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Some Remarks on Fuzzy k-Pseudometric Spaces

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Abstract. An important class of spaces was introduced by I.A. Bakhtin (under the name "metric-type") and independently rediscovered by S. Czerwik (under the name "b-metric"). Metric-type spaces generalize "classic" metric spaces by replacing the triangularity axiom with a more general axiom $d(x,z) \leq k \cdot (d(x,y) + d(y,z))$ for all $x, y, z \in X$ where $k \geq 1$ is a fixed constant. Recently R. Saadadi has introduced the fuzzy version of "metric-type" spaces. In this paper we consider topological and sequential properties of such spaces, illustrate them by several examples and prove a certain version of the Baire Category Theorem.

1. Introduction

Metric-type spaces were introduced in 1989 by I.A. Bakhtin [3] as a generalization of metric spaces; later they were independently rediscovered by S. Czerwik [5, 6] under the name of b-metric spaces. The class of metric-type spaces is essentially broader than the class of metric spaces and includes such natural and important for application metric-type structures as $\rho(x,y) =\mid x - y \mid^2$ on the real line or $\rho(f,g) = \int_a^b (f(x) - g(x))^2 dx$ on the class of Lebesgue measurable functions. Metric-type spaces were considered in several recently published papers, see, e.g. [2, 12, 13].

Basing on K. Menger's concept of a statistical metric [18], I. Kramosil and J. Michalek [15] introduced the notion of a fuzzy metric. Actually their fuzzy metrics are in a certain sense equivalent to Menger's statistical metrics, but the essential difference is in their definition and interpretation. While in Menger's theory "distance" is realized as a certain *probability*, Kramosil - Michalek's "distance" is described by a certain *fuzzy notion*. A. George and R. Veeramani [10] slightly modified Kramosil-Michalek's definition of a fuzzy metric. in order to make fuzzy metrics more appropriate for consideration of the induced topological structure. Recently R. Saadadi [20] has introduced the fuzzy version of metric-type spaces and considered some topological properties of such spaces.

Being fully respectful to the "founders" of this concept, I.A. Bakhtin and S. Czerwik, we prefer to use in this work the term a *k-(pseudo)metric space* and a *fuzzy k-(pseudo)metric space* for the following reasons. The term "a metric type (space)" is cumbrous and not convenient especially when being applied in the context of fuzzy structures and specified with other adjectives. On the other hand, the term "b-metric" has no justification for the letter "b" and says nothing about the constant "k" laid in the basis of the definition of such "metrics".

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The structure of the paper is as follows. In the next section we recall definitions related to fuzzy metric spaces and give a preliminary information about "ordinary" k-(pseudo)metric spaces. Section 3 contains basic definitions related to fuzzy k-(pseudo)metric spaces as well as a series of i examples of fuzzy k-(pseudo)metric spaces. Section 4 is devoted to topological properties of fuzzy k-(pseudo)metric spaces. We introduce two different structures induced by a fuzzy k-(pseudo)metric m: a topology $\mathcal{T}_m$ and a supratopology $\mathcal{S}_m$ and consider Hausdorfness, compactness, boundedness and density in such spaces. In Section 5 we consider sequential properties of fuzzy k-(pseudo)metric spaces, that is properties, which can be described by means of sequences. We mark out two, natural in our opinion, definitions of convergence for sequences in fuzzy k-(pseudo)metric spaces: σ- and τ-convergence, and prove that with respect to σ-convergence a fuzzy k-(pseudo)metric space is sequential. In Section 6 the property of completeness of a fuzzy k-(pseudo)metric space is defined. A restricted version of the Baire Category Theorem for fuzzy k-(pseudo)metric spaces is proved here. In the last, 7^{th} Section, we make a brief analysis of the obtained results and sketch out some directions in which we expect the development of the theory of fuzzy k-pseudometric spaces.

2. Preliminaries

2.1. Fuzzy Metric Spaces

Let $\mathbb{R}^+ = (0, \infty)$, X be a set and $* : [0,1] \times [0,1] \to [0,1]$ be a continuous t-norm, see, e.g. [14, 21].

Definition 2.1. ([10, 11]) A fuzzy metric on a set X is a pair $(m, *)$, where $m : X \times X \times \mathbb{R}^+ \to (0,1]$ is a mapping satisfying the following conditions for all $x, y, z \in X$, $s, t \in \mathbb{R}^+$:
(1FM) $m(x,y,t) > 0$;
(2FM) $m(x,y,t) = 1 \Longleftrightarrow x = y$;
(3FM) $m(x,y,t) = m(y,x,t)$;
(4FM) $m(x,z,t+s) \geq m(x,y,t) * m(y,z,s)$;
(5FM) $m(x,y,-) : \mathbb{R}^+ \to (0,1]$ is continuous.
The triple $(X, m, *)$ is a called *a fuzzy metric space*. If axiom (2FM) is replaced by a weaker axiom
(2FPM) $x = y \Longrightarrow m(x,y,t) = 1$
we get definitions of a fuzzy pseudometric and a fuzzy pseudometric space. In its turn, if axiom (4FM) is replaced by axiom (4FSM)
(4FSM) $m(x,z,t) \geq m(x,y,t) * m(y,z,t)$,
then $(m, *)$ is called a strong fuzzy metric.

We write just m and (X, m) if the t-norm $*$ is specified,.

2.2. k-(Pseudo)Metrics and k-(Pseudo)Metric Spaces

Let $k \geq 1$ be a constant and X be a set.

Definition 2.2. ([3, 5, 6]) A k-metric on a set X is a mapping $d : X \times X \to \mathbb{R}^+_0 = [0, +\infty)$ such that
(1Mk) $d(x,y) = 0 \Longleftrightarrow x = y$;
(2Mk) $d(x,y) = d(y,x)\ \forall x, y \in X$;
(3Mk) $d(x,z) \leq k \cdot (d(x,y) + d(y,z))\ \forall x, y, z \in X$.
In case the axiom (1Mk) is replaced by a weaker axiom
(1′Mk) $x = y \Longrightarrow d(x,y) = 0$;
we come to the concept of a k-pseudometric. The corresponding pair (X, d) is called a k-(pseudo)metric space.

Obviously, we return to the definition of a metric if k = 1, while in case k< 1 the definition makes no sense.

Remark 2.3. An essential difference between ordinary (pseudo)metric and its k-type version displays itself when applying it in case of more than tree points. Indeed let (X, ρ) be a metric space and $x, y, z, u \in X$. Then, applying the triangular inequality, we get $\rho(x, u) \leq \rho(x, y) + \rho(y, z) + \rho(z, u)$. On the other hand, in case of a k-(pseudo)metric space (X, d), we get the inequality $d(x, u) \leq kd(x, y) + k^2 d(y, z) + k^2 d(z, u)$. Thus, as different from an ordinary (pseudo)metric, a k-(pseudo)metric looses the "homogeneity" of the triangular axiom and has to use also constants, different from the original constant k.

Example 2.4. A series of k-pseudometrics can be obtained from an ordinary pseudometric by the following construction. Let $k \geq 1$ be a fixed constant and let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a continuous increasing mapping such that $\varphi(0) = 0$ and $\varphi(a + b) \leq k \cdot (\varphi(a) + \varphi(b))$ for all $a, b \in \mathbb{R}^+$. Further, let $\rho : X \times X \to \mathbb{R}_0^+$ be a pseudometric on a set X, Then by setting

$$d_{\rho\varphi}(x, y) = (\varphi \circ \rho)(x, y) \quad x, y \in X$$

we get a k-pseudometric $d_{\rho\varphi}$ on the set X. Indeed, the validity of axioms (1'Mk) and (2Mk) for $d_{\rho\varphi}$ is obvious. To verify axiom (3Mk) let $x, y, z \in X$. Then

$$d_{\rho\varphi}(x, y) = (\varphi \circ \rho)(x, y) \leq k \cdot ((\varphi \circ \rho)(x, z) + (\varphi \circ \rho)(z, y) = k \cdot (d_{\rho\varphi}(x, z) + d_{\rho\varphi}(z, y)).$$

Thus $d_{\rho\varphi} : X \times X \to \mathbb{R}_0^+$ is a k-pseudometric. In case ρ is a metric, $d_{\rho\varphi}$ obviously satisfies (1Mk) and hence is a k-metric.

In particular, one can take $\varphi(x) = x^p$ where $p > 1$. Moreover, as noticed in [20, Remark 1.6], since $(x+y)^p \leq \frac{x^p+y^p}{2}$ for a given pseudometric $d : X \times X \to \mathbb{R}_0^+$ the function $d^p : X \times X \to \mathbb{R}_0^+$ is a 2^{p-1}-pseudometric.

Example 2.5. Given Lebesgue measurable functions $f, g : [a, b] \to \mathbb{R}^+$ define

$$d(f, g) = \int_a^b \mid f(x) - g(x) \mid^2 dx.$$

In this way we get a 2-pseudometric on the set $L([a, b])$ of all Lebesgue measurable functions on $[a, b]$. This 2-pseudometric was mentioned in [13].

3. Fuzzy k-Pseudometric Spaces: Definitions and Examples

Let $k \geq 1$ be a fixed constant. A fuzzy k-pseudometric on a set X is defined by taking the first three and the last axioms as they are in Definition 2.1, but changing axiom (4FM) in order to reflect the role of the constant k.

Definition 3.1. ([20]) A fuzzy k-pseudometric on a set X is a pair $(m, *)$ where $*$ is a t-norm and $m : X \times X \times \mathbb{R}^+ \to (0, 1]$ is a mapping satisfying the following conditions for all $x, y, z \in X$, $s, t \in \mathbb{R}^+$:
(1FMk) $m(x, y, t) > 0$;
(2'FMk) $m(x, y, t) = 1 \Longrightarrow x = y$;
(3FMk) $m(x, y, t) = m(y, x, t)$;
(4FMk) $m(x, z, k(t + s)) \geq m(x, y, t) * m(y, z, s)$;
(5FMk) $m(x, y, -) : \mathbb{R}^+ \to (0, 1]$ is continuous.
The triple $(X, m, *)$ is a called *a fuzzy k-pseudometric space*. If axiom (2'FMk) is replaced by a stronger axiom
(2FMk) $x = y \Longleftrightarrow m(x, y, t) = 1$
we get definitions of a fuzzy k-metric, and a fuzzy k-metric space. If axiom (4FMk) is replaced by axiom (4FSMk)
(4FSMk) $m(x, z, kt) \geq m(x, y, t) * m(y, z, t)$;
then m is called a strong fuzzy k-(pseudo)metric.

Remark 3.2. As different from "ordinary" fuzzy (pseudo)metrics, fuzzy k-(pseudo)metrics need not be non-decreasing on the third argument: we can show only that $m(x,y,t) \leq m(x,y,t')$ whenever $t' \geq kt$. Indeed, by taking $s = \frac{t'}{k} - t$ we have $m(x,y,t) = m(x,y,t) * m(y,y,s) \leq m(x,y,k(t+s)) = m(x,y,t')$. Note also that to get the inequality $m(x,y,t) \leq m(x,y,t')$ in case of a strong fuzzy k-pseudometric the condition $t' \geq kt$ is not sufficient; instead we have to request $t' \geq 2kt$.

Remark 3.3. Let $*_1, *_2$ be continuous t-norms and $\alpha *_1 \beta \leq \alpha *_2 \beta$ for all $\alpha, \beta \in [0,1]$. If $(m, *_2)$ is a fuzzy k-(pseudo)metric on a set X, then $(m, *_1)$ is a fuzzy k-(pseudo)metric on X, too. In particular, if $(m, \wedge)$ is a fuzzy k-(pseudo)metric, then $(m, *)$ is a fuzzy k-(pseudo)metric for every continuous t-norm $*$.

We proceed with examples of fuzzy k-pseudometrics.

Example 3.4. Patterned after construction of the standard fuzzy pseudometric induced by a pseudometric in [10], we present here an analogous construction of a fuzzy k-pseudometric from a k-pseudometric. Let $d : X \times X \to \mathbb{R}_0^+$ be a k-pseudometric. Then the mapping $m_d : X \times X \times \mathbb{R}^+ \to (0,1]$ defined by $m_d(x,y,t) = \frac{t}{t+d(x,y)}$ is a fuzzy k-pseudometric for the minimum t-norm and hence for any other continuous t-norm.

The validity of axioms (1FMk), (2'FMk), (3FMk) and (5FMk) is obvious. Hence, to prove this statement, we have to verify (4FMk), that is to show that

$$\frac{t}{t+d(x,y)} \wedge \frac{s}{s+d(y,z)} \leq \frac{k(t+s)}{k(t+s)+d(x,z)} \quad \forall x,y,z \in X \text{ and } \forall s,t > 0.$$

Since d is a k-pseudometric and hence $d(x,z) \leq k(d(x,y) + d(y,z))$, we replace the inequality to be proved by a stronger inequality

$$\frac{t}{t+d(x,y)} \wedge \frac{s}{s+d(x,y)} \leq \frac{k(t+s)}{k(t+s)+k(d(x,y)+d(y,z))} = \frac{t+s}{(t+s)+(d(x,y)+d(y,z))}.$$

Without loss of generality we assume that $\frac{t}{t+d(x,y)} \leq \frac{s}{s+d(y,z)}$, and therefore we have to show that

$$\frac{t}{t+d(x,y)} \leq \frac{t+s}{t+s+(d(x,y)+d(y,z))}$$

We prove this inequality straightforward just by noticing that, as it follows from our assumption, $t \cdot (y,z) \leq s \cdot d(x,y)$.

In case $*$ is the product t-norm, m_d is also a strong k-metric. Indeed, the requested inequality $m_d(x,y,t) \cdot m_d(y,z,t) \leq k \cdot m_d(x,z,t)$ follows from the inequality

$$\frac{t}{t+a} \cdot \frac{t}{t+b} \leq k \cdot \frac{t}{t+k(a+b)}$$

which obviously holds for any k> 1, $a, b \in [0,\infty)$ and $t > 0$.

Example 3.5. Noticing that $(s+t)^n \geq s^n + t^n$ for all positive integers $n \in \mathbb{N}$ and all $t, s > 0$, we generalize the previous example as follows (cf example 2.9 in [10]):

Let $d : X \times X \to [0,\infty)$ be a k-metric and let $\alpha, \beta \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Then by setting

$$m_d(x,y,t) = \frac{\alpha t^n}{\alpha t^n + \beta d(x,y)},$$

we obtain a fuzzy k-metric for the minimum t-norm and hence for any other continuous t-norm.

Example 3.6. Here we describe a scheme allowing to construct different fuzzy k-pseudometrics, in particular, fuzzy k-pseudometrics with certain prescribed properties.

In case $*$ is the product t-norm this example can be found in [20]

Let ρ be a pseudometric and let a function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy properties requested in Example 2.4 with some $k > 1$. We define a mapping $m_{\rho\varphi} : X \times X \times \mathbb{R}^+ \to (0, 1]$ by setting

$$m_{\rho\varphi}(x, y, t) = \frac{t}{t + (\varphi \circ \rho)(x, y)} \text{ for all } x, y \in X.$$

Referring to construction described in Example 2.4, we know that $\varphi \circ \rho$ is a k-pseudometric on the set X, and further, referring to Example 3.4 we conclude that $m_{\rho\varphi} : X \times X \times \mathbb{R}^+ \to (0, 1]$ is a fuzzy k-pseudometric for the minimum t-norm and hence for any continuous t-norm. In case of the product t-norm, it is also a strong fuzzy k-pseudometric.

To have a concrete example of a standard fuzzy k-pseudometric which fails to be a fuzzy pseudometric, we can take any one of the k-pseudometrics $d_{\rho\varphi}$ constructed on a pseudometric space (X, ρ) with the appropriate choice of a function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and a constant k.

Example 3.7. Let $(X, \|\cdot\|)$ be a Banach space. Then from the inequality $\|a + b\|^2 \leq 2 \cdot (\|a\|^2 + \|b\|^2)$ we conclude that by setting $d(x, y) = \|x - y\|^2$ a 2-metric is obtained that fails to be a metric. We define the mapping $m_{\|\|^2} : X \times X \times \mathbb{R}^+ \to [0, 1]$ by setting

$$m_{\|\|}(x, y, t) = e^{\left(-\frac{\|x-y\|^2}{t}\right)} \ \forall x, y \in X, t > 0$$

and show that it is a fuzzy 2-metric for the product t-norm $* = \cdot$ and hence also for every weaker t-norm, in particular, for the Łukasiewicz t-norm.

The validity of axioms (1FMk), (2FMk), (3FMk) and (5FMk) is obvious. Hence, to prove this statement, we have to verify (4FMk). From the obvious inequality

$$\|x - z\|^2 \leq 2 \cdot \left(\tfrac{t+s}{t}\right) \|x - y\|^2 + 2 \cdot \left(\tfrac{t+s}{s}\right) \|y - z\|^2 \forall x, y, z \in X, s, t > 0,$$

we obtain

$$\tfrac{\|x-z\|^2}{2(t+s)} \leq \tfrac{\|x-y\|^2}{t} + \tfrac{\|y-z\|^2}{s} \forall x, y, z \in X, s, t > 0.$$

From this inequality we get $e^{-\frac{\|x-z\|^2}{2(t+s)}} \geq e^{-\frac{\|x-y\|^2}{t}} \cdot e^{-\frac{\|y-z\|^2}{s}}$ and hence

$$m_{\|\cdot\|^2}(x, z, 2(t + s)) \geq m_{\|\cdot\|^2}(x, y, t) \cdot m_{\|\cdot\|^2}(y, z, t).$$

Example 3.8. Applying the construction from Example 3.4 to the 2-pseudometric defined in Example 2.9 we get a fuzzy k-pseudometric on the set of all Lebesgue measurable functions on the interval $[a, b]$:

$$m_d(f, g) = \frac{t}{t + \int_a^b (f(x) - g(x))^2 dx}$$

Example 3.9. Here we present a construction allowing to obtain a new strong fuzzy k-metrics from a given one (cf similar example in case of strong fuzzy metrics, [19]).

Let $m : X \times X \times \mathbb{R}^+ \to (0, 1]$ be a strong fuzzy k-metric for the product t-norm. Then the mapping $n : X \times X \times \mathbb{R}^+ \to [0, 1]$ defined by

$$n(x, y, t) = \frac{t + m(x, y, t)}{t + 1} \ \forall x, y \in X, \ \forall t > 0$$

is a strong fuzzy k-metric, too. Since the validity of axioms (1FMk), (2FMk), (3FMk), and (5FMk), for $n : X \times X \to [0, 1]$ are ensured by the corresponding axioms for $m : X \times X \times \mathbb{R}^+ \to (0, 1]$, we have to establish only axiom (4FMk), that is the inequality

$$\frac{t + m(x, y, t)}{t + 1} \cdot \frac{t + m(y, z, t)}{t + 1} \leq \frac{kt + m(x, z, kt)}{kt + 1}.$$

It will follow from the stronger inequality

$$\frac{t+m(x,y,t)}{t+1}\cdot\frac{t+m(y,z,t)}{t+1}\leq\frac{t+m(x,y,kt)}{t+1},$$

which in its turn can be reduced to the inequality

$$t\cdot m(x,y,t)+t\cdot m(y,z,t)+m(x,y,t)\cdot m(y,z,t)\leq t+t\cdot m(x,z,kt)+m(x,z,kt).$$

The last inequality can be easily established recalling that $m(x,y,t)\cdot m(y,z,t)\leq m(x,z,kt)$ by axiom (4FSMk) and noticing that $m(x,y,t)+m(y,z,t)\leq 1+m(x,y,t)\cdot m(y,z,t)$.

4. Topological Structure of a Fuzzy k-Pseudometric Space

4.1. Topological Structures Induced by a k-Pseudometric

Although we are mainly interested in the topological structure in fuzzy k-pseudometric spaces, we start with consideration of topological structure induced by ordinary k-pseudometrics. Actually, the properties of topologies of fuzzy k-pseudometric spaces and the topologies of ordinary k-pseudometric spaces have much in common. In particular, the topology of a k-pseudometric space (X,d) and the topology of the standard fuzzy k-pseudometric space (X,m_d) coincide. Recall a similar situation in case of a pseudometric and the corresponding standard fuzzy pseudometric.

Let (X,d) be a k-pseudometric space. In accordance with the standard terminology in the theory of metric spaces we define the open ball with center $a\in X$ and radius $r>0$ in (X,d) by $B_d(a,\varepsilon)=\{x\in X: d(a,x)<\varepsilon\}$. Let $\mathcal{B}_d=\{B_d(a,\varepsilon): a\in X,\varepsilon>0\}$ be the family of all non-empty balls in (X,d). Taking open balls as the basis for topological considerations, we have the following two "natural" options.

1. **Supratopology $\mathcal{S}_d$**

   Let $\mathcal{S}_d$ be the family of all unions of open balls, that is

   $$\mathcal{S}_d=\{U\subseteq X:\exists B_d(a_i,\varepsilon_i), i\in I \text{ such that } U=\bigcup_{i\in I}B_d(a_i,\varepsilon_i)\}.$$

   The family $\mathcal{S}_d$ is obviously a suprartopology (see e.g. [17]), that is $\emptyset\in\mathcal{S}_d$ (as the union of an empty family of balls), $X\in\mathcal{S}_d$ and the union of any sets from $\mathcal{S}_d$ remains in $\mathcal{S}_d$. We call $\mathcal{S}_d$ the supratopology induced by the k-pseudometric d on X. $\mathcal{S}_d$ need not be a topology, since the intersection of even two elements $U_1,U_2\in\mathcal{S}_d$ need not be in $\mathcal{S}_d$. The problem is that, as the example below shows, given a ball $B_d(a,r)$ and a point $x\in B_d(a,r)$, we cannot guarantee that there exists a ball with the center x and with a radius $\varepsilon>0$ such that $B_d(x,\varepsilon)\subseteq B_d(a,r)$.

   Example 4.1. Let $s>0$ be a fixed constant and let $X=\{a\}\cup[b,c]$ where a is a point and $[b,c]$ is a closed interval of length s. Further, given $t\in(0,s)$, let d_t be a point in $[b,c]$ such that $d_t-b=t$ (and hence $c-d_t=s-t$). We introduce a 2-metric d on the set X as follows. The distance on the set $[b,c]$ is the usual Euclidean metric. We define the distance between a and other points as $d(a,b)=s$, $d(a,c)=2s$, $d(a,d_t)=2s-t$. The mapping $d: X\times X\to[0,\infty)$ thus defined is a 2-metric. Indeed,

   - $d(a,b)=s<d(a,d_t)+d(d_t,b)=2s;$
   - $d(b,d_t)\leq d(b,a)+d(a,d_t);$
   - $d(a,d_t)=2s-t\leq 2(d(d_t,b)+d(b,a))=2(s+t);$
   - $d(a,c)=2s\leq 2(d(a,d_t)+d(d_t,c))=2(2s-t+s-t)=2(3s-2t);$
   - $d(d_t,c)\leq d(a,c)+d(a,d_t);$
   - $d(a,c)=2s\leq 2(d(b,c)+d(a,b))=2s;$
   - $d(a,b)\leq d(a,c)+d(c,b);\ d(c,b)\leq d(a,b)+d(c,a).$

Now, let $\varepsilon > 0, \varepsilon < \frac{s}{2}$. Then $B_d(a, s+\varepsilon) = \{a, b\} \cup (d_{s-\varepsilon}, c\}$ while for any $\delta > 0$ $B_d(b, \delta) = (b, d_\delta) \subset [b, c]$ and hence $B_d(b, \delta) \not\subseteq B_d(a, s+\varepsilon)$.

2. **Topology $\mathcal{T}_d$**

   We call a set $U \subseteq X$ $\mathcal{T}_d$-open if for every $x \in X$ there exists $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq U$. One can easily notice that $U_1, U_2 \in \mathcal{T}_d \implies U_1 \cap U_2 \in \mathcal{T}_d$ and the union of any family of $\mathcal{T}_d$-open sets is $\mathcal{T}_d$-open. Thus $\mathcal{T}_d$ is indeed a topology on X. Obviously each $U \in \mathcal{T}_d$ can be expressed as a union of some open balls and for this reason it belongs to $\mathcal{S}_d$. Thus, $\mathcal{T}_d \subseteq \mathcal{S}_d$. On the other hand not every open ball $B(a, \varepsilon)$ needs to be $\mathcal{T}_d$-open (see Example 4.1) and hence generally $\mathcal{S}_d \neq \mathcal{T}_d$.

Theorem 4.2. *Let $\rho : X \times X \to \mathbb{R}^+_0$ be a pseudometric and $\varphi : [0, \infty) \to [0, \infty)$ be a function satisfying conditions in Example 2.4. and let $d_{\rho\varphi} = \varphi \circ \rho : X \times X \to \mathbb{R}^+_0$. Then the families of balls $\mathcal{B}_\rho$ and $\mathcal{B}_{d_{\rho\varphi}}$ generated by pseudometric ρ and k-pseudometric $d_{\rho\varphi}$ coincide.*

The proof follows from the next series of implications:
$x \in B_\rho(a, r) \Leftrightarrow \rho(a, x) < r \Leftrightarrow (\varphi \circ \rho)(a, x) < \varphi(r) \Leftrightarrow x \in B_{d_{\rho\varphi}}(a, \varphi(r))$;
$x \in B_{d_{\rho\varphi}}(a, \varphi(r)) \Leftrightarrow d_{\rho\varphi}(a, x) < \varphi(r) \Leftrightarrow (\varphi \circ \rho)(a, x) < \varphi(r) \Leftrightarrow x \in B_\rho(a, r)$.

Corollary 4.3. *Let ρ be a pseudometric on a set X and $d_{\rho\varphi}$ be the k-pseudometric constructed from ρ as above. Then $T_\rho = \mathcal{T}_{d_{\rho\varphi}} = \mathcal{S}_{d_{\rho\varphi}} = \mathcal{S}_\rho$.*

Remark 4.4. In case of a general k-pseudometric d on a set X we can prove only that if $y \in B_d(a, r)$ *and* $d(a, y) < \frac{r}{k}$, then there exists $\varepsilon > 0$ for which $B_d(y, \varepsilon) \subseteq B_d(a, r)$. One of the obstacles preventing to extend such statements for points which are "further" than $\frac{r}{k}$ from the point a is non-transitivity of the triangle equality for k-pseudometrics, see Remark 2.3.

4.2. Topological Structures Induced by a Fuzzy k-Pseudometric.

Let $(X, m, *)$ be a fuzzy k-pseudometric space. Patterned after the definition of an open ball in a fuzzy metric space [10, 11], we define an open ball with the center at a point $a \in X$, radius $r > 0$ and at the level $t > 0$ as

$$B(a, r, t) = \{x \in X : m(a, x, t) > 1 - r\}.$$

Let $\mathcal{B}_m = \{B(a, r, t) : a \in X, r \in (0, 1), t \in \mathbb{R}^+\}$ be the family of all open balls. As in case of k-pseudometrics, we can use the family $\mathcal{B}_m$ to define two topological structures on the set X:

1. **Supratopology $\mathcal{S}_m$**

   Let $\mathcal{S}_m$ be the family of all unions of open balls, that is

   $$\mathcal{S}_m = \{U \subseteq X : \exists B_m(a_i, r_i, t_i), i \in I \text{ such that } U = \bigcup_{i \in I} B_m(a_i, r_i, t_i)\}.$$

   The family $\mathcal{S}_m$ is obviously a suprartopology. As in the case of $\mathcal{S}_d$, it may fail to be a topology. We refer to the elements of $\mathcal{S}_m$ as $\mathcal{S}_m$-open sets in the fuzzy k-pseudometric space (X, m).

   A Digression: Some Remarks on Supratopologies

   Since the concept of a supratopology is important for our research, we shall give here a brief information about supratopologies. As far as we know, this concept for the first time is considered in the paper [17] and it was originally provoked by the need to study the families of generalized (in Levin's sense [16]) closed and open sets in a topological space. In a supertopological space, one naturally defines closed sets as complements of open, the closure cl$A$ of a set A as the intersection of all closed sets containing it, its interior A^0 as the union of all its open subsets, and all other topological concepts are verbatim transfered to the case of supratopology. However, one must be cautious: while some of the properties are equivalent to their topological prototypes, others can essentially differ. For us it will be important also that a supratopology of a space X can be defined by the closure operator $cl : 2^X \to 2^X$ satisfying the following properties:

(s1) $cl\emptyset = \emptyset$;
(s2) $A \subset clA$ for every $A \subseteq X$
(s3) $cl(A) \cup cl(B) \subseteq cl(A \cup B)$ for all $A, B \in 2^X$.
In case (s3) is replaced by a stronger axiom
(s3′) $cl(A) \cup cl(B) \subseteq cl(A \cup B)$ for all $A, B \in 2^X$
we come to the notion of a pretopological space. In such spaces a certain "rudiment" of the finite intersection axiom can be perceived. Finally, if we add idempotency axiom to the list (s1) - (s3) (in this case the axioms (s3) and (s3′) are equivalent)
(s4) $(cl(cl(A)) = A$ for every $A \subseteq X$.
we come to the classical concept of a topology. The fuzzy counterpart of supratopology first appeared, as far as we know, in [1].

2. **Topology $\mathcal{T}_m$** We call a set $U \subseteq X$ $\mathcal{T}_m$-open, if for every $x \in U$ there exist $r, t > 0$ such that $B_m(x, r, t) \subseteq U$. One can easily notice that the intersection of two $\mathcal{T}_m$-open sets is $\mathcal{T}_m$-open and the union of any family of $\mathcal{T}_m$-open sets is open and hence $\mathcal{T}_m$ is indeed a topology.
By the same arguments as in Subsection 2 we see that $\mathcal{T}_m \subseteq \mathcal{S}_m$, but generally $\mathcal{T}_m \neq \mathcal{S}_m$.

Remark 4.5. In [20] the author assumes that every "open" ball in a (fuzzy) k-metric space is open and hence the family of all open balls make a base for a topology. However, as we have seen, it is not always the case. This is the reason why we distinguish between the supratopology and topology induced by (fuzzy) k-pseudometrics and consequently, our results "argue" with the corresponding results in [20].

The counterpart of the following result for (ordinary) fuzzy pseudometric spaces is well-known [10, Remark 3.6].

Proposition 4.6. *Let $d : X \times X \to \mathbb{R}_0^+$ be a k-pseudometric and let $m_d : X \times X \times \mathbb{R}^+ \to (0, 1]$ be the standard fuzzy k-pseudometric induced by d, see Example 3.6. Then the families of balls in the k-pseudometric space (X, d) and the fuzzy k-pseudometric space (X, m_d) coincide: $\mathcal{B}_d = \mathcal{B}_{m_d}$.*

Proof. Let $B_{m_d}(a, r, t)$ be an open ball with center $a \in X$, radius $r > 0$ at the level $t \in (0, \infty)$. Then $x \in B_{m_d}(a, r, t) \iff m_d(a, x, t) > 1 - r \iff \frac{t}{t+d(a,x)} \iff \iff d(a, x) < \frac{tr}{1-r} =_{def} \varepsilon \iff x \in B_d(a, \varepsilon)$. Thus, every open ball $B_{m_d}(a, r, t)$ in a fuzzy k-pseudometric space (X, m_d) is also an open ball $B_d(a, \varepsilon)$ in the k-pseudometric space (X, d). Conversely, from the above series of implications it follows that each open ball $B_d(a, \varepsilon)$ coincides with the open ball $B_{m_d}(a, r, t)$ where $r = \frac{\varepsilon}{\varepsilon+t}$. However, this means that $\mathcal{B}_d = \mathcal{B}_{m_d}$ and hence $\mathcal{T}_d = \mathcal{T}_{m_d}$. □

Corollary 4.7. *Corresponding supratopologies and topologies induced by d and m_d, respectively, coincide: $\mathcal{S}_d = \mathcal{S}_{m_d}$; $\mathcal{T}_d = \mathcal{T}_{m_d}$.*

Theorem 4.8. *Supratopology $\mathcal{S}_m$ induced by a fuzzy k-metric m is Hausdorff.*

Let $m : X \times X \times \mathbb{R}^+ \to (0, 1]$ be a fuzzy k-metric, $x, y \in X, x \neq y$ and $m(x, y, t) = r$. Take some $r_0 \in (r, 1)$, and referring to continuity of the t-norm $*$, take $s \in [r_0, 1]$ such that $s * s = r_0$. To complete the proof we show that

$$B_m\left(x, 1 - s, \tfrac{t}{2k}\right) \bigcap B_m\left(y, 1 - s, \tfrac{t}{2k}\right) = \emptyset.$$

Indeed, assume that $z \in B_m\left(x, 1 - s, \frac{t}{2k}\right) \cap B_m\left(y, 1 - s, \frac{t}{2k}\right)$. Then $m\left(x, z, \frac{t}{2k}\right) > s,\ m\left(y, z, \frac{t}{2k}\right) > s$ and hence $m\left(x, z, \frac{t}{2k}\right) * m\left(z, y, \frac{t}{2k}\right) \geq s * s = r_0 > r$. On the other hand, from axiom (4FMk), we have

$$m\left(x, z, \tfrac{t}{2k}\right) * m\left(z, y, \tfrac{t}{2k}\right) \leq m\left(x, y, k\left(\tfrac{t}{2k} + \tfrac{t}{2k}\right)\right) = m(x, y, t) = r.$$

The obtained contradiction completes the proof.

Obviously, supratopologies induced by arbitrary fuzzy k-*pseudo*metrics need not be Hausdorff.

Remark 4.9. The topology $\mathcal{T}_m$ induced by a fuzzy k-metric need not be Hausdorff: one cannot guarantee that for any two points $x_1 \neq x_2$ there exist balls $B(x_1, r_1, t_1)$, $B(x_2, r_2, t_2)$ such that $B(x_1, r_1, t_1) \cap B(x_2, r_2, t_2) = \emptyset$.

The above considerations were partly inspired by the preprint [4]

4.3. Subsets of a Fuzzy k-Pseudometric Space

In this section, (X, m) is a k-pseudometric space and $(X, \mathcal{S}_m)$ and $(X, \mathcal{T}_m)$ the corresponding supratopological and topological spaces. We discuss and compare properties of compactness, boundedness and density in these spaces.

4.3.1. Compactness

Although the structures $\mathcal{S}_m$ and $\mathcal{T}_m$ are constructed from the same "bricks" - open balls, the compactness in the both structures may differ, since in the case of $\mathcal{S}_m$ all open balls may be used to constitute a cover, while in $\mathcal{T}_m$ there should be only those open balls whose unions form open sets. Hence the compactness of a set $A \subseteq X$ in $\mathcal{S}_m$ guarantees its compactness in $\mathcal{T}_m$. However, we do not know whether the converse is true.

4.3.2. Boundedness

As different from compactness, boundedness is a metric-type property, so its definition does not depend upon in which one of the structures $\mathcal{S}_m$ or $\mathcal{T}_m$ we are working.

Definition 4.10. A set $A \subseteq X$ is called D-bounded if there exist $t > 0$ and $r \in (0, 1)$ such that $m(x, y, t) > 1 - r$ for all $x, y \in A$. A set $A \subseteq X$ is called D-bounded on a level t or D_t-bounded if there exists $r \in (0, 1)$ such that $m(x, y, t) > 1 - r$ for all $x, y \in A$. A set $A \subseteq X$ is called strongly D-bounded if it is D-bounded on every level $t > 0$.

Theorem 4.11. *An $\mathcal{S}_m$-compact subset of a k-pseudometric space is strongly D-bounded.*

Proof. Let A be an $\mathcal{S}_m$-compact subset of a fuzzy k-pseudometric space (X, m). Fix some $t > 0$, $r \in (0, 1)$ and consider a cover $\{B_m(z, r, t) : z \in A\}$ of the set A. By compactness of A we can find a finite family of points $Z = \{z_1, \ldots, z_n\} \subseteq A$ such that $A \subseteq \bigcup_{i=1}^{n} B_m(z_i, r, t)$. Let

$$l = \min\{m(z_i, z_j, t) : z_i, z_j \in Z\}$$

Take any $x, y \in X$. Then there exist $z_p, z_q \in Z$ such that $x \in B_m(z_p, r, t)$ and $y \in B_m(z_q, r, t)$. Then $m(z_p, z_q, t) \geq l$ and hence

$$m(x, z_p, t) * m(z_p, z_q, t) * m(z_q, y, t) \geq (1 - r) * l * (1 - r).$$

Since $(1 - r) * l * (1 - r) < 1$, we can choose $s \in (0, 1)$ such that $(1 - r) * l * (1 - r) > 1 - s$ and hence $m(x, z_p, t) * m(z_p, z_q, t) * m(z_q, y, t) > 1 - s$. On the other hand,

$$m(x, z_p, t) * m(z_p, z_q, t) * m(z_q, y, t) \leq m(x, z_q, 2kt) * m(z_q, y, t) \leq m(x, y, k(2k + 1)t).$$

Thus $m(x, y, k(2k + 1)t) > 1 - s$ and hence A is k(2k+1)t bounded. Since $t \in (0, \infty)$ can be chosen arbitrary, we conclude that A is strongly D-bounded. □

Remark 4.12. We considered two types of boundedness for subsets of fuzzy metric spaces in [22]. Patterned after [22], we call a set A in a fuzzy k-metric space (X, m) B_t-bounded if there exist $a \in X$, $r \in (0, 1)$ such that $A \subseteq B_m(a, r, t)$. A set $A \subseteq X$ is called B-bounded if there exist $a \in X$, $r \in (0, 1)$ and $t > 0$ such that A is B_t-bounded. A set $A \subseteq X$ is called strongly B-bounded if it is B_t-bounded on every level t.

A D_t-bounded set is B_t-bounded. Indeed, let $A \subseteq X$ be D_t-bounded, then there exists $r > 0$ such that $m(x, y, t) > 1 - r$ for all $x, y \in X$. Let $x_0 \in A$, then $m(x_0, x, t) > 1 - r$ for every $x \in A$, that is $A \subseteq B_m(x_0, r, t)$.

Conversely, a B_t-bounded set is D_{2kt}-bounded. Indeed, let A be B_t bounded, then $A \subseteq B_m(x_0, r, t)$ for some $x_0 \in A$ and $r > 0$. Then $m(x_0, x, t) > 1 - r$ for every $x \in A$. Given two points $x, y \in A$, we have

$$m(x, y, 2kt) \geq m(x_0, x, t) * m(x_0, y, t) \geq (1 - r) * (1 - r) =_{def} 1 - s.$$

and hence A is D_{2kt}-bounded.

We can summarize the obtained results as follows:

Prefix D comes as an abbreviation for *diameter-type*
The prefix B comes as the abbreviation of *Ball-type*

Corollary 4.13. *A set A in a fuzzy k-pseudometric space is B-bounded if and only if it is D-bounded. A set A in a fuzzy k-pseudometric space is strongly B-bounded if and only if it is strongly D-bounded.*

4.3.3. Dense subsets

Concerning density, we again have to distinguish two cases.

Definition 4.14. A subset A of a fuzzy k-pseudometric space (X, m) is called $\mathcal{S}_m$-dense or dense in $\mathcal{S}_m$ if each open ball B(a,r,t) has a nonempty intersection with A:

$$A \cap B(a, r, t) \neq \emptyset \ \forall a \in X, \forall r \in (0, 1), \ \forall t > 0.$$

A subset A is called τ-dense if it is dense in topology $\mathcal{T}_m$

From a well-known fact of general topology we conclude that the union of two subsets is $\mathcal{T}_m$-dense if and only if at least one of them is $\mathcal{T}_m$-dense. Obviously, the union of two $\mathcal{S}_m$-dense subsets is $\mathcal{S}_m$-dense. However, it is not clear, whether the union of two non-σ-dense subsets can become σ-dense.

5. Sequences in Fuzzy k-Metric Spaces

5.1. Two Types of Convergence in Fuzzy k-Pseudometric Spaces

A certain similarity in the definitions of a fuzzy pseudometric and a fuzzy k-pseudometric space arises interest in the role of sequences in the context of fuzzy k-pseudometric spaces. In this section we will touch this problem.

Let (X, m) be a fuzzy k-pseudometrics space. Given a sequence $(x_n)_{n\in\mathbb{N}}$ and a point $x_0 \in X$, we say that $(x_n)_{n\in\mathbb{N}}$ σ-converges to x_0 and write $\lim^{\sigma}_{n\to\infty} x_n = x_0$ if $(x_n)_{n\in\mathbb{N}}$ converges in the supratopology $\mathcal{S}_m$: that is for every open ball $B(x_0, r, t)$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in B(x_0, r, t)$ for all $n \geq n_0$. We say that $x_0 \in X$ is a σ-accumulation point of a sequence $(x_n)_{n\in\mathbb{N}}$ if it is its accumulation point in $\mathcal{S}_m$: that is if each ball $B(x_0, r, t)$ contains infinitely many members of this sequence.

Given a sequence $(x_n)_{n\in\mathbb{N}}$ and a point $x_0 \in X$, we say that $(x_n)_{n\in\mathbb{N}}$ τ-converges to x_0 and write $\lim^{\tau}_{n\to\infty} x_n = x_0$ if it converges in $\mathcal{T}_m$: that is for every open set U containing x_0 there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$. We say that $x_0 \in X$ is a τ-accumulation point of a sequence $(x_n)_{n\in\mathbb{N}}$ if it is accumulation point in $\mathcal{T}_m$: that is each open neighborhood U of x_0 contains infinitely many members of this sequence.

From the definitions one can easily get the following

Theorem 5.1. *If* $\lim^{\sigma}_{n\to\infty} x_n = x_0$, *then* $\lim^{\tau}_{n\to\infty} x_n = x_0$*:*

Remark 5.2. Similar, as in case of topological structures induced by (fuzzy)-k-pseudometrics, see 4.5, we have to distinguish between two different types of convergence of sequences. And this is the reason why some of our results argue with the analogous statements in [20]

Theorem 5.3. *If x_0 is a σ-accumulation point of a sequence $(x_n)_{n\in\mathbb{N}}$, then it is also its τ-accumulation point.*

Theorem 5.4. *Let (X, m) be a fuzzy k-pseudometric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in this space. Then* $\lim^{\sigma}_{n\to\infty} x_n = a$ *if and only if* $\lim_{n\to\infty} m(a, x_n, t) = 1$ *for each* $t \in (0, \infty)$.

Proof. Assume that $\lim^{\sigma}_{n\to\infty} x_n = a$ and take an open ball $B(a, r, t)$l. Then we can choose $n_0 \in \mathbb{N}$ such that $x_n \in B(a, r, t)$ for all $n \geq n_0$, and hence $m(a, x_n, t) > 1 - r$ for all $n \geq n_0$. Since r and t were taken arbitrary, we conclude that $\lim_{n\to\infty} m(a, x_n, t) = 1$ for every $t > 0$

Assume now that $\lim^{\sigma}_{n\to\infty} x_n \neq a$. Then there exists a ball $B(a, r, t)$ such that $x_n \notin B(a, r, t)$ for infinitely many $n \in \mathbb{N}$. However, this means that $m(a, x_n, t) \leq 1 - r$ for infinitely many $n \in \mathbb{N}$, and hence either $\lim_{n\to\infty} m(a, x_n, t) \neq 1$, or $\lim_{n\to\infty} m(a, x_n, t)$ does not exist. □

From here and applying Theorem 5.1 we get

Corollary 5.5. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a fuzzy k-metric space (X, m). If* $\lim_{n\to\infty} m(a, x_n, t) = 1$ *for a point* $a \in X$, *then* $\lim^{\sigma}_{n\to\infty} x_n = \lim^{\tau}_{n\to\infty} x_n = a$.

5.2. Sequentiality Properties of Fuzzy k-Metric Spaces

Recall that a topological space (X, T) is called sequential if its subset A is closed whenever it contains the limits of all convergent sequence in this subset, see [7], [8], [9],. It is well-known and easy to prove, that each metric space is sequential. In a natural way we extend the concept of sequentially to the case of supratopological spaces.

Theorem 5.6. *Let (X, m) be a fuzzy k-pseudometric space. Then the induced supratopology $\mathcal{S}_m$ is sequential.*

Proof. Assume that A is a subset of the space (X, m) which is not closed. Then its complement is not open and hence there exists a point $a \in X \setminus A$ such that for every $t > 0$ and every $r \in (0, 1)$ it holds $B(a, r, t) \cap A \neq \emptyset$. We fix t and for every $n \in \mathbb{N}$ choose a point $x_n \in B(a, \frac{1}{n}, t) \cap A$. From the construction it is clear that $\{x_n : n \in \mathbb{N}\} \subseteq A$ and $\lim^{\sigma}_{n\to\infty} x_n = a \notin A$. □

We guess that the topology $\mathcal{T}_m$ need not be sequential. Unfortunately, at present we do not have corresponding examples.

5.3. Closed Balls and Diffusion of Limits of Convergent Sequences

By a closed ball with center $a \in X$, radius $r \in (0, 1)$ at the level $t \in (0, \infty)$ we call the set $\bar{B}(a, r, t) = \{x : m(a, x, t) \geq 1 - r\}$. The proof of the following proposition is obvious:

Proposition 5.7. *Let (X, m) be a fuzzy k-pseudometric space, $a \in X$ and $0 < r' < r < 1$. Then for every $t > 0$ it holds $\bar{B}(a, r', t) \subseteq B(a, r, t)$.*

Unfortunately, a closed ball need not be closed neither in supratopology $\mathcal{S}_m$ nor in topology $\mathcal{T}_m$. We can prove only the following related to closedness property of a closed ball:

Proposition 5.8. *Let $\bar{B}(a, r, t)$ be a closed ball in a fuzzy k-pseudometric space. Then for every $b \notin B(a, r, t)$ there exists $\varepsilon > 0$ such that*

$$\bar{B}\left(a, r, \frac{t}{2k}\right) \cap \bar{B}\left(b, \varepsilon, \frac{t}{2k}\right) = \emptyset.$$

Proof. Let $b \notin B(a, r, t)$, then $m(a, b, t) = s < 1 - r$. Since the t-norm $*$ is continuous, there exists $\varepsilon > 0$ such that $s < (1 - r) * (1 - \varepsilon)$. We claim that $\bar{B}\left(a, r, \frac{t}{2k}\right) \cap \bar{B}\left(a, \varepsilon, \frac{t}{2k}\right) = \emptyset$.. Indeed, assume that there exists $x \in \bar{B}\left(a, r, \frac{t}{2k}\right) \cap \bar{B}\left(a, \varepsilon, \frac{t}{2k}\right)$. Then $m\left(a, x, \frac{t}{2k}\right) \geq 1 - r$ and $m\left(b, x, \frac{t}{2k}\right) \geq 1 - \varepsilon$. From here we get

$$m(a, b, t) \geq m\left(a, x, \tfrac{t}{2k}\right) * m\left(b, x, \tfrac{t}{2k}\right) \geq (1 - r) * (1 - \varepsilon) > s.$$

The obtained contradiction completes the proof. □

Theorem 5.9. *Let (X, m) be a fuzzy k-pseudometric space and $\bar{B}(a, r, t)$ be a closed ball. Further, let $(x_n)_{n\in\mathbb{N}} \subset \bar{B}(a, r, t)$ be a σ-convergence sequence. Then $\lim^{\sigma}_{n\to\infty} x_n \in \bar{B}(a, r, 2kt)$. In other words, closed ball $\bar{B}(a, r, 2kt)$ contains the σ-limits of all σ-convergent sequences from the ball $\bar{B}(a, r, t)$*

Proof. Take any σ-convergent sequence $(x_n)_{n\in\mathbb{N}}$ contained in $\bar{B}(a, r, t)$ and assume that $\lim^{\sigma}_{n\to\infty} x_n = b \notin \bar{B}(a, r, 2kt)$. Then from Proposition 5.8 it follows that there exists $\varepsilon > 0$ such that $\bar{B}(a, r, t) \cap \bar{B}(a, \varepsilon, t) = \emptyset$. Since $\lim^{\sigma}_{n\to\infty} x_n = b$, we can find n_0 such that $x_n \in \bar{B}(b, \varepsilon, t)$ for all $n \geq n_0$. However this contradicts the assumption that $(x_n)_{n\in\mathbb{N}}$ is contained in $\bar{B}(a, r, t)$. □

6. Completeness of Fuzzy k-Pseudometric Spaces

6.1. Cauchy Sequences in Fuzzy k-Pseudometric Spaces

Definition 6.1. A sequence $(x_n)_{n\in\mathbb{N}}$ in a fuzzy k-pseudometric space (X, m) is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $m(x_n, x_k, t) > 1 - \varepsilon$ for all $n, k \geq n_0$.

Proposition 6.2. *If a sequence $(x_n)_{n\in\mathbb{N}}$ in a fuzzy k-pseudometric space (X, m) σ-converges, then it is Cauchy.*

Proof. Take $t > 0$ and $\varepsilon > 0$. Let $t' = \frac{t}{2k}$ and, by continuity of the t-norm, find $\delta > 0$ such that $(1-\delta)*(1-\delta) \geq 1-\varepsilon$. Now, let a sequence $(x_n)_{n\in\mathbb{N}}$ be σ-convergent and let $\lim_{n\to+\infty} = a$. Applying Theorem 5.4, we can find $n_0 \in \mathbb{N}$ such that $m(a, x_n, t') > 1 - \delta$ for all $n \geq n_0$. In the result, we have

$$m(x_n, x_p, t) \geq m(a, x_n, t') * m(a, x_p, t') \geq (1-\delta) * (1-\delta) > 1 - \varepsilon,$$

and hence the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. □

Definition 6.3. A fuzzy k-pseudometric space is called complete if each its Cauchy sequence σ-converges.

6.2. Fuzzy k-Pseudometric Version of a Baire Theorem

Impossibility to use the intersection axiom for open sets in case of the supratopology $\mathcal{S}_m$ on one hand, and the probable "non-openness" of open balls in case of topology $\mathcal{T}_m$ make it doubtful to get a full-bodied version of the Baire Category theory neither in $\mathcal{S}_m$ nor $\mathcal{T}_m$. To overcome this obstacle, we introduce the concept of a valuably open set and with its help get a certain restricted version of Baire category theorem.

Definition 6.4. Given a fuzzy k-pseudometric space (X, m), an open set U of the space $(X, \mathcal{S}_m)$ will be called *valuably open* if for every ball $B(x_0, r, t)$ having non-empty intersection with U there exists a ball $B(x_1, r_1, t_1) \subseteq B(x_0, r, t) \cap U$ for some $x_1 \in B(x_0, r, t), r_1 \in (0, 1)$ and $t_1 > 0$.

Theorem 6.5. [Baire theorem for fuzzy k-pseudometrics] *Let (X, m) be a complete fuzzy k-pseudometric space. Then the intersection of a countable family of σ-dense valuably open sets in the corresponding supratopological space $(X, \mathcal{S}_m)$ is open.*

Proof Let (X, m) be a complete fuzzy k-pseudometric space and $D_1 \supseteq D_2 \supseteq D_3 \supseteq \ldots D_n \ldots$ be a sequence of valuably open σ-dense subsets of this space. Further, let $U \subseteq X$ be an open subset of X. We have to prove that $U \cap (\bigcap_n D_n) \neq \emptyset$. Referring to Proposition 5.7 we can choose an open ball $B_0 = B(x_0, r_0, t_0)$ such that $\bar{B}(x_0, r_0, t_0) \subseteq U$. Without loss of generality we may assume that $r_0 \leq 1, t_0 \leq 1$. Let $t'_0 = \frac{t_0}{2k}$ and let $B'_0 = B(x_0, r_0, t'_0)$.

Since the set D_1 is σ-dense, $D_1 \cap B'_0 \neq \emptyset$, and since D_1 is valuably open, we can find $B(x_1, r_1, t_1) =_{def} B_1$ such that $\bar{B}(x_1, r_1, t_1) \subseteq D_1 \cap B_0$. Without loss of generality we assume that $r_1 \leq \frac{1}{2},\ t_1 \leq \frac{1}{2}$. Let now $t'_1 = \frac{t_1}{k}$ and $B'_1 = B(x_1, r_1, t'_1)$

Since the set D_2 is σ-dense, $D_2 \cap B'_1 \neq \emptyset$, and D_2 is valuably open, we can find $B(x_2, r_2, t_2) =_{def} B_2$ such that $\bar{B}(x_2, r_2, t_2) \subseteq D_2 \cap B'_1$. Without loss of generality we assume that $r_2 \leq \frac{1}{3},\ t_2 \leq \frac{1}{3}$. Let now $t'_2 = \frac{t_2}{2k}$ and $B'_2 = B(x_2, r_2, t'_2)$

We continue such procedure by induction and in the result obtain a sequence of points $x_0, x_1, x_2, \ldots, x_n, \ldots$ and two sequences of open balls

$$B_0(x_0, r_0, t_0) \supseteq B_1(x_1, r_1, t_1) \supseteq B_2(x_2, r_2, t_2) \supseteq \ldots \supseteq B_n(x_n, r_n, t_n) \ldots,$$

$$B_0(x_0, r_0, t'_0) \supseteq B_1(x_1, r_1, t'_1) \supseteq B_2(x_2, r_2, t'_2) \supseteq \ldots \supseteq B_n(x_n, r_n, t'_n) \ldots,$$

where $r_n \leq \frac{1}{n+1}, t_n \leq \frac{1}{n+1}$.

We claim that the constructed sequence $x_0, x_1, x_2, \ldots, x_n, \ldots$ is Cauchy. Indeed let $t > 0$ and $\varepsilon > 0$ be given. First, by continuity of the t-norm, find $\delta \in (0, 1)$ such that $(1-\delta)*(1-\delta) \geq 1-\varepsilon$. Further, find $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < t'$ and $\frac{1}{n_0} < \delta$. Then for $n, k \geq n_0$ we have

$$m(x_n, x_k, t) \geq m(x_{n_0}, x_n, t) * m(x_{n_0}, x_k, t) \geq (1-\delta) * (1-\delta) \geq 1 - \varepsilon \quad \forall n, k \geq n_0.$$

Thus the sequence $x_0, x_1, x_2, \ldots, x_n, \ldots$ is Cauchy. Since the fuzzy k-pseudometric space (X, m) is complete this sequence σ-coverges.

Let $\lim^{\sigma}_{n\to\infty} x_n = a$. Take some $n \in \mathbb{N}^+$. Since all elements x_k of this sequence for $k \geq n$ are contained in the closed ball $\bar{B}_n = \bar{B}(x_n, r_n, t'_n)$, referring to Theorem 5.9, we conclude that the limit is contained in $\bar{B}(x_{n-1}, r_{n-1}, t_{n-1}) \cap D_{n-1}$ for all $n \geq 1$. Hence $U \cap (\bigcap_n D_n) \neq \emptyset$, that is $\bigcap_n D_n$ is σ-dense in X. □

7. Conclusion

In this paper, we considered some topological and sequential properties of k-pseudometric and fuzzy k-pseudometric spaces. Two structures in such spaces were defined: a supratopology $\mathcal{S}_m$ and a topology $\mathcal{T}_m$. As our results show, the supratopology $\mathcal{S}_m$ and the corresponding σ-convergence much better reflect the properties of the fuzzy k-pseudometric m than the topology $\mathcal{T}_m$ and τ-convergence, see e.g. theorems 4.8, 5.4, 5.6, 6.2. So we assume that in future research when dealing with topological and sequential structure of a fuzzy k-metric space, one should work in the framework of the supratopology and σ-convergence. Below we indicate some directions where in our opinion, the work should be done.

1. A chalenging problem for the future research is to find sone necessary and/or sufficient conditions for (fuzzy) k-pseudometrics for which the "open" balls are indeed open and hence the supratopology and the topology coincide.
2. As an interesting and important direction for the further research, we anticipate the study of fuzzy k-pseudometric spaces and its continuous mappings as a category. Are there non-trivial relations between the category of fuzzy k-pseudometric spaces on one side and the categories of fuzzy metric saces and of k-pseudometric spaces on the other?
3. It would be interesting to find a criteria for a supratopology or topology, which can be generated by a (fuzzy) k-pseudometric.
4. We expect the study of fixed point property for mappings of fuzzy k-pseudometric spaces in future.

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