 \le t|||P_1X||| + (1-t)|||XP_2|||,
$$

where $r = \min\{t, 1-t\}$.

Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$. Put $P_1 = A^*A, P_2 = B^*B$ in Theorem 2.2 thus we get the following function

$$
f(t,s) = |||(A^*A)^tX(B^*B)^s|||\,|||(A^*A)^{1-t}X(B^*B)^{1-s}|||, \quad t, s \in [0,1]. \tag{7}
$$

By Theorem 2.2 the function $f(t,s)$ is log-convex in each of its variables. Therefore $f(t,s)$ is convex in each of its variables. By Young inequality and Corollary 2.6 we get

$$\begin{aligned} f(1/2,1/2) &\le f(\alpha t_1+\beta t_2,s) \\ &\le f(t_1,s)^{\alpha} f(t_2,s)^{\beta} \\ &\le \alpha f(t_1,s)+\beta f(t_2,s), \end{aligned}$$

where $t_1,t_2,s,\alpha,\beta\in[0,1]$ and $\alpha+\beta=1$.

Theorem 2.11. Let $A,B,X\in\mathbb{M}_n(\mathbb{C})$ and $t\in[0,1]$. We have

$$\begin{aligned} |||AXB^*|||^2 &= f(1/2,1/2) \\ &\le f(t,1-t) \\ &\le (t|||A^*AX|||+(1-t)|||XB^*B|||-r(\sqrt{|||A^*AX|||}-\sqrt{|||XB^*B|||})^2) \\ &\quad\times((1-t)|||A^*AX|||+t|||XB^*B|||-r(\sqrt{|||A^*AX|||}-\sqrt{|||XB^*B|||})^2), \end{aligned}$$

where $r=\min\{t,1-t\}$.

Proof. Let $t\in[0,1]$ and $r=\min\{t,1-t\}$. By Lemma 2.10 (set $P_1=A^*A$ and $P_2=B^*B$), we get

$$\begin{aligned} f(t,1-t) &= |||(A^*A)^tX(B^*B)^{1-t}|||\,|||(A^*A)^{1-t}X(B^*B)^t||| \\ &\le (t|||A^*AX|||+(1-t)|||XB^*B|||-r(\sqrt{|||A^*AX|||}-\sqrt{|||XB^*B|||})^2) \\ &\quad\times((1-t)|||A^*AX|||+t|||XB^*B|||-r(\sqrt{|||A^*AX|||}-\sqrt{|||XB^*B|||})^2). \end{aligned}$$

Put $A=U|A|$ and $B=V|B|$ be the polar decompositions of A and B with unitarily matrices U and V. Thus we get

$$\begin{aligned} f(1/2,1/2) &= |||(A^*A)^{1/2}X(B^*B)^{1/2}|||^2 \\ &= |||\,|A|X|B|\,|||^2 \\ &= |||U|A|X|B|V^*|||^2 \\ &= |||AXB^*|||^2. \end{aligned}$$

This completes the proof. □

Remark 2.12. Theorem 2.11 can be seen as a refinement of Theorem 2.4 in [4].

Lemma 2.13. [11] Let $P_1,P_2,X\in\mathbb{M}_n(\mathbb{C})$ such that P_1 and P_2 are positive semidefinite. If $t\in[0,1]$ and $N\ge 2$ be an integer, then

$$\begin{aligned} &\|P_1^tXP_2^{1-t}\|_2^2+s_1^2(t)\|P_1X-XP_2\|_2^2 \\ &\quad+\sum_{j=2}^{N}\|P_1^{\frac{k_j(t)}{2^{j-1}}}XP_2^{1-\frac{k_j(t)}{2^{j-1}}}-P_1^{\frac{k_j(t)+1}{2^{j-1}}}XP_2^{1-\frac{k_j(t)+1}{2^{j-1}}}\|_2^2 \\ &\quad\le\|tP_1X+(1-t)XP_2\|_2^2, \end{aligned}$$

where $k_j(t)=[2^{j-1}t]$, $r_j(t)=[2^jt]$ and $s_j(t)=(-1)^{r_j(t)}2^{j-1}t+(-1)^{r_j(t)+1}[\frac{r_j(t)+1}{2}]$.

We point out that Lemma 2.13 is a refinement of Lemma 2.2 in [4].

When we restrict the unitarily invariant norm in (7) to the particularly important Hilbert-Schmidt norm $\|\cdot\|_2$, we reset the function as below

$$g(t,s) = \|(A^*A)^t X(B^*B)^s\|_2\,\|(A^*A)^{1-t}X(B^*B)^{1-s}\|_2, \quad t,s\in[0,1].$$

Theorem 2.14. Let $A,B,X\in\mathbb{M}_n(\mathbb{C})$ and $t\in[0,1]$. We have

$$\begin{aligned}
\|AXB^*\|_2^2 &= g(1/2,1/2)\\
&\le g(t,1-t)\\
&\le \Big(\|t(A^*A)X+(1-t)X(B^*B)\|_2^2-\sum_{j=2}^{N}\|(A^*A)^{\frac{k_j(t)}{2^{j-1}}}X(B^*B)^{1-\frac{k_j(t)}{2^{j-1}}}\\
&-(A^*A)^{\frac{k_j(t)+1}{2^{j-1}}}X(B^*B)^{1-\frac{k_j(t)+1}{2^{j-1}}}\|_2^2-s_1^2(t)\|(A^*A)X-X(B^*B)\|_2^2\Big)^{\frac12}\\
&\times\Big(\|t(A^*A)X+(1-t)X(B^*B)\|_2^2-\sum_{j=2}^{N}\|(A^*A)^{\frac{k_j(t)}{2^{j-1}}}X(B^*B)^{1-\frac{k_j(t)}{2^{j-1}}}\\
&-(A^*A)^{\frac{k_j(t)+1}{2^{j-1}}}X(B^*B)^{1-\frac{k_j(t)+1}{2^{j-1}}}\|_2^2-s_1^2(t)\|(A^*A)X-X(B^*B)\|_2^2\Big)^{\frac12}.
\end{aligned}$$

Proof. Let $t\in[0,1]$ and $r=\min\{t,1-t\}$. By Lemma 2.13 (set $P_1=A^*A$ and $P_2=B^*B$), we get

$$\begin{aligned}
g(t,1-t) &= \|(A^*A)^tX(B^*B)^{1-t}\|_2\,\|(A^*A)^{1-t}X(B^*B)^t\|_2\\
&\le \Big(\|t(A^*A)X+(1-t)X(B^*B)\|_2^2-\sum_{j=2}^{N}\|(A^*A)^{\frac{k_j(t)}{2^{j-1}}}X(B^*B)^{1-\frac{k_j(t)}{2^{j-1}}}\\
&-(A^*A)^{\frac{k_j(t)+1}{2^{j-1}}}X(B^*B)^{1-\frac{k_j(t)+1}{2^{j-1}}}\|_2^2-s_1^2(t)\|(A^*A)X-X(B^*B)\|_2^2\Big)^{\frac12}\\
&\times\Big(\|t(A^*A)X+(1-t)X(B^*B)\|_2^2-\sum_{j=2}^{N}\|(A^*A)^{\frac{k_j(t)}{2^{j-1}}}X(B^*B)^{1-\frac{k_j(t)}{2^{j-1}}}\\
&-(A^*A)^{\frac{k_j(t)+1}{2^{j-1}}}X(B^*B)^{1-\frac{k_j(t)+1}{2^{j-1}}}\|_2^2-s_1^2(t)\|(A^*A)X-X(B^*B)\|_2^2\Big)^{\frac12}.
\end{aligned}$$

Put $A=U|A|$ and $B=V|B|$ be the polar decompositions of A and B with unitarily matrices U and V. Thus we get

$$\begin{aligned}
g(1/2,1/2) &= \|(A^*A)^{1/2}X(B^*B)^{1/2}\|_2^2\\
&= \||A|X|B|\|_2^2\\
&= \|U|A|X|B|V^*\|_2^2\\
&= \|AXB^*\|_2^2.
\end{aligned}$$

This completes the proof. □

Remark 2.15. We remark that Theorem 2.14 is a refinement of Theorem 2.5 in [4].

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