Filomat 32:10 (2018), 3657–3665
https://doi.org/10.2298/FIL1810657A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Graph of Modules Over Commutative Rings II

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Abstract. Let M be a module over a commutative ring R. In this paper, we continue our study about the quasi-Zariski topology-graph $G(\tau^*_T)$ which was introduced in (On the graph of modules over commutative rings, Rocky Mountain J. Math. 46(3) (2016), 1–19). For a non-empty subset T of $Spec(M)$, we obtain useful characterizations for those modules M for which $G(\tau^*_T)$ is a bipartite graph. Also, we prove that if $G(\tau^*_T)$ is a tree, then $G(\tau^*_T)$ is a star graph. Moreover, we study coloring of quasi-Zariski topology-graphs and investigate the interplay between $\chi(G(\tau^*_T))$ and $\omega(G(\tau^*_T))$.

1. Introduction

Throughout this paper R is a commutative ring with a non-zero identity and M is a unital R-module. By $N \leq M$ (resp. $N < M$) we mean that N is a submodule (resp. proper submodule) of M.

Define $(N :_R M)$ or simply $(N : M) = \{r \in R | \, rM \subseteq N\}$ for any $N \leq M$. We denote $((0) : M)$ by $Ann_R(M)$ or simply $Ann(M)$. M is said to be faithful if $Ann(M) = (0)$.

Let $N, K \leq M$. Then the product of N and K, denoted by NK, is defined by $(N : M)(K : M)M$ (see [3]).

A prime submodule of M is a submodule $P \neq M$ such that whenever $re \in P$ for some $r \in R$ and $e \in M$, we have $r \in (P : M)$ or $e \in P$ [13].

The prime spectrum of M is the set of all prime submodules of M and denoted by $Spec(M)$.

There are many papers on assigning graphs to rings or modules (see, for example, [1, 4–7, 9, 16]). In [5], the present authors introduced and studied the graph $G(\tau^*_T)$ (resp. $AG(M)$), called the *quasi-Zariski topology-graph* (resp. *the annihilating-submodule graph*), where T is a non-empty subset of $Spec(M)$.

$AG(M)$ is an undirected graph with vertices $V(AG(M))$= $\{N \leq M |$ there exists $(0) \neq K < M$ with $NK = (0)\}$. In this graph, distinct vertices $N, L \in V(AG(M))$ are adjacent if and only if $NL = (0)$. Let $AG(M)^*$ be the subgraph of $AG(M)$ with vertices $V(AG(M)^*) = \{N < M$ with $(N : M) \neq Ann(M) |$ there exists a submodule $K < M$ with $(K : M) \neq Ann(M)$ and $NK = (0)\}$. By [4, Theorem 3.4], one conclude that $AG(M)^*$ is a connected subgraph.

$G(\tau^*_T)$ is an undirected graph with vertices $V(G(\tau^*_T))$= $\{N < M |$ there exists $K < M$ such that $V^*(N) \cup V^*(K) = T$ and $V^*(N), V^*(K) \neq T\}$ and distinct vertices N and L are adjacent if and only if $V^*(N) \cup V^*(L) = T$ (see [5, Definition 2.1]).

2010 *Mathematics Subject Classification*. Primary 13C13; Secondary 13C99
Keywords. prime submodule, top module, quasi-Zariski topology, graph, vertices, annihilating-submodule
Received: 03 October 2017; Accepted: 30 April 2018
Communicated by Dijana Mosić
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For any submodule N of M, $V^*(N)$ is the set of all prime submodules of M containing N. Of course, $V^*(M)$ is the empty set and $V^*(0)$ is $Spec(M)$. Note that for any family of submodules N_i ($i \in I$) of M, $\cap V^*(N_i) = V^*(\Sigma_{i\in I} N_i)$. Thus if $Z^*(M)$ denotes the collection of all subsets $V^*(N)$ of $Spec(M)$, then $Z^*(M)$ contains the empty set and $Spec(M)$, and $Z^*(M)$ is closed under arbitrary intersections. If $Z^*(M)$ is closed under finite unions, i.e. for any submodules N and K of M, there exists a submodule L of M such that $V^*(N) \cup V^*(K) = V^*(L)$, for in this case $Z^*(M)$ satisfies the axioms for the closed subsets of a topological space and M is called a top module for short. The *quasi-Zariski topology* on $X = Spec(M)$ is the topology τ^*_M described by taking the set $Z^*(M) = \{V^*(N) | N$ is a submodule of $M\}$ as the set of closed sets of $Spec_R(M)$, where $V^*(N) = \{P \in X | P \supseteq N\}$ [15].

If $Spec(M) \neq \emptyset$, the mapping $\psi : Spec(M) \to Spec(R/Ann(M))$ such that $\psi(P) = (P : M)/Ann(M)$ for every $P \in Spec(M)$, is called the *natural map* of $Spec(M)$ [14].

A topological space X is irreducible if for any decomposition $X = X_1 \cup X_2$ with closed subsets X_i of X with $i = 1, 2$, we have $X = X_1$ or $X = X_2$

The prime radical $\sqrt{N}$ is defined to be the intersection of all prime submodules of M containing N, and in case N is not contained in any prime submodule, $\sqrt{N}$ is defined to be M [13].

We recall that $N < M$ is said to be a semiprime submodule of M if for every ideal I of R and every submodule K of M with $I^2K \subseteq N$ implies that $IK \subseteq N$. Further M is called a semiprime module if $(0) \subseteq M$ is a semiprime submodule. Every intersection of prime submodules is a semiprime submodule (see [17]).

The notations $Nil(R)$, $Min(M)$, and $Min(T)$ will denote the set of all nilpotent elements of R and the set of all minimal prime submodules of M, and the set of minimal members of T, respectively.

A clique of a graph is a complete subgraph and the supremum of the sizes of cliques in G, denoted by $\omega(G)$, is called the clique number of G. Let $\chi(G)$ denote the chromatic number of the graph G, that is, the minimal number of colors needed to color the vertices of G so that no two adjacent vertices have the same color. Obviously $\chi(G) \geq \omega(G)$.

In this article, we continue our studying about $G(\tau^*_T)$ and $AG(M)$ and we try to relate the combinatorial properties of the above mentioned graphs to the algebraic properties of M.

In section 2 of this paper, we state some properties related to the quasi-Zariski topology-graph that are basic or needed in the later sections. In section 3, we study the bipartite quasi-Zariski topology-graphs of modules over commutative rings (see Proposition 3.1). Also, we prove that if $G(\tau^*_T)$ is a tree, then $G(\tau^*_T)$ is a star graph (see Theorem 3.5). In section 4, we study coloring of the quasi-Zariski topology-graph of modules and investigate the interplay between $\chi(G(\tau^*_T))$ and $\omega(G(\tau^*_T))$. We show that under condition over minimal submodules of $M/\cap_{P\in T} P$, we have $\omega(G(\tau^*_T)) = \chi(G(\tau^*_T))$ (see Theorem 4.1). Moreover, we investigate some relations between the existence of cycles in the quasi-Zariski topology-graph of a cyclic module and the number of its minimal members of T (see Proposition 4.9).

Let us introduce some graphical notions and denotations that are used in what follows: A graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set of vertices, $V(G)$, a set $E(G)$ of edges, and an incident function ψ_G that associates an unordered pair of distinct vertices with each edge. The edge e joins x and y if $\psi_G(e) = \{x, y\}$, and we say x and y are adjacent. A path in graph G is a finite sequence of vertices $\{x_0, x_1, \ldots, x_n\}$, where x_{i-1} and x_i are adjacent for each $1 \leq i \leq n$ and we denote $x_{i-1} - x_i$ for existing an edge between x_{i-1} and x_i.

A graph H is a subgraph of G, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$. A bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V; that is, U and V are each independent sets and complete bipartite graph on n and m vertices, denoted by $K_{n,m}$, where V and U are of size n and m, respectively, and $E(G)$ connects every vertex in V with all vertices in U. Note that a graph $K_{1,m}$ is called a star graph and the vertex in the singleton partition is called the center of the graph. For some $U \subseteq V(G)$, we denote by $N(U)$, the set of all vertices of $G \setminus U$ adjacent to at least one vertex of U. For every vertex $v \in V(G)$, the size of $N(v)$ is denoted by $deg(v)$. If all the vertices of G have the same degree k, then G is called k-regular, or simply regular. We denote by C_n a cycle of order n. Let G and G' be two graphs. A graph homomorphism from G to G' is a mapping $\phi : V(G) \longrightarrow V(G')$ such that for every edge $\{u, v\}$ of G, $\{\phi(u), \phi(v)\}$ is an edge of G'. A retract of G is a subgraph H of G such that there exists a homomorphism $\phi : G \longrightarrow H$ such that $\phi(x) = x$, for every vertex x of H. The homomorphism ϕ is called the retract (graph) homomorphism (see [10]).

Throughout the rest of this paper, we denote: T is a non-empty subset of $Spec(M)$, $Q := \cap_{P\in T} P$, $\bar{M} := M/Q$, $\bar{N} := N/Q$, $\bar{m} := m + Q$, and $\bar{I} := I/(Q : M)$, where N is a submodule of M containing Q, $m \in M$, and I is an ideal of R containing $(Q : M)$.

2. Auxiliary results

In this section, we provide some properties related to the quasi-Zariski topology-graph that are basic or needed in the sequel. Throughout this paper M is a top module and by [15, Theorem 3.5], every multiplication module is a top module.

Remark 2.1. *By [15, Lemma 2.1], if M is a top module, then for every pair of submodules N and L of M, we have $V^*(N) \cup V^*(L) = V^*(\sqrt{N}) \cup V^*(\sqrt{L}) = V^*(\sqrt{N} \cap \sqrt{L})$. By [5, Proposition 2.3], we have T is a closed subset of $Spec(M)$ if and only if $T = V^*(\cap_{P\in T} P)$ and $G(\tau^*_T) \neq \emptyset$ if and only if $T = V^*(\cap_{P\in T} P)$ and T is not irreducible. So if N and K are adjacent in $G(\tau^*_T)$, then $\sqrt{N} \cap \sqrt{K} = \cap_{P\in T} P$. Therefore $\cap_{P\in T} P \subseteq \sqrt{N}, \sqrt{K}$.*

Lemma 2.2. *(See [2, Proposition 7.6].) Let $R_1, R_2, \ldots, R_n$ be non-zero ideals of R. Then the following statements are equivalent:*

(a) $R = R_1 \oplus \ldots \oplus R_n$;

(b) As an abelian group R is the direct sum of $R_1, \ldots, R_n$;

(c) There exist pairwise orthogonal idempotents $e_1, \ldots, e_n$ with $1 = e_1 + \ldots + e_n$, and $R_i = Re_i$, $i = 1, \ldots, n$.

Proposition 2.3. *Suppose that e is an idempotent element of R. We have the following statements.*

(a) $R = R_1 \oplus R_2$, where $R_1 = eR$ and $R_2 = (1 - e)R$.

(b) $M = M_1 \oplus M_2$, where $M_1 = eM$ and $M_2 = (1 - e)M$.

(c) For every submodule N of M, $N = N_1 \oplus N_2$ such that N_1 is an R_1-submodule M_1, N_2 is an R_2-submodule M_2, and $(N :_R M) = (N_1 :_{R_1} M_1) \oplus (N_2 :_{R_2} M_2)$.

(d) For submodules N and K of M, $NK = N_1K_1 \oplus N_2K_2$, $N \cap K = N_1 \cap K_1 \oplus N_2 \cap K_2$ such that $N = N_1 \oplus N_2$ and $K = K_1 \oplus K_2$.

(e) Prime submodules of M are $P \oplus M_2$ and $M_1 \oplus Q$, where P and Q are prime submodules of M_1 and M_2, respectively.

(f) For submodule N of M, we have $\sqrt{N} = \sqrt{N_1 \oplus N_2} = \sqrt{N_1} \oplus \sqrt{N_2}$, where $N = N_1 \oplus N_2$.

Proof. This is clear. □

An ideal $I < R$ is said to be nil if I consist of nilpotent elements.

Lemma 2.4. *(See [12, Theorem 21.28].) Let I be a nil ideal in R and $u \in R$ be such that $u + I$ is an idempotent in R/I. Then there exists an idempotent e in uR such that $e - u \in I$.*

Lemma 2.5. *(See [6, Lemma 2.4].) Let N be a minimal submodule of M and let $Ann(M)$ be a nil ideal. Then we have $N^2 = (0)$ or $N = eM$ for some idempotent $e \in R$.*

Lemma 2.6. *Assume that T is a closed subset of $Spec(M)$ and $\bar{M}$ is a multiplication module. Then $AG(\bar{M})$ is isomorphic with a (an induced) subgraph of $G(\tau^*_T)$.*

Proof. Let $\bar{N} \in V(AG(\bar{M}))$. Then there exists a nonzero submodule $\bar{K}$ of $\bar{M}$ such that it is adjacent to $\bar{N}$. So we have $NK \subseteq Q$. Hence $V^*(NK) = T$. If $V^*(N) = T$, then $N = Q$, a contradiction. Hence N is a vertex in $G(\tau^*_T)$ which is adjacent to K. □

Lemma 2.7. *If $\bar{M}$ is a faithful multiplication module, then $G(\tau^*_{Spec(M)})$ and $AG(M)$ are the same.*

Proof. $\bar{M}$ is a faithful module so that $T = Spec(M)$. If $G(\tau^*_{Spec(M)}) \neq \emptyset$, then there exist non-trivial submodules N and K of M which is adjacent in $G(\tau^*_{Spec(M)})$. Hence $V^*(NK) = Spec(M)$ which implies that $NK = (0)$ so that $AG(M) \neq \emptyset$. By Lemma 2.6, $AG(M)$ is isomorphic with a subgraph of $G(\tau^*_{Spec(M)})$. One can see that the vertex map $\phi : V(G(\tau^*_{Spec(M)})) \longrightarrow V(AG(M))$, defined by $N \longrightarrow N$ is an isomorphism. □

Recall that $\Delta(G(\tau^*_T))$ is the maximum degree of $G(\tau^*_T)$ and the length of an R-module M, is denoted by $l_R(M)$.

Lemma 2.8. *Let every nontrivial submodule of M be a vertex in $G(\tau^*_T)$. If $\Delta(G(\tau^*_T)) < \infty$, then $l_R(M) \leq \Delta(G(\tau^*_T)) + 1$. Also, every non-trivial submodule of M has finitely many submodules.*

Proof. Straightforward. □

Theorem 2.9. *Let $\bar{M}$ be a multiplication module and $G(\tau^*_T) \neq \emptyset$. Then M has acc (resp. dcc) on vertices of $G(\tau^*_T)$ if and only if $\bar{M}$ is a Noetherian (resp. an Artinian) module.*

Proof. Suppose that $G(\tau^*_T)$ has acc (resp. dcc) on vertices. By [5, Proposition 2.3 (iii)], $\bar{M}$ is not a prime module and hence there exist $r \in R$ and $\bar{m} \in \bar{M}$ such that $r\bar{m} = \bar{0}$ but $\bar{m} \neq \bar{0}$ and $r \notin Ann(\bar{M})$. Now $\overline{rM} \cong \bar{M}/(\bar{0} :_{\bar{M}} r)$. Further, $\overline{rM}$ and $(\bar{0} :_{\bar{M}} r)$ are vertices because $(\bar{0} :_{\bar{M}} r)(\overline{rM}) = ((\bar{0} :_{\bar{M}} r) : \bar{M})(\overline{rM} : \bar{M})\bar{M} \subseteq \overline{rM}((\bar{0} :_{\bar{M}} r) : \bar{M}) \subseteq r(\bar{0} :_{\bar{M}} r) = \bar{0}$. Then $\{\bar{N} | \bar{N} \leq \bar{M}, \bar{N} \subseteq \overline{rM}\} \cup \{\bar{N} | \bar{N} \leq \bar{M}, \bar{N} \subseteq (\bar{0} :_{\bar{M}} r)\} \subseteq V(G(\tau^*_T))$. It follows that the R-modules $\overline{rM}$ and $(\bar{0} :_{\bar{M}} r)$ have acc (resp. dcc) on submodules. Since $\overline{rM} \cong \bar{M}/(\bar{0} :_{\bar{M}} r)$, $\bar{M}$ has acc on submodules and the proof is completed. □

3. Quasi-Zariski topology-graph of modules

First, in this section we give the more notation to be used throughout the remainder of this article. Suppose that e $(e \neq 0, 1)$ is an idempotent element of R. Let $M_1 := eM, M_2 := (1 - e)M, T_1 := \{P_1 \in Spec(M_1) | P_1 \oplus M_2 \in T\}$, $T_2 := \{P_2 \in Spec(M_2) | M_1 \oplus P_2 \in T\}$, $Q_1 := \cap_{P_1 \in T_1} P_1, Q_2 := \cap_{P_2 \in T_2} P_2$, $\bar{M_1} = \overline{eM} = eM/Q_1$, and $\bar{M_2} = \overline{(e-1)M} = (e-1)M/Q_2$. Consequently we have, $Q = Q_1 \oplus Q_2$, where $Q = \cap_{P \in T} P$ and $\bar{M} \cong \bar{M_1} \oplus \bar{M_2}$

We recall that a submodule N of M is a prime R-module if and only if it is a prime $R/Ann(M)$-module (see [4, Result 1.2]).

Proposition 3.1. *Suppose that $\bar{M}$ is a multiplication module. Then the following statements hold.*

(a) *If there exists a vertex of $G(\tau^*_T)$ which is adjacent to every other vertex, then $\bar{M} \cong \bar{M_1} \oplus \bar{M_2}$, where $\bar{M_1}$ is a simple module and $\bar{M_2}$ is a prime module for some idempotent element $e \in R$.*

(b) *If $\bar{M_1}$ and $\bar{M_2}$ are prime modules for some idempotent element $e \in R$, then $G(\tau^*_T)$ is a complete bipartite graph.*

Proof. (*a*) Suppose that N is adjacent to every other vertex of $G(\tau^*_T)$. Since $V^*(N) = V^*(\sqrt{N})$, we have $N = \sqrt{N}$. It is clear that $\bar{N}$ is a minimal submodule of $\bar{M}$. We have $(\bar{N})^2 \neq (0)$ because $V^*(N) \neq T$. Then Lemma 2.5, implies that $\bar{M} \cong \overline{eM} \oplus \overline{(e-1)M}$ for some idempotent element e of R. Without loss of generality we may assume that $M_1 \oplus Q_2$ is adjacent to every other vertex. We claim that $\bar{M_1}$ is a simple module and $\bar{M_2}$ is a prime module. Let $Q_1 \subsetneq K < M_1$. We have $V^*(K \oplus Q_2) \neq T$ because $Q_1 \oplus Q_2 \subsetneq K \oplus Q_2$. Since $V^*(K \oplus Q_2) \cup V^*(Q_1 \oplus M_2) = T$, we have $K \oplus Q_2$ is a vertex and hence is adjacent to $M_1 \oplus Q_2$. Therefore $V^*(K \oplus Q_2) \cup V^*(M_1 \oplus Q_2) = V^*(K \oplus Q_2) = T$, a contradiction. It implies that $\bar{M_1}$ is a simple module. Now, we show that $\bar{M}_2$ is a prime module. It is enough to show that is a prime $R/(Q_2 : M_2)$-module. Otherwise, $\bar{I}\bar{K} = (\bar{0})$, where $(Q_2 : M_2) \subsetneq I < R$ and $Q_2 \subsetneq K < M$. It follows that $V^*(M_1 \oplus K) \cup V^*(Q_1 \oplus IM_2) = V^*(Q_1 \oplus K(IM_2)) = T$ because $K(IM_2) \subseteq IK \subseteq Q_2$ and $(Q_2 : M_2)^2 M_2 \subseteq K(IM_2)$. Therefore $V^*(M_1 \oplus K) \cup V^*(M_1 \oplus Q_2) = T = V^*(M_1 \oplus Q_2)$, a contradiction.

(b) Assume that $N_1 \oplus N_2$ is adjacent to $K_1 \oplus K_2$. One can see that $\sqrt{N_1K_1} \oplus \sqrt{N_2K_2} = \sqrt{Q_1} \oplus \sqrt{Q_2}$. It implies that $\overline{(\sqrt{(K_1 : M_1)M_1} : M_1)}\ \overline{\sqrt{(N_1 : M_1)M_1}} = (\bar{0})$ and $\overline{(\sqrt{(K_2 : M_2)M_2} : M_2)}\ \overline{\sqrt{(N_2 : M_2)M_2}} = (\bar{0})$. Since $\bar{M}_1$ and $\bar{M}_2$ are prime modules, $(\sqrt{(K_1 : M_1)M_1} : M_1) = (Q_1 : M_1)$ or $\sqrt{(N_1 : M_1)M_1} = Q_1$ and $(\sqrt{(K_2 : M_2)M_2} : M_2) = (Q_2 : M_2)$ or $\sqrt{(N_2 : M_2)M_2} = Q_2$. Therefore $G(\tau^*_T)$ is a complete bipartite graph with two parts U and V such that $N \in U$ if and only if $V^*(N) = V^*(M_1 \oplus Q_2)$ and $K \in V$ if and only if $V^*(K) = V^*(Q_1 \oplus M_2)$. □

Corollary 3.2. *Let $\bar{M}$ be a faithful multiplication module. Then the following statements are equivalent.*

(a) *There is a vertex of $G(\tau^*_{Spec(M)})$ which is adjacent to every other vertex of $G(\tau^*_{Spec(M)})$.*

(b) *$G(\tau^*_{Spec(M)})$ is a star graph.*

(c) *$M = F \oplus D$, where F is a simple module and D is a prime module.*

Proof. (a) $\Rightarrow$ (b) Let $\bar{M}$ be a faithful module. Then $Q = (0)$ and we have $T = Spec(M)$. By Proposition 3.1, $M = M_1 \oplus M_2$, where M_1 is a simple module and M_2 is a prime module. Then every non-zero submodule of M is of the form $M_1 \oplus N_2$ and $(0) \oplus N_2$, where N_2 is a non-zero submodule of M_2. By our hypothesis, we can not have any vertex of the form $M_1 \oplus N_2$, where N_2 is a non-zero proper submodule of M_2. Also $M_1 \oplus (0)$ is adjacent to every other vertex, and non of the submodules of the form $(0) \oplus N_2$ can be adjacent to each other. So $G(\tau^*_{Spec(M)})$ is a star graph.

(b) $\Rightarrow$ (c) This follows by Proposition 3.1 (a).

(c) $\Rightarrow$ (a) Assume that $M = F \oplus D$, where F is a simple module and D is a prime module. It is easy to see that for some minimal submodule N of M, we have $N^2 \neq (0)$. Since M is a faithful module, Lemma 2.5 implies that $F \cong eM$, where e is an idempotent element of R. Finally Proposition 3.1 (a) completes the proof. □

Lemma 3.3. *Let $e \in R$ be an idempotent element of R and let $\bar{M}$ be a multiplication module. If $G(\tau^*_T)$ is a triangle-free graph, then both $\bar{M}_1$ and $\bar{M}_2$ are prime R-modules. Moreover, if $G(\tau^*_T)$ has no cycle, then $\bar{M}_1$ is a simple module and $\bar{M}_2$ is a prime module.*

Proof. Without loss of generality, we can assume that $\bar{M}_1$ is a prime module. Then $\bar{I}\bar{K} = (\bar{0})$, where $(Q_2 : M_2) \subsetneq I < R$ and $Q_2 \subsetneq K < M$. It follows that $V^*(M_1 \oplus K) \cup V^*(Q_1 \oplus IM_2) = V^*(Q_1 \oplus K(IM_2)) = T$ (if $IM_2 = K$, then $V^*(Q_1 \oplus K) = V^*(Q_1 \oplus K^2) = V^*(Q_1 \oplus K(IM_2)) = T$, a contradiction). So both $\bar{M}_1$ and $\bar{M}_2$ are prime R-modules. Now suppose that $G(\tau^*_T)$ has no cycle. If none of $\bar{M}_1$ and $\bar{M}_2$ is a simple module, then we choose non-trivial submodules N_i in M_i for some $i = 1, 2$. So $N_1 \oplus Q_2$, $Q_1 \oplus N_2$, $M_1 \oplus Q_2$, and $Q_1 \oplus M_2$ form a cycle, a contradiction. □

Corollary 3.4. *Assume that $\bar{M}$ is a multiplication module. Then $G(\tau^*_T)$ is a star graph if and only if $\bar{M}_1$ is a simple module and $\bar{M}_2$ is a prime module for some idempotent $e \in R$.*

Proof. The necessity is clear by Proposition 3.1 (a). For the converse, assume that $\bar{M} = \bar{M}_1 \oplus \bar{M}_2$, where $\bar{M}_1$ is a simple module and $\bar{M}_2$ is a prime for some idempotent $e \in R$. Using the Proposition 3.1 (b), $G(\tau^*_T)$ is a complete bipartite graph with two parts U and V such that $N \in U$ if and only if $V^*(N) = V^*(M_1 \oplus Q_2)$ and $K \in V$ if and only if $V^*(K) = V^*(Q_1 \oplus M_2)$. We claim that $|U| = 1$. Otherwise, $V^*(M_1 \oplus Q_2) = V^*(N_1 \oplus Q_2)$, where $Q_1 \neq N_1 < M_1$. It follows that $\sqrt{(N_1 : M_1)M_1} = M_1$, a contradiction (note that if M is a multiplication module, then $\sqrt{N} \neq M$, where $N < M$). So $G(\tau^*_T)$ is a star graph. □

Theorem 3.5. *If $G(\tau^*_T)$ is a tree, then $G(\tau^*_T)$ is a star graph.*

Proof. Suppose that $G(\tau^*_T)$ is not a star graph. Then $G(\tau^*_T)$ has at least four vertices. Obviously, there are two adjacent vertices L and K of $G(\tau^*_T)$ such that $|N(L) \setminus \{K\}| \geq 1$ and $|N(K) \setminus \{L\}| \geq 1$. Let $N(L) \setminus \{K\} = \{L_i\}_{i\in\Lambda}$ and $N(K) \setminus \{L\} = \{K_j\}_{j\in\Gamma}$. Since $G(\tau^*_T)$ is a tree, we have $N(L) \cap N(K) = \emptyset$. By [5, Theorem 2.6], $diam(G(\tau^*_T)) \leq 3$. So every edge of $G(\tau^*_T)$ is of the form $\{L, K\}$, $\{L, L_i\}$ or $\{K, K_j\}$, for some $i \in \Lambda$ and $j \in \Gamma$. Now, Pick $p \in \Lambda$ and $q \in \Gamma$. Since $G(\tau^*_T)$ is a tree, $\sqrt{L_p} \cap \sqrt{K_q}$ is a vertex of $G(\tau^*_T)$. If $\sqrt{L_p} \cap \sqrt{K_q} = L_u$ for some $u \in \Lambda$, then $V^*(K) \cup V^*(L_u) = T$, a contradiction. If $\sqrt{L_p} \cap \sqrt{K_q} = K_v$, for some $v \in \Gamma$, then $V^*(L) \cup V^*(K_v) = T$, a contradiction. If $\sqrt{L_p} \cap \sqrt{K_q} = L$ or $\sqrt{L_p} \cap \sqrt{K_q} = K$, then $V^*(L) = T$ or $V^*(K) = T$, respectively, a contradiction. So the claim is proved. □

Proposition 3.6. *Let $\bar{M}$ be a multiplication module. Then in each case of the following statements, $|T| = 2$ and $G(\tau^*_T) \cong K_2$.*

(a) R be an Artinian ring and $G(\tau^*_T)$ is a bipartite graph.

(b) $Ann(\bar{M})$ is a nil ideal of R and $G(\tau^*_T)$ is a finite bipartite graph.

(c) $Ann(\bar{M})$ is a nil ideal of R and $G(\tau^*_T)$ is a regular graph of finite degree.

Proof. (a) First we may assume that $G(\tau^*_T)$ is not empty. Then R can not be a local ring. Otherwise, $T = V^*(mM)$, where m is the unique maximal ideal of R. Therefore [5, Proposition 2.3] implies that $mM = M$ and hence T is empty, a contradiction. Hence by [8, Theorem 8.9], $R = R_1 \oplus \ldots \oplus R_n$, where R_i is an Artinian local ring for $i = 1, \ldots, n$ and $n \geq 2$. By Lemma 2.2 and Proposition 2.3, since $G(\tau^*_T)$ is a bipartite graph, we have $n = 2$ and hence $\bar{M} \cong \bar{M}_1 \oplus \bar{M}_2$ for some idempotent $e \in R$. If $\bar{M}_1$ is a prime module, then it is easy to see that $\bar{M}_1$ is a vector space over $R/Ann(\bar{M}_1)$ and so is a semisimple R-module. A Similar argument as we did in proof of Corollary 3.4 implies that $|T| = 2$ and $G(\tau^*_T) \cong K_2$.

(b) By Theorem 2.9, $\bar{M}$ is an Artinian and Noetherian module so that $R/Ann(\bar{M})$ is an Artinian ring. A similar arguments in part (a) says that, $R/Ann(\bar{M})$ is a non-local ring. So by [8, Theorem 8.9] and Lemma 2.2, there exist pairwise orthogonal idempotents modulo $Ann(\bar{M})$. By lemma 2.4, $\bar{M} \cong \bar{M}_1 \oplus \bar{M}_2$, for some idempotent e of R. Now, the proof that $G(\tau^*_T) \cong K_2$ is similar to the proof of Corollary 3.4.

(c) We may assume that $G(\tau^*_T)$ is not empty. So $\bar{M}$ is not a prime module by [5, Proposition 2.3] and a similar manner in proof of Theorem 2.9, shows that $\bar{M}$ has a finite length so that $R/Ann(\bar{M})$ is an Artinian ring. As in the proof of part (b), $\bar{M} \cong \bar{M}_1 \oplus \bar{M}_2$ for some idempotent $e \in R$. If $\bar{M}_1$ has one non-trivial submodule N, then $deg(Q_1 \oplus M_2) > deg(N \oplus M_2)$ (we note that by [7, Proposition 2.5], $\bar{N}\bar{K} = (\bar{0})$ for some $(\bar{0}) \neq \bar{K} < \bar{M}_1$) and this contradicts the regularity of $G(\tau^*_T)$. Hence $\bar{M}_1$ is a simple module. Finally a similar argument as we have seen in Corollary 3.4 gives $G(\tau^*_T) \cong K_2$. □

Theorem 3.7. *Assume that $\bar{M}$ is a multiplication module and $|Min(\bar{M})| \geq 3$. Then $G(\tau^*_T)$ contains a cycle.*

Proof. If $G(\tau^*_T)$ is a tree, then by Theorem 3.5, $G(\tau^*_T)$ is a star graph. Suppose that $G(\tau^*_T)$ is a star graph. Then by Corollary 3.4, $\bar{M} \cong \bar{M}_1 \oplus \bar{M}_2$, where $\bar{M}_1$ is a simple module and $\bar{M}_2$ is a prime module and hence by Proposition 2.3 (e), $Min(\bar{M}) = \{\bar{0} \oplus \bar{M}_2, \bar{M}_1 \oplus \bar{0}\}$, that is $|Min(\bar{M})| = 2$, a contradiction. Therefore $G(\tau^*_T)$ contains a cycle.

□

4. Coloring of the quasi-Zariski topology-graph of modules

The purpose of this section is to study of coloring of the quasi-Zariski topology-graph of modules and investigate the interplay between $\chi(G(\tau^*_T))$ and $\omega(G(\tau^*_T))$. We note that since $E(G(\tau^*_T)) \geq 1$ when $G(\tau^*_T) \neq \emptyset$, then $\chi(G(\tau^*_T))) \geq 2$.

Theorem 4.1. *Let $\bar{M}$ be an Artinian module such that for every minimal submodule $\bar{N}$ of $\bar{M}$, N is a vertex in $G(\tau^*_T)$. Then $\omega(G(\tau^*_T)) = \chi(G(\tau^*_T))$.*

Proof. $\bar{M}$ is Artinian, so it contains a minimal submodule. Clearly, for every minimal submodule $\bar{N}$ of $\bar{M}$, $V^*(N) \neq T$. Also, $N \cap L = Q$, where $\bar{N}$ and $\bar{L}$ are minimal submodules of $\bar{M}$. It follows that N and L are adjacent in $G(\tau^*_T)$, where $\bar{N}$ and $\bar{L}$ are minimal submodules of $\bar{M}$. First, suppose that $\bar{M}$ has infinitely many minimal submodules. Then $\omega(G(\tau^*_T)) = \infty$ and there is nothing to prove. Next, assume that $\bar{M}$ has k minimal submodules, where k is finite. We conclude that $\chi(G(\tau^*_T)) = k = \omega(G(\tau^*_T))$. Obviously, $\omega(G(\tau^*_T)) \geq k$. If possible, assume that $\omega(G(\tau^*_T)) > k$. Let $\Sigma = \{N_\lambda\}_{\lambda\in I}$I, where $|I| = \omega(G(\tau^*_T))$ be a maximum clique in $G(\tau^*_T)$. As every $N_\lambda \in \omega$, $\overline{\sqrt{N_\lambda}}$ contains a minimal submodule, there exists a minimal submodule $\bar{K}$ and submodules N_i and N_j in ω, such that $\bar{K} \subset \overline{\sqrt{N_i}} \cap \overline{\sqrt{N_j}}$, and hence $V^*(K) = T$, a contradiction. Hence $\omega(G(\tau^*_T)) = k$. Next, we claim that $G(\tau^*_T)$ is k-colorable. In order to prove, put $A = \{\bar{K}_1, \ldots, \bar{K}_k\}$ be the set of all minimal submodules of $\bar{M}$. Now, we define a coloring f on $G(\tau^*_T)$ by setting $f(N) = min\{i | K_i \subseteq \sqrt{N}\}$ for every vertex N of $G(\tau^*_T)$. Let N and L be adjacent in $G(\tau^*_T)$ and $f(N) = f(L) = j$. Thus $K_j \subseteq \sqrt{N} \cap \sqrt{L}$, a contradiction. It implies that f is a proper k coloring of $G(\tau^*_T)$ and hence $\chi(G(\tau^*_T)) \leq k = \omega(G(\tau^*_T))$, as desired. □

Theorem 4.2. *Assume that $\bar{M}$ is a faithful multiplication module. Then the following statements are equivalent.*

(a) $\chi(G(\tau^*_{Spec(M)})) = 2$.

(b) $G(\tau^*_{Spec(M)})$ *is a bipartite graph.*

(c) $G(\tau^*_{Spec(M)})$ *is a complete bipartite graph.*

(d) Either R is a reduced ring with exactly two minimal prime ideals or $G(\tau^*_{Spec(M)})$ is a star graph with more than one vertex.

Proof. By using Lemma 2.7, $G(\tau^*_{Spec(M)})$ and $AG(M)$ are the same and so [6, Theorem 3.2] completes the proof. □

Lemma 4.3. *Assume that T is a finite closed subset of $Spec(M)$. Then $\chi(G(\tau^*_T)))$ is finite. In particular, $\omega(G(\tau^*_T)))$ is finite.*

Proof. Suppose that $T = \{P_1, P_2, \ldots, P_k\}$ is a finite set of distinct prime submodules of M. Define a coloring $f(N) = min\{n \in \mathbb{N} | P_n \notin V^*(N)\}$, where N is a vertex of $G(\tau^*_T)$. We can see that $\chi(G(\tau^*_T))) \leq k$. □

Corollary 4.4. *Assume that $e \in R$ is an idempotent element and $\bar{M}$ is a multiplication module. Then $G(\tau^*_T)$ is a complete bipartite graph if and only if $\bar{M}_1$ and $\bar{M}_2$ are prime modules.*

Proof. Assume that $G(\tau^*_T)$ is a complete bipartite graph. Therefore $G(\tau^*_T)$ is a triangle-free graph. So Lemma 3.3 follows that $\bar{M}_1$ and $\bar{M}_2$ are prime modules. The conversely holds by Proposition 3.1 (b). □

Remark 4.5. *Assume that S is a multiplicatively closed subset of R such that $S \cap (\cup_{P\in T}(P : M)) = \emptyset$. Let $T_S = \{S^{-1}P : P \in T\}$. One can see that $V^*(N) = T$ if and only if $V^*(S^{-1}N) = T_S$, where M is a finitely generated module.*

Theorem 4.6. *Let S be a multiplicatively closed subset of R defined in Remark 4.5 and M is a finitely generated module. Then $G(\tau^*_{T_S})$ is a retract of $G(\tau^*_T)$ and $\omega(G(\tau^*_{T_S})) = \omega(G(\tau^*_T))$.*

Proof. Consider a vertex map $\phi : V(G(\tau^*_T)) \longrightarrow V(G(\tau^*_{T_S})), N \longrightarrow N_S$. Clearly, $N_S \neq K_S$ implies that $N \neq K$ and $V^*(N) \cup V^*(K) = T$ if and only if $V^*(N_S) \cup V^*(K_S) = T_S$. Thus ϕ is surjective and hence $\omega(G(\tau^*_{T_S})) \leq \omega(G(\tau^*_T))$. If $N \neq K$ and $V^*(N) \cup V^*(K) = T$, then we show that $N_S \neq K_S$. On the contrary suppose that $N_S = K_S$. Then $V^*(N_S) = V^*(\sqrt{N_S}) = V^*(\sqrt{N_S} \cap \sqrt{K_S}) = V^*(N_S) \cup V^*(K_S) = T_S$ and so $V^*(N) = T$, a contradiction. This shows that the map ϕ is a graph homomorphism. Now, for any vertex N_S of $G(\tau^*_{T_S})$, we can choice a fixed vertex N of $G(\tau^*_T)$. Then ϕ is a retract (graph) homomorphism which clearly implies that $\omega(G(\tau^*_{T_S})) = \omega(G(\tau^*_T))$ under the assumption. □

Corollary 4.7. *Let S be a multiplicatively closed subset of R defined in Remark 4.5 and let M be a finitely generated module. Then $\chi(AG(M_S)) = \chi(AG(M))$.*

Corollary 4.8. *Assume that M is a semiprime module and $AG(M)^*$ does not have an infinite clique. Then M is a faithful module and $0 = (P_1 \cap \ldots \cap P_k : M)$, where P_i is a prime submodule of M for $i = 1, \ldots, k$.*

Proof. By [6, Theorem 3.7 (b)], M is a faithful module and the last assertion follows directly from the proof of [6, Theorem 3.7 (b)]. □

Proposition 4.9. *Let $\bar{M}$ be a cyclic module and let T be a closed subset of $Spec(M)$. We have the following statements.*

(a) If $\{P_1, \ldots, P_n\} \subseteq Min(T)$, then there exists a clique of size n in $G(\tau^*_T)$.

(b) We have $\omega(G(\tau^*_T)) \geq |Min(T)|$ and if $|Min(T)| \geq 3$, then $gr(G(\tau^*_T)) = 3$.

(c) If $\sqrt{(\bar{0})} = (\bar{0})$, then $\chi(G(\tau_{Spec(M)})) = \omega(G(\tau^*_{Spec(M)})) = |Min(T)|$.

Proof. (a) The proof is straightforward by the facts that $AG(\bar{M}) = AG(\bar{M})^*$ has a clique of size n by [7, Theorem 2.18] and $AG(\bar{M})$ is isomorphic with a subgraph of $G(\tau^*_T)$ by Lemma 2.6.

(b) This is clear by item (a).

(c) If $|Min(T)| = \infty$, then by Proposition 4.9 (b), there is nothing to prove. Otherwise, [7, Theorem 2.20] implies that $AG(\bar{M})$ does not have an infinite clique. So $\bar{M}$ is a faithful module by Corollary 4.8. Next, Lemma 2.7 says that $G(\tau^*_{Spec(M)})$ and $AG(M)$ are the same. Now the result follows by [7, Theorem 2.20]. □

Lemma 4.10. *Assume that $\bar{M}$ is a semiprime multiplication module. Then the following statements are equivalent.*

(a) $\chi(G(\tau^*_{Spec(M)})))$ is finite.

(b) $\omega(G(\tau_{Spec(M)})))$ is finite.

(c) $G(\tau^*_{Spec(M)}))$ does not have an infinite clique.

Proof. $(a) \Longrightarrow (b) \Longrightarrow (c)$ is clear.

$(c) \Longrightarrow (d)$ Suppose that $G(\tau^*_{Spec(M)}))$ does not have an infinite clique. By Lemma 2.6, $AG(\bar{M})$ does not have an infinite clique and so by Corollary 4.8, there exists a finite number of prime submodules $P_1, ..., P_k$ of M such that $\cap_{P \in T} P = P_1 \cap \ldots \cap P_k$. Define a coloring $f(N) = min\{n \in \mathbb{N} | \ P_n \notin V^*(N)\}$, where N is a vertex of $G(\tau^*_T)$. Then we have $\chi(G(\tau^*_{Spec(M)}))) \leq k$. □

Corollary 4.11. *Assume that $\bar{M}$ is a multiplication module and $AG(\bar{M})$ does not have an infinite clique. Then $G(\tau^*_{Spec(M)})$ and $AG(M)^*$ are the same. Also, $\chi(G(\tau^*_{Spec(M)})))$ is finite.*

Proof. Since $\bar{M}$ is a semiprime module, by Corollary 4.8, $\bar{M}$ is a faithful module and there exists a finite number of prime submodules $P_1, ..., P_k$ of M such that $\cap_{P \in T} P = P_1 \cap \ldots \cap P_k$. So the result follows by Lemma 2.7 and from the proof of $(c) \Longrightarrow (d)$ of Lemma 4.10. □

Proposition 4.12. *Suppose that $\sqrt{(\bar{0})} = (\bar{0})$ and $\bar{M}$ is a multiplication module. Then the following statements are equivalent.*

(a) $\chi(G(\tau^*_{Spec(M)}))$ is finite.

(b) $\omega(G(\tau^*_{Spec(M)}))$ is finite.

(c) $G(\tau^*_{Spec(M)})$ does not have an infinite clique.

(d) $Min(T)$ is a finite set.

Proof. $(a) \Longrightarrow (b) \Longrightarrow (c)$ is clear.

$(c) \Longrightarrow (d)$ Suppose $G(\tau^*_{Spec(M)})$ does not have an infinite clique. By Lemma 2.6, $AG(\bar{M})$ does not have an infinite clique and hence by Corollary 4.8, there exists a finite number of prime submodules $P_1, ..., P_k$ of M such that $\cap_{P \in T} P = P_1 \cap P_2 \cap ... \cap P_k$. By assumptions, one can see that $Min(T)$ is a finite set.

$(d) \Longrightarrow (a)$ Assume that $Min(T)$ is a finite set (equivalently, $\bar{M}$ has a finite number of minimal prime submodules) so that $\cap_{P \in T} P = P_1 \cap P_2 \cap ... \cap P_k$, where $Min(T) = \{P_1, ..., P_k\}$. Define a coloring $f(N) = min\{n \in N | P_n \notin V^*(N)\}$, where N is a vertex of $G(\tau^*_{Spec(M)})$. Then we have $\chi(G(\tau^*_{Spec(M)})) \leq k$. □

Proposition 4.13. *Assume that* $\sqrt{(\bar{0})} = (\bar{0})$ *and* $\bar{M}$ *is a faithful multiplication module. Then the following statements are equivalent.*

(a) $\chi(G(\tau^*_{Spec(M)}))$ *is finite.*

(b) $\omega(G(\tau^*_{Spec(M)}))$ *is finite.*

(c) $G(\tau^*_{Spec(M)})$ *does not have an infinite clique.*

(d) R has a finite number of minimal prime ideals.

(e) $\chi(G(\tau^*_{Spec(M)})) = \omega(G(\tau^*_{Spec(M)})) = |Min(R)| = k$, *where k is finite.*

Proof. This is clear by Lemma 2.7, [6, Proposition 3.11], and [6, Corollary 3.12]. □

Acknowledgments. We would like to thank the referees for valuable comments and the careful reading of our manuscript.

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