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Existence of Solutions for a Class of Variational-Hemivariational-Like Inequalities in Banach Spaces

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Abstract. This paper is devoted to study the existence of solutions for a class of variational-hemivariational-like inequalities in reflexive Banach spaces. Using the notion of the stable (ϕ, η)-quasimonotonicity, the properties of Clarke's generalized directional derivative and Clarke's generalized gradient, we establish some existence results of solutions when the constrained set is nonempty, bounded (or unbounded), closed and convex. Moreover, a sufficient condition to the boundedness of the solution set and a necessary and sufficient condition to the existence of solutions are also derived.

1. Introduction

In the early 1980s, Panagiotopulos introduced and studied the hemivariational inequalities as variational expressions for several classes of mechanical problems with nonsmooth and nonconvex energy superpotentials; see e.g., [27–29] and the references therein. The derivative of hemivariational inequality is base on the mathematical notion of the generalized gradient of Clarke ([11]). The hemivariational inequalities appear in a variety of mechanical problems, for example, the unilateral contact problems in nonlinear elasticity, the problems describing the adhesive and frictional effects, the nonconvex semipermeability problems, the masonry structures, and the delamination problems in multilayered composites ([5–7, 28, 29]). Carl [1], Carl et al. [2, 3] and Xiao and Huang [35] studied the existence of solutions of some kinds of hemivariational inequalities using the method of sub-super solutions. Migorski and Ochal [25], and Park and Ha [31, 32] studied the problem using the regularized approximating method. Goelevan et al. [16] and Liu [22] proved the existence of solutions using the method of the first eigenfunction. For more related works regarding the existence of solutions for hemivariational inequalities, we refer to [4, 9, 10, 13, 21, 26, 28–30, 36, 38–45] and the references therein.

Let K be a nonempty, closed and convex subset of a real reflexive Banach space X. Let $\eta : X \times X \to X$ be a mapping, $F : K \to X^*$ be a nonlinear operator and $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper and lower semicontinuous function such that $K_\phi := K \cap \mathrm{dom}\phi \neq \emptyset$, where $\mathrm{dom}\phi := \{x \in X : \phi(x) < +\infty\}$ is the effective domain of ϕ.

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Let Ω be a bounded open set in $\mathbf{R}^N$ and $j : \Omega \times \mathbf{R}^k \to \mathbf{R}$ be a function. Let $T : X \to L^p(\Omega; \mathbf{R}^k)$ be a linear and continuous mapping, where $1 < p < \infty$. We shall denote $\hat{u} := Tu$ and denote by $j^\circ(x, y; h)$ Clarke's generalized directional derivative of a locally Lipschitz mapping $j(x, \cdot)$ at the point $y \in \mathbf{R}^k$ with respect to the direction $h \in \mathbf{R}^k$, where $x \in \Omega$.

In 2000, Motreanu and Radulescu [26] introduced and studied the following variational-hemivariational inequality problem: find $u \in K$ such that

$$\langle F(u), v - u\rangle + \phi(v) - \phi(u) + \int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \ \forall v \in K. \tag{1}$$

Using Mosco's theorem, they proved that problem (1) admits a solution if the operator F is monotone and hemicontinuous (see [26, Theorem 2]).

In 2010, applying KKM theorem, Costea and Lupu [13] extended the result above from the case of single-valued to that of set-valued. If $F : K \to 2^{X^*}$ is a set-valued mapping, then problem (1) reduces to the following variational-hemivariational inequality problem:

Find $u \in K_\phi$ and $u^* \in F(u)$ such that

$$\text{Problem } \mathbf{(P)}: \quad \langle u^*, v - u\rangle + \phi(v) - \phi(u) + \int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \ \forall v \in K.$$

To weaken the hypotheses of monotonicity, Costea and Radulescu [12] investigated a special case of problem (1) with $\phi = I_K$, where I_K is the indicator function over the set K, i.e., $I_K(x) = 0$ if $x \in K$ and $I_K(x) = +\infty$ otherwise. The problem is formulated as finding $u \in K$ such that

$$\langle F(u), v - u\rangle + \int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \ \forall v \in K. \tag{2}$$

Using the notion of stable pseudomonotonicity introduced by He [17, 18] and KKM theorem, the authors obtained some existence results for problem (2).

In 2011, by introducing the notion of stable quasimonotonicity, Zhang and He [46] considered the following hemivariational inequality: find $u \in K$ and $u^* \in F(u)$ such that

$$\langle u^*, v - u\rangle + \int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \ \forall v \in K. \tag{3}$$

It is clear that problems (1)-(3) are special cases of problem (P). In brief, for a suitable choice of F, ϕ and T one can obtain a wide class of inequality problems, including mixed variational inequality and Stampacchia variational inequality. This shows that problem (P) is quite general and unifying in the same time.

Very recently, Tang and Huang [34] investigated the existence of solutions for problem (P) in reflexive Banach spaces. Using the notion of the stable ϕ-quasimonotonicity, the properties of Clarke's generalized directional derivative and Clarke's generalized gradient, they established some existence results of solutions when the constrained set is nonempty, bounded (or unbounded), closed and convex. Also, they gave a sufficient condition to the boundedness of the solution set and a necessary and sufficient condition to the existence of solutions.

Moreover, many interesting problems in mechanics and applied mathematics lead to other types of convex functionals ϕ other than the indicator function I_K over the set K ([3]). Marano and Papageorgiou [24] considered an elliptic variational-hemivariational inequality with $\phi(x) = \int_\Omega G(x, v(x))dx$, where $\xi \mapsto G(x, \xi)$, $\xi \in \mathbf{R}$ is nonnegative, proper, convex and lower semicontinuous for almost every $x \in \Omega$. As pointed out in Liu and Motreanu [19], problem (P) at resonance as well as nonresonance has a striking theoretic interest and a strong motivation due to its applications in mechanics and engineering. Recently, problem (P) has been studied by many authors, see e.g., [2, 5, 15, 21, 23, 26] and the references therein.

Now let us recall some important techniques for mixed variational(-like) inequality problems. Let H be a real Hilbert space. In 2007, Schaible, Yao and Zeng [33] extended the auxiliary principle technique to develop an iterative algorithm for mixed variational-like inequality problem (MVLIP) in H by using the following hypotheses on the mapping $\eta : H \times H \to H$ involved by the (MVLIP):

(a) η is Lipschitz continuous with constant $\lambda > 0$, i.e., $\|\eta(u, v)\| \leq \lambda\|u - v\|, \ \forall u, v \in H$;
(b) $\eta(u, v) + \eta(v, u) = 0, \ \forall u, v \in H$;
(c) $\eta(u, v) = \eta(u, w) + \eta(w, v), \ \forall u, v, w \in H$;
(d) η is affine in the first variable.

Furthermore, by employing the notion of ϕ-pseudomonotonicity of the operator, Zhong and Huang [47] studied the stability of Minty mixed variational inequality. Indeed, various kinds of generalized monotonicity of the operator play an important role in the theory of variational inequalities. For example, we can refer to [8, 12, 33, 37] for more details.

Motivated and inspired by the research work mentioned above, in this paper, we investigate the existence of solutions for a class of variational-hemivariational-like inequality problems in reflexive Banach spaces, that is, the following variational-hemivariational-like inequality problem:

(VHVLIP) Find $u \in K_\phi$ and $u^* \in F(u)$ such that

$$\langle u^*, \eta(v, u)\rangle + \phi(v) - \phi(u) + \int_\Omega j^\circ(x, \hat{u}(x); (T\eta(v, u))(x))dx \geq 0, \ \forall v \in K.$$

Using the notion of the stable (ϕ, η)-quasimonotonicity, the properties of Clarke's generalized directional derivative and Clarke's generalized gradient, some existence results of solutions are proved when the constrained set is a nonempty, bounded (or unbounded), closed and convex set. Moreover, a sufficient condition to the boundedness of the set of solutions and a necessary and sufficient condition to the existence of solutions are also derived. The results presented in this paper generalize and improve some known results.

The rest of the paper is organized in the following way. In the next section, we recall some definitions and necessary materials. In Sect. 3, we recall and introduce some kinds of generalized η-monotonicity of the operator. Moreover, we discuss the relations of these generalized η-monotonicity in details. Section 4 is devoted to proving our main results. We show the existence of solutions in the case when the constraint set K is bounded and unbounded in Theorems 3.2 and 3.5, respectively. Theorem 3.7 provides a sufficient condition to the boundedness of the solution set. Theorem 3.8 gives a necessary and sufficient condition to the existence of solutions of (VHVLIP).

2. Preliminaries

Let X be a real reflexive Banach space with the norm denoted by $\|\cdot\|$ and X^* be its dual space. Let u_0 and $\{u_n\}$ be a point and a sequence in X, respectively. We use the notations $u_n \to u_0$ and $u_n \rightharpoonup u_0$ to indicate the strong convergence of $\{u_n\}$ to u_0 and the weak convergence of $\{u_n\}$ to u_0, respectively. Moreover, we denote by $\langle\cdot, \cdot\rangle$ the duality pairing between X and its dual X^*. For a nonempty, closed and convex subset K of X and every $r > 0$, we define

$$B_r := \{u \in K : \|u\| \leq r\}.$$

Let $\eta : X \times X \to X$ be a mapping. η is said to be skew if $\eta(u, v) + \eta(v, u) = 0$ for all $u, v \in X$.

Let $T : X \to L^p(\Omega; \mathbf{R}^k)$ be a linear compact operator, where $1 < p < \infty$ and $k \geq 1$, and Ω be a bounded open set in $\mathbf{R}^N$. Denote by q the conjugated exponent of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Let $j : \Omega \times \mathbf{R}^k \to \mathbf{R}$ be a function where the mapping

$$j(\cdot, y) : \Omega \to \mathbf{R} \text{ is measurable, for every } y \in \mathbf{R}^k. \tag{4}$$

We assume that at least one of the following conditions holds: either there exists $l \in L^q(\Omega; \mathbf{R})$ such that

$$|j(x, y_1) - j(x, y_2)| \leq l(x)|y_1 - y_2|, \ \forall x \in \Omega, \forall y_1, y_2 \in \mathbf{R}^k, \tag{5}$$

or

$$\text{the mapping } j(x, \cdot) \text{ is locally Lipschitz, } \forall x \in \Omega, \tag{6}$$

and there exists $C > 0$ such that

$$|z| \leq C(1 + |y|^{p-1}), \ \forall x \in \Omega, \forall z \in \partial j(x, y). \tag{7}$$

Recall that $f^\circ(x; v)$ denotes Clarke's generalized directional derivative of the locally Lipschitz function $f : X \to \mathbf{R}$ at the point $x \in X$ with respect to the direction $v \in X$, while $\partial f(x)$ is the Clarke's generalized gradient of f at $x \in X$ ([11]), i.e.,

$$f^\circ(x; v) = \limsup_{y \to x, t \to 0^+} \frac{f(y + tv) - f(y)}{t}$$

and

$$\partial f(x) = \{\xi \in X^* : \langle \xi, v \rangle \leq f^\circ(x; v), \ \forall v \in X\}.$$

Let $J : L^p(\Omega; \mathbf{R}^k) \to \mathbf{R}$ be an arbitrary locally Lipschitz functional. For each $u \in X$ there exists (see e.g., [11]) $z_u \in \partial J(\hat{u})$ such that

$$J^\circ(\hat{u}; \xi) = \langle z_u, \xi \rangle = \max\{\langle w, \xi \rangle : w \in \partial J(\hat{u})\}. \tag{8}$$

Denoting by $T^* : L^q(\Omega; \mathbf{R}^k) \to X^*$ the adjoint operator of T, we define the subset $U(J, T)$ of X^* as follows:

$$U(J, T) = \{-z_u^* : u \in K, \ z_u^* = T^* z_u\}. \tag{9}$$

It follows from (8) that $J^\circ(\hat{u}; \hat{v}) = \langle z_u^*, v \rangle$.

Lemma 2.1. *([11, Proposition 2.1.1]) Let $f : X \to \mathbf{R}$ be Lipschitz of rank M near x. Then*

(i) the function $v \mapsto f^\circ(x; v)$ is finite, positively homogeneous and subadditive on X, and satisfies

$$|f^\circ(x; v)| \leq M \|v\|;$$

(ii) $f^\circ(x; v)$ is upper semicontinuous as a function of (x, v) and, as a function of v alone, is Lipschitz of rank M on X;

(iii) $f^\circ(x; -v) = (-f)^\circ(x; v)$.

Lemma 2.2. *([11, Proposition 2.7.5]) If $J(\varphi) = \int_\Omega j(x, \varphi(x))dx$, and j satisfies the conditions* (4) *and* (5) *or* (4) *and* (6)-(7), *then J is uniformly Lipschitz on bounded subsets, and one has*

$$\partial J(\varphi) \subset \int_\Omega \partial j(x, \varphi(x))dx.$$

Further, if j is regular at $(x, \varphi(x))$ then J is regular at φ and equality holds.

Lemma 2.3. *([14]) Let K be a nonempty subset of a Hausdorff topological vector space E and let $G : K \to 2^E$ be a set-valued mapping satisfying the following properties:*

(i) G is a KKM mapping;

(ii) $G(x)$ is closed in E for every $x \in K$;

(iii) $G(x_0)$ is compact in E for some $x_0 \in K$.

Then $\bigcap_{x \in K} G(x) \neq \emptyset$.

Definition 2.4. *The mapping $F : K \to 2^{X^*}$ is said to be*

(i) lower semicontinuous at u_0 if, for any $u_0^* \in F(u_0)$ and sequence $\{u_n\} \subset K$ with $u_n \to u_0$, a sequence $u_n^* \in F(u_n)$ can be determined which converges to u_0^*;

(ii) *lower hemicontinuous if, the restriction of F to every line segment of K is lower semicontinuous with respect to the weak topology in X^*.*

Remark 2.5. *If F is lower hemicontinuous, then it is known that for any $u, v \in K$ and $u^* \in F(u)$, a sequence $u_n^* \in F(u_n)$ can be determined which converges weakly to u^*, where $u_n := u + \frac{1}{n}(v - u)$ for all $n \geq 1$.*

Definition 2.6. *Let $\eta : X \times X \to X$ be a mapping and $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function such that $K_\phi \neq \emptyset$. The mapping $F : K \to 2^{X^*}$ is said to be*

(i) *η-monotone if, for each pair of points $u, v \in K$,*

$$\langle v^* - u^*, \eta(v, u)\rangle \geq 0, \ \forall u^* \in F(u) \text{ and } v^* \in F(v);$$

(ii) *η-pseudomonotone if, for each pair of points $u, v \in K$,*

$$\langle u^*, \eta(v, u)\rangle \geq 0 \ \Rightarrow \ \langle v^*, \eta(v, u)\rangle \geq 0, \ \forall u^* \in F(u) \text{ and } v^* \in F(v);$$

(iii) *η-quasimonotone if, for each pair of points $u, v \in K$,*

$$\langle u^*, \eta(v, u)\rangle > 0 \ \Rightarrow \ \langle v^*, \eta(v, u)\rangle \geq 0, \ \forall u^* \in F(u) \text{ and } v^* \in F(v);$$

(iv) *stably η-pseudomonotone with respect to the set $U \subset X^*$ if, F and $F(\cdot) - \xi$ are η-pseudomonotone for every $\xi \in U$;*

(v) *stably η-quasimonotone with respect to the set $U \subset X^*$ if, F and $F(\cdot) - \xi$ are η-quasimonotone for every $\xi \in U$;*

(vi) *(ϕ, η)-pseudomonotone if, for each pair of points $u, v \in K$,*

$$\langle u^*, \eta(v, u)\rangle + \phi(v) - \phi(u) \geq 0 \ \Rightarrow \ \langle v^*, \eta(v, u)\rangle + \phi(v) - \phi(u) \geq 0, \ \forall u^* \in F(u) \text{ and } v^* \in F(v);$$

(vii) *(ϕ, η)-quasimonotone if, for each pair of points $u, v \in K$,*

$$\langle u^*, \eta(v, u)\rangle + \phi(v) - \phi(u) > 0 \ \Rightarrow \ \langle v^*, \eta(v, u)\rangle + \phi(v) - \phi(u) \geq 0, \ \forall u^* \in F(u) \text{ and } v^* \in F(v);$$

(viii) *stably (ϕ, η)-pseudomonotone with respect to the set $U \subset X^*$ if, F and $F(\cdot) - \xi$ are (ϕ, η)-pseudomonotone for every $\xi \in U$;*

(ix) *stably (ϕ, η)-quasimonotone with respect to the set $U \subset X^*$ if, F and $F(\cdot) - \xi$ are (ϕ, η)-quasimonotone for every $\xi \in U$.*

3. Existence theorems

Let $J : L^p(\Omega; \mathbf{R}^k) \to \mathbf{R}$ be the function $J(\varphi) = \int_\Omega j(x, \varphi(x))dx$, and $T : X \to L^p(\Omega; \mathbf{R}^k)$ be a linear compact operator, where $1 < p < \infty$, $k \geq 1$ and Ω is a bounded open set in $\mathbf{R}^N$.

Definition 3.1. *The Clarke's generalized directional derivative $J^\circ(\cdot; \cdot)$ is said to be bi-sequentially weakly upper semicontinuous w.r.t. T if, for any $\{u_n\}$ and $\{v_n\}$ in X with $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$, one has $\limsup_{n\to\infty} J^\circ(\hat{u}_n; \hat{v}_n) \leq J^\circ(\hat{u}; \hat{v})$.*

Theorem 3.2. *Let $\eta : X \times X \to X$ be a skew mapping that is affine in the first variable and weakly continuous in the second variable. Let the Clarke's generalized directional derivative $J^\circ(\cdot; \cdot)$ be bi-sequentially weakly upper semicontinuous w.r.t. linear compact operator T. Assume that K is a nonempty, closed, bounded and convex subset of X. Let $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function such that $K_\phi \neq \emptyset$. Let $F : K \to 2^{X^*}$ be a lower hemicontinuous set-valued mapping and stably (ϕ, η)-quasimonotone w.r.t. the set $U(J, T)$, where $U(J, T)$ is defined as (9). Further, we suppose j satisfies the conditions (4) and (5) or (4) and (6)-(7). Then the (VHVLIP) admits at least one solution.*

Proof. For any $v \in K_\phi$ define a set-valued mapping $G : K_\phi \to 2^X$ as follows:

$$G(v) := \{u \in K_\phi : \inf_{v^* \in F(v)} \langle v^*, \eta(v,u)\rangle + \phi(v) - \phi(u) + J^\circ(\hat{u}; T\eta(v,u)) \geq 0\}.$$

Consider two cases regarding G:

(i) G is not a KKM mapping, and
(ii) G is a KKM mapping.

(i) If G is not a KKM mapping, then, by the definition of KKM mapping, there exist $u_i \in K_\phi$ and $\lambda_i \in [0,1]$, $i = 1,2,...,n$ with $\sum_{i=1}^n \lambda_i = 1$ and $u_0 := \sum_{i=1}^n \lambda_i u_i \in \mathrm{co}\{u_1,...,u_n\}$ such that $u_0 \notin \bigcup_{i=1}^n G(u_i)$, that is,

$$\inf_{u_i^* \in F(u_i)} \langle u_i^*, \eta(u_i,u_0)\rangle + \phi(u_i) - \phi(u_0) + J^\circ(\hat{u}_0; T\eta(u_i,u_0)) < 0, \ \forall i \in \{1,2,...,n\}. \tag{10}$$

In addition, from $u_i \in K_\phi$, $i = 1,2,...,n$ and the convexity of ϕ, it follows that $\phi(u_0) \leq \sum_{i=1}^n \lambda_i \phi(u_i) < \infty$, which leads to $u_0 \in K_\phi$.

We claim that there exists a neighborhood U of u_0 such that for all $v \in U \cap K_\phi$,

$$\inf_{u_i^* \in F(u_i)} \langle u_i^*, \eta(u_i,v)\rangle + \phi(u_i) - \phi(v) + J^\circ(\hat{v}; T\eta(u_i,v)) < 0, \ \forall i \in \{1,2,...,n\}. \tag{11}$$

If not, for any neighborhood U of u_0, there exist $v_0 \in U \cap K_\phi$ and $i_0 \in \{1,2,...,n\}$ such that

$$\inf_{u_{i_0}^* \in F(u_{i_0})} \langle u_{i_0}^*, \eta(u_{i_0},v_0)\rangle + \phi(u_{i_0}) - \phi(v_0) + J^\circ(\hat{v}_0; T\eta(u_{i_0},v_0)) \geq 0.$$

Taking $U = B(u_0, \frac{1}{n})$, we obtain that there exists $v_n \in B(u_0, \frac{1}{n}) \cap K_\phi$ such that

$$\inf_{u_{i_0}^* \in F(u_{i_0})} \langle u_{i_0}^*, \eta(u_{i_0},v_n)\rangle + \phi(u_{i_0}) - \phi(v_n) + J^\circ(\hat{v}_n; T\eta(u_{i_0},v_n)) \geq 0.$$

By the bi-sequentially weakly upper semicontinuity of $J^\circ(\cdot,\cdot)$ w.r.t. operator T, the weak continuity of η in the second variable, $v_n \to u_0$ and the lower semicontinuity of ϕ, we have

$$\inf_{u_{i_0}^* \in F(u_{i_0})} \langle u_{i_0}^*, \eta(u_{i_0},u_0)\rangle + \phi(u_{i_0}) - \phi(u_0) + J^\circ(\hat{u}_0; T\eta(u_{i_0},u_0)) \geq 0.$$

This contradicts (10) and so the claim holds.

From (11), there exists a neighborhood U of u_0 such that for all $v \in U \cap K_\phi$, for all i

$$\inf_{u_i^* \in F(u_i)} \langle u_i^*, \eta(u_i,v)\rangle + \phi(u_i) - \phi(v) + J^\circ(\hat{v}; T\eta(u_i,v)) = \inf_{u_i^* \in F(u_i)} \langle u_i^*, \eta(u_i,v)\rangle + \phi(u_i) - \phi(v) + \langle z_v^*, \eta(u_i,v)\rangle < 0,$$

which can be rewritten as

$$\inf_{u_i^* \in F(u_i)} \langle u_i^* - (-z_v^*), \eta(u_i,v)\rangle + \phi(u_i) - \phi(v) < 0.$$

Using the stable (ϕ,η)-quasimonotonicity of F w.r.t. the set $U(J,T)$ and the skew property of η, we get that

$$\sup_{v^* \in F(v)} \langle v^* - (-z_v^*), \eta(u_i,v)\rangle + \phi(u_i) - \phi(v) \leq 0, \ \forall i \in \{1,2,...,n\}.$$

It is equivalent to

$$\sup_{v^* \in F(v)} \langle v^*, \eta(u_i,v)\rangle + \phi(u_i) - \phi(v) + J^\circ(\hat{v}; T\eta(u_i,v)) \leq 0, \ \forall i \in \{1,2,...,n\}.$$

By (i) of Lemma 2.1, the affinity of η in the first variable and the convexity of ϕ, taking into account that $u_0 = \sum_{i=1}^n \lambda_i u_i \in \mathrm{co}\{u_1, ..., u_n\}$ and T is a linear operator, we deduce that

$$\begin{aligned}
&\sup_{v^*\in F(v)} \langle v^*, \eta(u_0, v)\rangle + \phi(u_0) - \phi(v) + J^\circ(\hat{v}; T\eta(u_0, v)) \\
&= \sup_{v^*\in F(v)} \langle v^*, \eta(\sum_{i=1}^n \lambda_i u_i, v)\rangle + \phi(\sum_{i=1}^n \lambda_i u_i) - \phi(v) + J^\circ(\hat{v}; T\eta(\sum_{i=1}^n \lambda_i u_i, v)) \\
&\le \sup_{v^*\in F(v)} \langle v^*, \sum_{i=1}^n \lambda_i \eta(u_i, v)\rangle + \sum_{i=1}^n \lambda_i(\phi(u_i) - \phi(v)) + J^\circ(\hat{v}; \sum_{i=1}^n \lambda_i T\eta(u_i, v)) \\
&\le \sum_{i=1}^n \lambda_i [\sup_{v^*\in F(v)} \langle v^*, \eta(u_i, v)\rangle + \phi(u_i) - \phi(v) + J^\circ(\hat{v}; T\eta(u_i, v))] \\
&\le 0.
\end{aligned} \tag{12}$$

Again from (i) of Lemma 2.1 and the skew property of η and the linearity of T, it follows that

$$J^\circ(\hat{v}; T\eta(v, u_0)) + J^\circ(\hat{v}; T\eta(u_0, v)) \ge J^\circ(\hat{v}; T(\eta(v, u_0) + \eta(u_0, v))) = J^\circ(\hat{v}; 0) = 0,$$

which together with (12) and the skew property of η, yields

$$\inf_{v^*\in F(v)} \langle v^*, \eta(v, u_0)\rangle + \phi(v) - \phi(u_0) + J^\circ(\hat{v}; T\eta(v, u_0)) \ge 0, \ \forall v \in U \cap K_\phi. \tag{13}$$

Let $v' \in K_\phi$ be arbitrarily fixed and define $u_n = u_0 + \frac{1}{n}(v' - u_0)$. Then there exists $N \in \mathbf{N}$ such that for any $n \ge N$, we have $u_n \in U \cap K_\phi$. For any $u_0^* \in F(u_0)$, since F is lower hemicontinuous, a sequence $u_n^* \in F(u_n)$ can be determined which converges weakly star to u_0^*. It follows from (13) that for any $n \ge N$,

$$\langle u_n^*, \eta(u_n, u_0)\rangle + \phi(u_n) - \phi(u_0) + J^\circ(\hat{u}_n; T\eta(u_n, u_0)) \ge 0.$$

Taking into account that T is a linear operator and η is affine in the first variable with $\eta(u_0, u_0) = 0$, by (i) of Lemma 2.1 and the convexity of ϕ, we obtain

$$\begin{aligned}
0 &\le \langle u_n^*, \eta(u_0 + \frac{1}{n}(v' - u_0), u_0)\rangle + \phi(u_0 + \frac{1}{n}(v' - u_0)) - \phi(u_0) + J^\circ(\hat{u}_n; T\eta(u_0 + \frac{1}{n}(v' - u_0), u_0)) \\
&= \langle u_n^*, \frac{1}{n}\eta(v', u_0)\rangle + \phi(u_0 + \frac{1}{n}(v' - u_0)) - \phi(u_0) + J^\circ(\hat{u}_n; \frac{1}{n}T\eta(v', u_0)) \\
&\le \frac{1}{n}[\langle u_n^*, \eta(v', u_0)\rangle + \phi(v') - \phi(u_0) + J^\circ(\hat{u}_n; T\eta(v', u_0))].
\end{aligned}$$

Multiplying the inequality above by n and passing to the limit as $n \to \infty$, from (ii) of Lemma 2.1, we have

$$\langle u_0^*, \eta(v', u_0)\rangle + \phi(v') - \phi(u_0) + J^\circ(\hat{u}_0; T\eta(v', u_0)) \ge 0, \ \forall v' \in K_\phi. \tag{14}$$

Since $J(\varphi) = \int_\Omega j(x, \varphi(x))dx$ and j satisfies the conditions (4) and (5) or (4) and (6)-(7), by Lemma 2.2, we have

$$\int_\Omega j^\circ(x, \hat{u}(x); (T\eta(v, u))(x))dx \ge J^\circ(\hat{u}; T\eta(v, u)), \ \forall u, v \in X$$

and so

$$\langle u_0^*, \eta(v', u_0)\rangle + \phi(v') - \phi(u_0) + \int_\Omega j^\circ(x, \hat{u}_0(x); (T\eta(v', u_0))(x))dx \ge 0, \ \forall v' \in K_\phi. \tag{15}$$

If $v' \in K \setminus \mathrm{dom}\phi$, then $\phi(v') = +\infty$ and thus the inequality in (15) holds automatically. This together with (15) shows that $u_0 \in K_\phi$ is a solution of (VHVLIP).

(ii) G is a KKM mapping. Since ϕ is convex and lower semicontinuous, we know that it is weakly lower semicontinuous. For any $v \in K_\phi$, by the bi-sequentially weakly upper semicontinuity of $J^\circ(\cdot, \cdot)$ w.r.t. operator T, and the weak continuity of η in the second variable, we conclude that $u \mapsto \inf_{v^* \in F(v)} \langle v^*, \eta(v,u)\rangle + \phi(v) - \phi(u) + J^\circ(\hat{u}; T\eta(v,u))$ is weakly upper semicontinuous. Then $G(v)$ is weakly closed. It follows from the convexity and lower semicontinuity of ϕ that $\mathrm{dom}\phi$ is convex and closed. This, together with the fact that K is a bounded, closed and convex subset in reflexive Banach space, implies that $K_\phi = \mathrm{dom}\phi \cap K \neq \emptyset$ is weakly compact. Since $G(v) \subset K_\phi$, we obtain that $G(v)$ is weakly compact for each $v \in K_\phi$. Thus, all conditions of Lemma 2.3 are satisfied in the weak topology and so we have $\cap_{v \in K_\phi} G(v) \neq \emptyset$.

Taking $u_0 \in \cap_{v\in K_\phi} G(v)$, we get $u_0 \in G(v)$ for each $v \in K_\phi$. Thus,

$$\inf_{v^* \in F(v)} \langle v^*, \eta(v, u_0)\rangle + \phi(v) - \phi(u_0) + J^\circ(\hat{u}_0, T\eta(v, u_0)) \geq 0, \ \forall v \in K_\phi. \tag{16}$$

Let $v' \in K_\phi$ be arbitrarily fixed and define $u_n := u_0 + \frac{1}{n}(v' - u_0)$. Then it follows from the convexity of K_ϕ that $u_n \in K_\phi$ for all $n \in \mathbf{N}$. For any $u_0^* \in F(u_0)$, since F is lower hemicontinuous, a sequence $u_n^* \in F(u_n)$ can be determined which converges weakly star to u_0^*.

From (16), for any $n \in \mathbf{N}$, we have

$$\langle u_n^*, \eta(u_n, u_0)\rangle + \phi(u_n) - \phi(u_0) + J^\circ(\hat{u}_0, T\eta(u_n, u_0)) \geq 0.$$

Taking into account that T is a linear operator and ϕ is a convex function, by (i) of Lemma 2.1 and the affinity of η in the first variable with $\eta(u_0, u_0) = 0$, we obtain

$$\begin{aligned}
0 &\leq \langle u_n^*, \eta(u_0 + \frac{1}{n}(v' - u_0), u_0)\rangle + \phi(u_0 + \frac{1}{n}(v' - u_0)) - \phi(u_0) + J^\circ(\hat{u}_0; T\eta(u_0 + \frac{1}{n}(v' - u_0), u_0)) \\
&= \langle u_n^*, \frac{1}{n}\eta(v', u_0)\rangle + \phi(u_0 + \frac{1}{n}(v' - u_0)) - \phi(u_0) + J^\circ(\hat{u}_0; \frac{1}{n}T\eta(v', u_0)) \\
&\leq \frac{1}{n}[\langle u_n^*, \eta(v', u_0)\rangle + \phi(v') - \phi(u_0) + J^\circ(\hat{u}_0; T\eta(v', u_0))].
\end{aligned}$$

Multiplying the inequality above by n and passing to the limit as $n \to \infty$, we get

$$\langle u_0^*, \eta(v', u_0)\rangle + \phi(v') - \phi(u_0) + J^\circ(\hat{u}_0; T\eta(v', u_0)) \geq 0, \ \forall v' \in K_\phi. \tag{17}$$

Since $J(\varphi) = \int_\Omega j(x, \varphi(x))dx$ and j satisfies the conditions (4) and (5) or (4) and (6)-(7), by Lemma 2.2, we have

$$\int_\Omega j^\circ(x, \hat{u}(x); (T\eta(v,u))(x))dx \geq J^\circ(\hat{u}; T\eta(v,u)), \ \forall u, v \in X.$$

Combining with (17), we obtain

$$\langle u_0^*, \eta(v', u_0)\rangle + \phi(v') - \phi(u_0) + \int_\Omega j^\circ(x, \hat{u}_0(x); (T\eta(v', u_0))(x))dx \geq 0, \ \forall v' \in K_\phi. \tag{18}$$

If $v' \in K \setminus \mathrm{dom}\phi$, then $\phi(v') = +\infty$ and thus the inequality in (18) holds automatically. This together with (18) shows that $u_0 \in K_\phi$ is a solution of (VHVLIP). □

Remark 3.3. *Theorem 3.2 generalizes and improves some recent results. In fact,*

(i) Theorem 3.2 generalizes and extends Theorem 4.1 of Tang and Huang [34] from the variational-hemivariational inequality problem (VHVIP) to the variational-hemivariational-like inequality problem (VHVLIP);

(ii) Theorem 3.2 generalizes and extends Theorem 3.1 of Zhang and He [46] from the hemivariational inequality problem (HVIP) to the variational-hemivariational-like inequality problem (VHVLIP);

(iii) Compared with [13], the stable (ϕ, η)-quasimonotonicity of F in Theorem 3.2 is more general than the η-monotonicity of F (with $\eta(u,v) = u - v, \forall u, v \in X$) in Theorem 2 of [13];

(iv) Theorem 3.2 generalizes and extends Corollary 1 of Costea and Radulescu [12] from the hemivariational inequality problem (HVIP) to the variational-hemivariational-like inequality problem (VHVLIP) and by relaxing the stable η-pseudomonotonicity of F (with $\eta(u,v) = u - v, \forall u, v \in X$) in Corollary 1 of [12] to F being stably (ϕ, η)-quasimonotone;

(v) Theorem 3.2 generalizes and improves Theorem 2 of Motreanu and Radulescu [26] by extending F from the single-valued case to the set-valued one and by relaxing the monotonicity of F in Theorem 2 of [26] to F being stably (ϕ, η)-quasimonotone;

(vi) If $\eta(u,v) = u - v, \forall u, v \in X, \quad \phi = I_K$ and the mapping F is single-valued, then Theorem 3.2 generalizes and improves Theorem 2 of [30] by relaxing the monotonicity of F in [30] to F being stably (ϕ, η)-quasimonotone and by extending F from the single-valued case to the set-valued one.

Relaxing the constraint set K to the unbounded case, we need to introduce some notions of coercivity.

Proposition 3.4. *Let $\eta : X \times X \to X$ be a skew mapping. Consider the following coercivity conditions:*

(A) There exists a nonempty subset V_0 contained in a weakly compact subset V_1 of K_ϕ such that the set

$$D = \{u \in K_\phi : \inf_{v^* \in F(v)} \langle v^*, \eta(v,u)\rangle + \phi(v) - \phi(u) + J^\circ(\hat{u}; T\eta(v,u)) \geq 0, \ v \in V_0\}$$

is weakly compact or empty.

(B) There exists $n_0 \in \mathbf{N}$ such that for every $u \in K_\phi \setminus B_{n_0}$, there exists some $v \in K_\phi$ with $\|v\| < \|u\|$ such that

$$\sup_{u^* \in F(u)} \langle u^*, \eta(v,u)\rangle + \phi(v) - \phi(u) + J^\circ(\hat{u}; T\eta(v,u)) \leq 0.$$

(C) There exists $n_0 \in \mathbf{N}$ such that for every $u \in K_\phi \setminus B_{n_0}$, there exists some $v \in K_\phi$ with $\|v\| < \|u\|$ such that

$$\sup_{u^* \in F(u)} \langle u^*, \eta(v,u)\rangle + \phi(v) - \phi(u) + \int_\Omega j^\circ(x, \hat{u}(x); T\eta(v,u)(x)) < 0.$$

Then we have

(i) (A)⇒(B), if F is stably (ϕ, η)-quasimonotone with respect to the set $U(J, T)$.

(ii) (C)⇒(B), if $J(\varphi) = \int_\Omega j(x, \varphi(x))dx$, j satisfies the conditions (4) *and* (5) *or* (4) *and* (6)-(7).

Proof. (i) If $D = \emptyset$, since V_0 is nonempty and contained in a weakly compact subset V_1 of K_ϕ, then there exists a number number $M < \infty$ such that $\|z\| < M$ for all $z \in V_0$. Taking $n_0 = M$, we obtain that for every $u \in K_\phi \setminus B_{n_0}$, there exists some $v \in V_0 \neq \emptyset$ such that $v \in B_{n_0}$ and

$$\inf_{v^* \in F(v)} \langle v^*, \eta(v,u)\rangle + \phi(v) - \phi(u) + J^\circ(\hat{u}; T\eta(v,u)) < 0. \tag{19}$$

If $D \neq \emptyset$, then D is weakly compact. Since $D \cup V_0 \subset D \cup V_1$, which is a weakly compact subset, we conclude that there exists a number number $M < \infty$ such that $\|z\| < M$ for all $z \in D \cup V_0$. Taking $n_0 = M$, for every $u \in K_\phi \setminus B_{n_0}$, we deduce that $u \notin D$ and so there exists some $v \in V_0 \neq \emptyset$ such that $v \in B_{n_0}$ and (19) holds.

Hence, no matter the set D is empty or not, there exists $n_0 \in \mathbf{N}$ such that, for any $u \in K_\phi \setminus B_{n_0}$, there exists $v \in B_{n_0}$ such that (19) holds. Therefore,

$$\inf_{v^* \in F(v)} \langle v^*, \eta(v,u)\rangle + \phi(v) - \phi(u) + J^\circ(\hat{u}; T\eta(v,u)) = \inf_{v^* \in F(v)} \langle v^*, \eta(v,u)\rangle + \phi(v) - \phi(u) + \langle z_u^*, \eta(v,u)\rangle < 0, \tag{20}$$

which together with the skew property of η, implies that

$$\sup_{v^* \in F(v)} \langle v^* - (-z_u^*), \eta(u,v)\rangle + \phi(u) - \phi(v) > 0.$$

Using the stable (ϕ, η)-quasimonotonicity of F with respect to the set $U(J, T)$, we have

$$\inf_{u^* \in F(u)} \langle u^* - (-z_u^*), \eta(u, v)\rangle + \phi(u) - \phi(v) \geq 0,$$

which together with the skew property of η, implies that

$$\sup_{u^* \in F(u)} \langle u^*, \eta(v, u)\rangle + \phi(v) - \phi(u) + J^\circ(\hat{u}; T\eta(v, u)) \leq 0.$$

This verifies (B).

(ii) By Lemma 2.2, we have

$$\int_\Omega j^\circ(x, \hat{u}(x); T\eta(v, u)(x))dx \geq J^\circ(\hat{u}; T\eta(v, u)), \ \forall u, v \in X.$$

Combining with (C), we get (B). □

Theorem 3.5. *Let $\eta : X \times X \to X$ be a skew mapping that is affine in the first variable and weakly continuous in the second variable. Let the Clarke's generalized directional derivative $J^\circ(\cdot, \cdot)$ be bi-sequentially weakly upper semicontinuous w.r.t. linear compact operator T. Assume that K is a nonempty, closed, unbounded and convex subset of X and $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function such that $K_\phi \neq \emptyset$. Let $F : K \to 2^{X^*}$ be a lower hemicontinuous set-valued mapping and stably (ϕ, η)-quasimonotone w.r.t. the set $U(J, T)$, where $U(J, T)$ is defined as* (9)*(2.6). Further, we suppose j satisfies the conditions* (4) *and* (5) *or* (4) *and* (6)-(7)*. If the condition (B) holds, then the (VHVLIP) admits at least one solution.*

Proof. Take $m > n_0$. Since B_m is bounded and convex, from (14) or (17) in Theorem 3.2, we conclude that there exist $u_m \in B_m \cap \mathrm{dom}\phi$ and $u_m^* \in F(u_m)$ such that

$$\langle u_m^*, \eta(v, u_m)\rangle + \phi(v) - \phi(u_m) + J^\circ(\hat{u}_m; T\eta(v, u_m)) \geq 0, \ \forall v \in B_m \cap \mathrm{dom}\phi. \tag{21}$$

(i) If $\|u_m\| = m$, then $\|u_m\| > n_0$. Since the condition (B) holds, there is some $v_0 \in K_\phi$ with $\|v_0\| < \|u_m\|$ such that

$$\langle u_m^*, \eta(v_0, u_m)\rangle + \phi(v_0) - \phi(u_m) + J^\circ(\hat{u}_m; T\eta(v_0, u_m)) \leq 0. \tag{22}$$

Let $v \in K_\phi$ be arbitrarily fixed. Since $\|v_0\| < \|u_m\| = m$, there is $t \in (0, 1)$ such that $v_t := v_0 + t(v - v_0) \in B_m \cap \mathrm{dom}\phi$. Note that T is a linear mapping and ϕ is convex. It follows from (21), (22), Lemma 2.1 (i) and the affinity of η in the first variable that

$$\begin{aligned}
0 &\leq \langle u_m^*, \eta(v_t, u_m)\rangle + \phi(v_t) - \phi(u_m) + J^\circ(\hat{u}_m; T\eta(v_t, u_m)) \\
&= \langle u_m^*, \eta(v_0 + t(v - v_0), u_m)\rangle + \phi(v_0 + t(v - v_0)) - \phi(u_m) + J^\circ(\hat{u}_m; T\eta(v_0 + t(v - v_0), u_m)) \\
&\leq (1 - t)[\langle u_m^*, \eta(v_0, u_m)\rangle + \phi(v_0) - \phi(u_m) + J^\circ(\hat{u}_m; T\eta(v_0, u_m))] \\
&\quad + t[\langle u_m^*, \eta(v, u_m)\rangle + \phi(v) - \phi(u_m) + J^\circ(\hat{u}_m; T\eta(v, u_m))] \\
&\leq t[\langle u_m^*, \eta(v, u_m)\rangle + \phi(v) - \phi(u_m) + J^\circ(\hat{u}_m; T\eta(v, u_m))], \ \forall v \in K_\phi.
\end{aligned} \tag{23}$$

Therefore, this together with $t \in (0, 1)$ implies that

$$\langle u_m^*, \eta(v, u_m)\rangle + \phi(v) - \phi(u_m) + J^\circ(\hat{u}_m; T\eta(v, u_m)) \geq 0, \ \forall v \in K_\phi. \tag{24}$$

(ii) If $\|u_m\| < m$, then for any $v \in K_\phi$, there is some $t \in (0, 1)$ such that $v_t := u_m + t(v - u_m) \in B_m \cap \mathrm{dom}\phi$. Note that T is a linear mapping and ϕ is a convex function. It follows from (21) and (i) of Lemma 2.1, that

$$\begin{aligned}
0 &\leq \langle u_m^*, \eta(v_t, u_m)\rangle + \phi(v_t) - \phi(u_m) + J^\circ(\hat{u}_m; T\eta(v_t, u_m)) \\
&\leq t[\langle u_m^*, \eta(v, u_m)\rangle + \phi(v) - \phi(u_m) + J^\circ(\hat{u}_m; T\eta(v, u_m))], \ \forall v \in K_\phi.
\end{aligned}$$

Therefore, this together with $t \in (0, 1)$ implies that (24) also holds.

Since $J(\varphi) = \int_\Omega j(x, \varphi(x))dx$ and j satisfies the conditions (4) and (5) or (4) and (6)-(7), by Lemma 2.2, we have

$$\int_\Omega j^\circ(x, \hat{u}(x); (T\eta(v, u))(x))dx \geq J^\circ(\hat{u}; T\eta(v, u)), \ \forall u, v \in X,$$

and so

$$\begin{aligned}&\langle u_m^*, \eta(v, u_m)\rangle + \phi(v) - \phi(u_m) + \int_\Omega j^\circ(x, \hat{u}_m(x); (T\eta(v, u_m))(x))dx \\ &\geq \langle u_m^*, \eta(v, u_m)\rangle + \phi(v) - \phi(u_m) + J^\circ(\hat{u}_m; T\eta(v, u_m)) \\ &\geq 0, \ \forall v \in K_\phi.\end{aligned} \tag{25}$$

If $v' \in K \setminus \mathrm{dom}\phi$, then $\phi(v') = +\infty$ and thus the inequality in (25) holds automatically. This together with (25) shows that $u_m \in K_\phi$ is a solution of (VHVLIP). □

Remark 3.6. *Theorem 3.5 generalizes and extends Theorem 4.2 of [34] from the variational-hemivariational inequality problem (VHVIP) to the variational-hemivariational-like inequality problem (VHVLIP). It also generalizes and extends Theorem 3.2 of [46] from the hemivariational inequality problem (HVIP) to the variational-hemivariational-like inequality problem (VHVLIP). In addition, Theorem 3.5 also generalizes and improves Theorem 2 of [12] by extending F from single-valued case to set-valued one and relaxing the corresponding coercivity condition and stable η-pseudomonotonicity of the operator in [12] with $\eta(u, v) = u - v, \forall u, v \in X$.*

If the constraint set K is bounded, then the solution set of the (VHVLIP) is obviously bounded. In the case when the constraint set K is unbounded, the solution set of the (VHVLIP) may be unbounded. In the sequel, we provide a sufficient condition to the boundedness of the solution set of the (VHVLIP), when K is unbounded. The following theorem generalizes and extends Theorem 4.3 of [34] from the variational-hemivariational inequality problem (VHVIP) to the variational-hemivariational-like inequality problem (VHVLIP). Meantime, it also generalizes and extends Theorem 3.3 of [46].

Theorem 3.7. *Let $\eta : X \times X \to X$ be a skew mapping that is affine in the first variable and weakly continuous in the second variable. Let the Clarke's generalized directional derivative $J^\circ(\cdot, \cdot)$ be bi-sequentially weakly upper semicontinuous w.r.t. linear compact operator T. Assume that K is a nonempty, closed, unbounded and convex subset of X and $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function such that $K_\phi \neq \emptyset$. Let $F : K \to 2^{X^*}$ be a lower hemicontinuous set-valued mapping and stably (ϕ, η)-quasimonotone w.r.t. the set $U(J, T)$, where $U(J, T)$ is defined as (9). Further, we suppose j satisfies the conditions* (4) *and* (5) *or* (4) *and* (6)-(7)*. If the condition (C) holds, then the solution set of (VHVLIP) is nonempty and bounded.*

Proof. From Proposition 3.4, we get (C)⇒(B). By Theorem 3.5, we know that the solution set of (VHVLIP) is nonempty. If the solution set is unbounded, then there exist $u_0 \in K_\phi$ and $u_0^* \in F(u_0)$ such that $\|u_0\| > n_0$ and

$$\langle u_0^*, \eta(v, u_0)\rangle + \phi(v) - \phi(u_0) + \int_\Omega j^\circ(x, \hat{u}_0(x); (T\eta(v, u_0))(x))dx \geq 0, \quad \forall v \in K. \tag{26}$$

Since $\|u_0\| > n_0$, by the condition (C), there exists $v_0 \in K_\phi$ with $\|v_0\| < \|u_0\|$ such that

$$\sup_{u^* \in F(u_0)} \langle u^*, \eta(v_0, u_0)\rangle + \phi(v_0) - \phi(u_0) + \int_\Omega j^\circ(x, \hat{u}_0(x); (T\eta(v_0, u_0))(x))dx < 0,$$

which contradicts (26). Hence, the solution set is bounded. □

Using a similar technique to that used in [6, 30], we are also able to provide a necessary and sufficient condition for the (VHVLIP).

Theorem 3.8. *Let $\eta : X \times X \to X$ be a skew mapping that is affine in the first variable. Let $T : X \to L^p(\Omega; \mathbf{R}^k)$ be a linear compact operator, where $1 < p < \infty$, $k \geq 1$ and Ω is a bounded open set in $\mathbf{R}^N$. Assume that K is a nonempty, closed and convex subset of X and $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function such that $K_\phi \neq \emptyset$. Further, we suppose j satisfies the conditions (4) and (5) or (4) and (6)-(7). Then a necessary and sufficient condition for the (VHVLIP) to have a solution is that there exists $r > 0$ with the property that at least one solution of the problem:*

find $u_r \in K_\phi \cap B_r$ and $u_r^* \in F(u_r)$ such that

$$\text{Problem } P_r: \quad \langle u_r^*, \eta(v, u_r)\rangle + \phi(v) - \phi(u_r) + \int_\Omega j^\circ(x, \hat{u}_r(x); (T\eta(v, u_r))(x))dx \geq 0, \ \forall v \in K \cap B_r, \tag{27}$$

satisfies the inequality $\|u_r\| < r$.

Proof. The necessity is evident.

Let us now suppose that there exists a solution u_r of the problem (P_r) with $\|u_r\| < r$. We now prove that u_r is a solution of the (VHVLIP). For any fixed $v \in K$, we choose $\varepsilon > 0$ enough small so that $w = u_r + \varepsilon(v - u_r)$ satisfies $\|w\| < r$. Hence, it follows from (27), the linearity of T and the affinity of η in the first variable with $\eta(u_r, u_r) = 0$ that

$$\begin{aligned} 0 &\leq \langle u_r^*, \eta(w, u_r)\rangle + \phi(w) - \phi(u_r) + \int_\Omega j^\circ(x, \hat{u}_r(x); (T\eta(w, u_r))(x))dx \\ &= \langle u_r^*, \varepsilon\eta(v, u_r)\rangle + \phi(u_r + \varepsilon(v - u_r)) - \phi(u_r) + \int_\Omega j^\circ(x, \hat{u}_r(x); \varepsilon(T\eta(v, u_r))(x))dx. \end{aligned}$$

Using the convexity of ϕ and the positive homogeneity of the map $v \mapsto j^\circ(u; v)$, the conclusion follows. □

Remark 3.9. *If $\eta(u, v) = u - v$ for all $u, v \in X$, then Theorem 3.8 reduces to Theorem 4.4 of [34]. If F is single-valued and $\phi = I_K$ additionally, then Theorem 3.8 reduces to Theorem 3 of Panagiotopoulos et al. [30]. Theorem 3.8 is also a generalization of Theorem 3.1.7 in [20].*

4. Concluding remarks

In this paper, we study the existence of solutions for a class of variational-hemivariational-like inequalities in reflexive Banach spaces. Using the notion of the stable (ϕ, η)-quasimonotonicity and the properties of Clarke's generalized directional derivative and Clarke's generalized gradient, we prove some existence results of solutions when the constrained set is nonempty, bounded (or unbounded), closed and convex. On the other hand, a sufficient condition to the boundedness of the solution set and a necessary and sufficient condition to the existence of solutions are also derived. The results presented in this paper generalize and improve some known results in the earlier and recent literature.

The purpose of this paper is to generalize and extend the main results (i.e., Theorems 4.1-4.4) of Tang and Huang [34] from the variational-hemivariational inequality problem (VHVIP) to the variational-hemivariational-like inequality problem (VHVLIP). To accomplish this end, we make some requirements which are more general than those in Theorems 4.1-4.4 of [34], for example, let $\eta : X \times X \to X$ be a skew mapping that is affine in the first variable and weakly continuous in the second variable; and let $F : K \to 2^{X^*}$ be a stably (ϕ, η)-quasimonotone w.r.t. the set $U(J, T)$, where $U(J, T)$ is defined as (9). In addition, by assuming the bi-sequentially weakly upper semicontinuity of $J^\circ(\cdot, \cdot)$ w.r.t. linear compact operator T, for Case (ii) in the proof of Theorem 3.2, we make sure that the mapping $u \mapsto \inf_{v^* \in F(v)}\langle v^*, \eta(v, u)\rangle + \phi(v) - \phi(u) + J^\circ(\hat{u}; T\eta(v, u))$ is weakly upper semicontinuous. So, it is known that $G(v)$ is weakly compact for each $v \in K_\phi$. Therefore, the requirement that the Clarke's generalized directional derivative $J^\circ(\cdot, \cdot)$ is bi-sequentially weakly upper semicontinuous w.r.t. linear compact operator T, plays a crucial role in the generalization of Theorems 4.1-4.4 of Tang and Huang [34],

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References

[1] S. Carl, Existence of extremal solutions of boundary hemivariational inequalities, J. Differ. Equ. 171(2001) 370–396.
[2] S. Carl, V. K. Le, D. Motreanu, Existence and comparison principles for general quasilinear variational-hemivariational inequalities, J. Math. Anal. Appl. 302(2005) 65–83.
[3] S. Carl, V. K. Le, D. Motreanu, Nonsmooth Variational Problems and Their Inequalities: Comparison Principles and Applications, Springer, New York, 2007.
[4] L.C. Ceng, H. Gupta, C.F. Wen, Well-posedness by perturbations of variational-hemivariational inequalities with perturbations, Filomat 26(2012) 881–895.
[5] L.C. Ceng, Y.C. Liou, J.C. Yao, Y.H. Yao, C.H. Lo, Well-Posedness for systems of generalized mixed quasivariational inclusion problems and optimization problems with constraints, J. Nonlinear Sci. Appl. 10(2017) 5373-5392.
[6] L.C. Ceng, Y.C. Liou, J.C. Yao, Y.H. Yao, Well-posedness for systems of time-dependent hemivariational inequalities in Banach spaces, J. Nonlinear Sci. Appl. 10(2017) 4318-4336.
[7] L.C. Ceng, A. Petruşel, J.C. Yao, Y.H. Yao, Hybrid viscosity extragradient method for systems of variational inequalities, fixed Points of nonexpansive mappings, zero points of accretive operators in Banach spaces, Fixed Point Theory, in press.
[8] L.C. Ceng, C.F. Wen, Well-posedness by perturbations of generalized mixed variational inequalities in Banach spaces, J. Appl. Math. 2012(2012), Art. ID 194509, 38 pp.
[9] L.C. Ceng, N.C. Wong, J.C. Yao, Well-posedness for a class of strongly mixed variational-hemivariational inequalities with perturbations, J. Appl. Math. 2012(2012), Art. ID 712306, 21 pp.
[10] S.Y. Cho, X. Qin, J.C. Yao, Y.H. Yao, Viscosity approximation splitting methods for monotone and nonexpansive operators in Hilbert spaces, J. Nonlinear Convex Anal, in press.
[11] F.H. Clarke, Optimization and nonsmooth analysis, Wiley, New York, 1983.
[12] N. Costea, V. Radulescu, Hartman-Stampacchia results for stably pseudomonotone operators and non-linear hemivariational inequalities, Appl. Anal. 89(2010) 175–188.
[13] N. Costea, C. Lupu, On a class of variational-hemivariational inequalities involving set valued mappings, Adv. Pure Appl. Math. 1(2010) 233–246.
[14] K. Fan, Some properties of convex sets related to fixed point theorems, Math. Ann. 266(1984) 519–537.
[15] D. Goeleven, M. Thera, Semicoercive variational inequalities, J. Global Optim. 6(1995) 367–381.
[16] D. Goeleven D. Motreanu, P.D. Panagiotopoulos, Eigenvalue problems for variational-hemivariational inequalities at resonance, Nonlinear Anal. 33(1998) 161–180.
[17] Y.R. He, Asymptotic analysis and error bounds in optimization, Ph.D. Thesis, The Chinese University of Hong Kong, 2001.
[18] Y.R. He, Stable pseudomonotone variational inequality in reflexive Banach spaces, J. Math. Anal. Appl. 330(2007) 352–363.
[19] Y.R. He, K.F. Ng, Strict feasibility of generalized complementarity problems, J. Aust. Math. Soc. 81(2006) 15–20.
[20] D. Kinderlehrer, G. Stampacchia, An introduction to variational inequalities and their applications, Academic Press, New York, 1980.
[21] Z.H. Liu, Elliptic variational hemivariational inequalities, Appl. Math. Lett. 16(2003) 871–876.
[22] Z.H. Liu, Existence results for quasilinear parabolic hemivariational inequalities, J. Differ. Equ. 244(2008) 1395–1409.
[23] Z.H. Liu, D. Motreanu, A class of variational-hemivariational inequalities of elliptic type, Nonlinearity 23(2010) 1741–1752.
[24] S.A. Marano, N.S. Papageorgiou, On some elliptic hemivariational and variational-hemivariational inequalities, Nonlinear Anal. 62(2005) 757–774.
[25] S. Migorski, A. Ochal, Boundary hemivariational inequality of parabolic type, Nonlinear Anal. 57(2004) 579–596.
[26] D. Motreanu, V. Radulescu, Existence results for inequality problems with lack of convexity, Numer. Funct. Anal. Optim. 21(2000) 869–884.
[27] P.D. Panagiotopoulos, Nonconvex energy functions, hemivariational inequalities and substationarity principles, Acta Mech. 48(1983) 111–130.
[28] P.D. Panagiotopoulos, Inequality problems in mechanics and applications, Convex and Nonconvex Energy Functions, Birkhauser, Basel, 1985.
[29] P.D. Panagiotopoulos, Hemivariational inequalities, applications in mechanics and engineering, Springer, Berlin, 1993.
[30] P.D. Panagiotopoulos, M. Fundo, V. Radulescu, Existence theorems of Hartman-Stampacchia type for hemivariational inequalities and applications, J. Global Optim. 15(1999) 41–54.
[31] J.Y. Park, T.G. Ha, Existence of antiperiodic solutions for hemivariational inequalities, Nonlinear Anal. 68(2008) 747–767.
[32] J.Y. Park, T.G. Ha, Existence of anti-periodic solutions for quasilinear parabolic hemivariational inequalities, Nonlinear Anal. 71(2009) 3203–3217.
[33] S. Schaible, J.C. Yao, L.C. Zeng, On the existence and convergence of approximate solutions for mixed variational-like inequalities, Optim. 56(2007) 105–114.
[34] G.J. Tang, N.J. Huang, Existence theorems of the variational-hemivariational inequalities, J. Global Optim. 56(2013) 605–622.
[35] Y.B. Xiao, N.J. Huang, Sub-super-solution method for a class of higher order evolution hemivariational inequalities, Nonlinear Anal. 71(2009) 558–570.

[36] Y.B. Xiao, N.J. Huang, Well-posedness for a class of variational-hemivariational inequalities with perturbations, J. Optim. Theory Appl. 151(2011) 33–51.
[37] H.Y. Yin, C.X. Xu, Z.X. Zhang, The *F*-complementarity problem and its equivalence with the least element problem, Acta Math. Sin. 44(2001) 679–686.
[38] Y.H. Yao, Y.C. Liou, J.C. Yao, Iterative algorithms for the split variational inequality and fixed point problems under nonlinear transformations, J. Nonlinear Sci. Appl. 10(2017) 843-854.
[39] Y.H. Yao, M.A. Noor, Y.C. Liou, S.M. Kang, Iterative algorithms for general multivalued variational inequalities, Abstr. Appl. Anal., 2012(2012), Article ID 768272.
[40] Y.H. Yao, A. Petrusel, X. Qin, An improved algorithm based on Korpelevich's method for variational inequalities in Banach spaces, J. Nonlinear Convex Anal, in press.
[41] Y.H. Yao, M. Postolache, Y.C. Liou, Z.S. Yao, Construction algorithms for a class of monotone variational inequalities, Optim. Lett. 10(2016) 1519-1528.
[42] Y.H. Yao, X. Qin, J.-C. Yao, Projection methods for firmly type nonexpansive operators, J. Nonlinear Convex Anal., in press.
[43] Y.H. Yao, N. Shahzad, Strong convergence of a proximal point algorithm with general errors, Optim. Lett. 6(2012) 621-628.
[44] Y.H. Yao, N. Shahzad, An algorithmic approach to the split variational inequality and fixed point problem, J. Nonlinear Convex Anal. 18(2017) 977-991.
[45] H. Zegeye, N. Shahzad, Y.H. Yao, Minimum-norm solution of variational inequality and fixed point problem in Banach spaces, Optim. 64(2015) 453-471.
[46] Y.L. Zhang, Y.R. He, On stably quasimonotone hemivariational inequalities, Nonlinear Anal. 74(2011) 3324–3332.
[47] R.Y. Zhong, N.J. Huang, Stability analysis for Minty mixed variational inequality in reflexive Banach spaces, J. Optim. Theory Appl. 147(2010) 454–472.