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On Non-Linear Contractions via Extended C_F-Simulation Functions

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Abstract. On grounds of the notion of simulation functions, in this manuscript, we bring in the concept of an extended C_F-simulation function and conceive a few common fixed point results via such kind of contractions on complete metric spaces. These class of auxiliary functions generalize, improve and extend those of simulation functions, extended simulation functions and C_F-simulation functions. However, as applications of the aforesaid results, we figure out some related consequences of it on the said spaces. Our findings are authenticated by the aid of some competent, non-trivial and constructive examples.

1. Introduction

The genesis of metric fixed point theory on complete metric spaces is allied with Banach contraction principle due to Stefan Banach [8], presented in 1922. This principle is one of very pre-eminent tests for the existence and uniqueness of the solution of elementary problems emerging in mathematics. Because of its potential implications in mathematical sciences, this theorem has been considered, discussed, improved and generalized in many different approaches (see [6, 7, 11, 13, 16, 22]).

In 2015, Khojasteh et al. [19] got the notion of simulation functions rolling and revealed a large class of functions, $\mathcal{Z}$-contraction, using a specific simulation function. Motivated by this dynamic concept, in 2016, A.F. Roldán and Bessem Samet [26] proposed the concept of extended simulation functions and acquired a φ-admissibility result concerning such kind of control functions. The obtained result is then implemented to secure some fixed point theorems, where X is equipped with a partial metric p.

Again, with a sense of purpose to generalize many fixed point theorems and enrich the literature, of late, Ansari [2] brought about the idea of C-class functions. Subsequently, Liu et al. [21] banked on these functions to extend the idea of simulation functions, marked them as C_F-simulation functions and enquired into the existence and uniqueness of coincidence points for two non-linear operators. In recent years, the notion of simulation functions, C_F-simulation functions have implicated wide-ranging fascination from mathematicians, more than ever from fixed point theorists [4, 5, 9, 17, 18, 20].

The intent of our draft is to make use of the theories from [19] and needless to say, the idea of extended simulation functions [26] to furnish a couple of related coincidence point results in the framework of metric

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spaces. To achieve these results, we conceive the notion of extended C_F-simulation functions and illustrate the definition by some non-trivial examples. Besides, we construct pertinent examples and deduce several related and existing results to demonstrate the applicability of our obtained theorem.

2. Preliminaries

We get under way with a brief recollection of elemental notions and some results compiled from [2, 12, 14, 19, 21, 23, 26]. Precisely, all through this paper, $\mathbb{N}$ will represent the set of all positive integers and $\mathbb{R}$ will mean the set of all real numbers.

In 2015, Khojasteh et al. [19] presented the notion of a simulation function. Afterwards, A.F. Roldán and Bessem Samet [26] instigated the class of extended simulation functions which reasonably enlarge the collection obtained in [19]. Here we come up with the definition.

Definition 2.1. *[26] An extended simulation function is a mapping $\theta : [0,\infty)^2 \to \mathbb{R}$ such that the following conditions hold:*

($\theta1$) $\theta(t,s) < s - t$ for each $t,s > 0$;

($\theta2$) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0,\infty),$$

and $s_n > \ell$, $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} \theta(t_n, s_n) < 0;$$

($\theta3$) let $\{t_n\}$ be a sequence in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \ell \in [0,\infty), \ \ \theta(t_n,\ell) \geq 0, \ \ n \in \mathbb{N},$$

then

$$\ell = 0.$$

The family of all extended simulation functions is denoted by $\varepsilon_{\mathcal{Z}}$. Every simulation function is also an extended simulation function. But the converse is not true.

Example 2.2. *We note down a couple of examples of extended simulation functions from the existing literature here.*

1. $\theta_1(t,s) = \psi(s) - t$ *for all* $t,s \in [0,\infty)$, *where* $\psi : [0,\infty) \to [0,\infty)$ *is upper semi-continuous from the right such that* $\psi(t) < t$ *for all* $t > 0$.
2. $\theta_2(t,s) = \frac{3}{4}s - t$ *for all* $t,s \in [0,\infty)$.
3. $$\theta_3(t,s) = \begin{cases} 1 - t, & \text{where } s = 0; \\ \frac{s}{2} - t, & s > 0, \end{cases}$$

   where $t,s \in [0,\infty)$.

For detailed terminologies, examples and more relevant results the readers are referred to [26].

Ansari [2], in his manuscript, first attempted to define the C-class functions. We put it down here.

Definition 2.3. *[2] A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and satisfies following axioms:*

1. $F(s,t) \leq s$,

2. *$F(s,t) = s$ implies that either $s = 0$ or $t = 0$, for all $s, t \in [0,\infty)$.*

Here we must note that for some F, we consider that $F(0,0) = 0$. The collection of C-class functions is denoted as $\mathcal{C}$.

Example 2.4. *The following functions $F_i : [0,\infty)^2 \to \mathbb{R}$ are some members of $\mathcal{C}$.*

1. $F_1(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$,
2. $F_2(s,t) = (s+l)^{\frac{1}{(1+t)^r}} - l$, *where* $l > 1$, $r \in (0,\infty)$.

For many more examples of C-class functions, see [2, 3].

Definition 2.5. *[21] A mapping $F : [0,\infty)^2 \to \mathbb{R}$ a has property C_F, if there exists a $C_F \geq 0$ such that*

1. $F(s,t) > C_F \Rightarrow s > t$,
2. $F(t,t) \leq C_F$, *for all* $s, t \in [0,\infty)$.

Example 2.6. *The following functions $F_i : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$ with property C_F, for all $s, t \in [0,\infty)$.*

1. $F_1(s,t) = \frac{s}{1+t}$, $C_F = 1, 2$.
2. $F_2(s,t) = \frac{s}{(1+t)^r}$, $r \in (0,\infty)$, $C_F = 1$.
3. $F_3(s,t) = s - t$, $C_F = r$, $r \in [0,\infty)$.

Now we present the notion of a C_F-simulation function using the C-class functions with property C_F. This is a proper generalization of the idea of the simulation functions coined by Khojasteh et al. in [19].

Definition 2.7. *[21] A C_F-simulation function is a mapping $\zeta : [0,\infty)^2 \to \mathbb{R}$ satisfying the following conditions:*

(ζ_a) $\zeta(0,0) = 0$,

(ζ_b) $\zeta(t,s) < F(s,t)$ *for all* $t, s > 0$, *where* $F \in \mathcal{C}$ *with property* C_F,

(ζ_c) *if* $\{t_n\}, \{s_n\}$ *are sequences in* $(0,\infty)$ *such that*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0,$$

and $t_n < s_n$, *then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < C_F.$$

The family of all C_F-simulation functions is denoted by $\mathcal{Z}_F$. Every simulation function is also a C_F-simulation function. The reverse inference may not be true, in general.

Example 2.8. *[21] Let $\zeta : [0,\infty)^2 \to \mathbb{R}$ be a function defined by $\zeta(t,s) = kF(s,t)$, where $t, s \in [0,\infty)$ and $k \in \mathbb{R}$ be such that $k < 1$ and for all $t, s \in [0,\infty)$. Considering $C_F = 1$, ζ is a C_F-simulation function.*

Choosing $F(s,t) = \frac{s}{1+t}$, we obtain $\zeta(t,s) = \frac{ks}{1+t}$ is also a C_F-simulation function with $C_F = 1$.

Example 2.9. *[21] Let $F : [0,\infty)^2 \to \mathbb{R}$ be a C-class function such that*

$$F(\psi(s), \varphi(t)) - t < F(s,t), \quad \psi(t) < t,$$

and let $\zeta : [0,\infty)^2 \to \mathbb{R}$ be the function defined as

$$\zeta(t,s) = F(\psi(s), \varphi(t)) - t.$$

Then $\zeta(t,s)$ is a C_F-simulation function with $C_F = 0$.

Now we recall the definition of a weakly compatible map.

Definition 2.10. *[15] Two self-maps f and g on a metric space (X, d) are said to be weakly compatible if $fgx = gfx$ for all x where $fx = gx$.*

Now, here we make a note of the following well-known result due to Abbas and Jungck [1] which is playing a crucial role in this sequel.

Theorem 2.11. *Let T and S be weakly compatible self-maps defined on a non-empty set X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is a unique common fixed point of T and S.*

Here we put forward the notions of Geraghty functions and Geraghty contractions which were discussed by Geraghty [12].

Definition 2.12. *[12] A function $\beta : [0, \infty) \to (0, 1)$ is called a Geraghty function if $\{r_n\} \subset [0, \infty)$ and $\lim_{n\to\infty} \beta(r_n) = 1^-$ implies $r_n \to 0^+$ as $n \to \infty$.*

Definition 2.13. *[12] A mapping $T : X \to X$ is called a Geraghty contraction if there exists a Geraghty function β such that*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y) \tag{1}$$

for all $x, y \in X$.

3. Common fixed point results via extended C_F-simulation functions

To take this section forward, we firstly illustrate the definition of an extended C_F-simulation function. Subsequently, we demonstrate several common fixed point result via such kind of control functions in the framework of complete metric spaces.

At the very beginning, we introduce the notion of an extended C_F-simulation function which is as follows:

Definition 3.1. *An extended C_F-simulation function is a mapping $\zeta : [0, \infty)^2 \to \mathbb{R}$ satisfying the following conditions:*

(θ1) $\theta(t, s) < F(s, t)$ for all $t, s > 0$, where $F \in \mathcal{C}$ with property C_F;

(θ2) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell,$$

where $\ell \in (0, \infty)$ and $s_n > \ell$ for all $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} \theta(t_n, s_n) < C_F;$$

(θ3) if $\{t_n\}$ be a sequence in $(0, \infty)$, such that

$$\lim_{n\to\infty} t_n = \ell \in [0, \infty), \ \ \theta(t_n, \ell) \geq C_F \Rightarrow \ell = 0.$$

The class of extended C_F-simulation functions is denoted by $\mathcal{E}_{(\mathcal{Z},F)}$. The subsequent example substantiates our previous definition.

Example 3.2. *Let $\theta : [0, \infty)^2 \to \mathbb{R}$ be a function defined by $\theta(t, s) = \frac{3}{4}s - t$, where $t, s \in [0, \infty)$. Considering $F(s, t) = s - t$ with $C_F = 1$, for all $t, s \in [0, \infty)$, we confirm that (θ1) is verified.*

Now if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l > 0$$

and $s_n > l$ for all $n \in \mathbb{N}$, then we obtain

$$\begin{aligned}\limsup_{n\to\infty} \theta(t_n, s_n) =& \limsup_{n\to\infty} \left[\frac{3}{4}s_n - t_n\right]\\ =& \frac{-l}{4}\\ <& C_F = 1.\end{aligned}$$

Therefore $\theta(t,s) = \frac{3}{4}s - t$ meets the condition ($\theta2$). We now check for ($\theta3$).

We choose a sequence $\{t_n\}$ in $(0, \infty)$ with

$$\lim_{n\to\infty} t_n = l \geq 0$$

for all $n \in \mathbb{N}$ such that

$$\begin{aligned}\theta(t_n, l) \geq C_F =& 1\\ =& \frac{3}{4}l - t_n \geq 1\\ \Rightarrow t_n \leq& \frac{3}{4}l - 1.\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} l \leq& \frac{3}{4}l - 1\\ \Rightarrow \frac{l}{4} \leq& -1\\ \Rightarrow l \leq& -4\end{aligned}$$

which is a contradiction to the fact that $l \geq 0$. Hence $\theta(t,s) = \frac{3}{4}s - t$ satisfies all the criteria of Definition 3.1 and so is an extended C_F-simulation function.

Here we put down two consequential propositions and establish the correlation between simulation functions, extended simulation functions and extended C_F-simulation functions.

Proposition 3.3. *A simulation function is an extended C_F-simulation function.*

Proof. Let $\zeta : [0, \infty)^2 \to \mathbb{R}$ be a simulation function. Then choosing $F(s,t) = s - t$ with $C_F = 0$, we get

$$\zeta(t,s) < F(s,t)$$

for all $t, s \in [0, \infty)$. This implies ($\theta1$) is verified.

Given two sequences $\{t_n\}, \{s_n\}$ in $(0, \infty)$ with

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l > 0$$

and $s_n > l$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\limsup_{n\to\infty} \zeta(t_n, s_n) <& 0\\ =& C_F,\end{aligned}$$

as $F(s,t) = s - t$ is a C-class function with $C_F = 0$. Hence ($\theta2$) is verified for $\zeta(t,s)$.

Now we pick a sequence $\{t_n\}$ in $(0, \infty)$ with

$$\lim_{n\to\infty} t_n = l \geq 0$$

for all $n \in \mathbb{N}$ such that

$$\begin{aligned} \zeta(t_n, l) &\geq C_F = 0 \\ \Rightarrow \limsup_{n\to\infty} \zeta(t_n, l) &\geq 0, \end{aligned}$$

which contradicts with the earlier discussions that says

$$\limsup_{n\to\infty} \zeta(t_n, l) < 0.$$

This means $l = 0$. Therefore, $\zeta(t, s)$ is an extended C_F-simulation function. □

But the converse is not true, in general. The Example 3.6 endorses our claim.

Proposition 3.4. *An extended simulation function is an extended C_F-simulation function.*

Proof. An extended simulation function is an extended C_F-simulation function with $C_F = 0$. □

The converse to this proposition is not true always and we illustrate this claim by means of Example 3.6.

Proposition 3.5. *A C_F-simulation function is an extended C_F-simulation function.*

Proof. Since this proof is similar to that of Proposition 3.4, we skip it. □

The following example confirms that the reverse implication of the previous claim may not hold in general.

Example 3.6. *Let $\theta : [0, \infty)^2 \to \mathbb{R}$ be a function defined by*

$$\theta(t, s) = \begin{cases} 1 - \frac{t}{2}, & \text{where } s = 0; \\ \frac{ks}{1+t}, & s > 0, \end{cases}$$

where $t, s \in [0, \infty)$ and $k \in [0, 1)$.

Since $\theta(0, 0) = 1$, it is neither a simulation function nor a C_F-simulation function.

Taking $F(s, t) = \frac{s}{1+t}$ with $C_F = 1$, for all $t, s \in [0, \infty)$, we observe that $\theta(t, s)$ attains (θ1).

For two given sequences $\{t_n\}, \{s_n\}$ in $(0, \infty)$ with

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l > 0$$

and $s_n > l$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \limsup_{n\to\infty} \theta(t_n, s_n) &= \limsup_{n\to\infty} \left[\frac{ks_n}{1 + t_n}\right] \\ &= \frac{kl}{1 + l} \\ &< \frac{l}{1 + l} \\ &< 1 = C_F. \end{aligned}$$

Therefore $\theta(t, s)$ achieves (θ2). But it is very easy to inspect that

$$\begin{aligned} \limsup_{n\to\infty} \theta(t_n, s_n) &= \limsup_{n\to\infty} \left[\frac{ks_n}{1 + t_n}\right] \\ &= \frac{kl}{1 + l} \\ &< \frac{l}{1 + l} \\ &\nless 0. \end{aligned}$$

Therefore $\theta(t,s)$ is not an extended simulation function. We now check for ($\theta 3$).

Now we choose a sequence $\{t_n\}$ in $(0,\infty)$ with

$$\lim_{n\to\infty} t_n = l \geq 0$$

for all $n \in \mathbb{N}$ such that

$$\begin{aligned} \theta(t_n, l) &\geq C_F = 1 \\ \Rightarrow \frac{kl}{1+t_n} &> 1 \\ \Rightarrow 1 + t_n &< kl \\ \Rightarrow t_n &< kl - 1. \end{aligned}$$

As $n \to \infty$, we get,

$$\begin{aligned} l &< kl - 1 \\ \Rightarrow (1-k)l &< -1 \\ \Rightarrow l &< \frac{-1}{1-k} \\ &< 0 \end{aligned}$$

which is impossible as $l \geq 0$. Hence $\theta(t,s)$ satisfies all the hypotheses of Definition 3.1 and rightly so an extended C_F-simulation function.

Now we are in a position to state our one of the main results involving extended C_F-simulation functions.

Theorem 3.7. *Assume that $T, S : X \to X$ are two self-maps on a complete metric space (X,d) such that $T(X) \subseteq S(X)$ and the following conditions hold:*

(i) *there exists an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ such that for each $(x,y) \in X \times X$*

$$\theta(d(Tx,Ty), M(x,y)) \geq C_F \tag{2}$$

holds with $Sx \neq Sy$, where

$$M(x,y) = \max\left\{ d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \frac{d(Sx,Ty) + d(Tx,Sy)}{2} \right\},$$

(ii) *$(S(X),d)$ $(or(T(X),d))$ is closed.*

Then T and S have a unique coincidence point. And if T and S are weakly compatible, then these mappings possess a unique common fixed point.

Proof. We formulate the iteration of Picard-Jungck in X such that $Sx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Without loss of generality, we consider $Sx_n \neq Sx_{n+1}$ for all $n \in \mathbb{N}$. Since, if $Sx_n = Sx_{n+1}$, for some $n \in \mathbb{N}$, then x_n is a coincidence point and the statement is verified.

Firstly, we prove that

$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = 0. \tag{3}$$

Employing (2) and ($\theta 1$), with $x = x_n$ and $y = x_{n+1}$, we obtain

$$\begin{aligned} C_F &\leq \theta(d(Tx_n, Tx_{n+1}), M(x_n, x_{n+1})) \\ &< F(M(x_n, x_{n+1}), d(Tx_n, Tx_{n+1})) \\ \Rightarrow M(x_n, x_{n+1}) &> d(Tx_n, Tx_{n+1}) \end{aligned} \tag{4}$$

where

$$\begin{aligned}M(x_n, x_{n+1}) =& \max\Big\{d(Sx_n, Sx_{n+1}), d(Sx_n, Tx_n), d(Sx_{n+1}, Tx_{n+1}),\\ &\frac{d(Tx_n, Sx_{n+1}) + d(Tx_n, Sx_{n+1})}{2}\Big\}\\ =& \max\Big\{d(Sx_n, Sx_{n+1}), d(Sx_n, Sx_{n+1}), d(Sx_{n+1}, Sx_{n+2}),\\ &\frac{d(Sx_{n+1}, Sx_{n+1}) + d(Tx_n, Sx_{n+1})}{2}\Big\}\\ =& \max\Big\{d(Sx_n, Sx_{n+1}), d(Sx_{n+1}, Sx_{n+2}), \frac{d(Tx_n, Sx_{n+1})}{2}\Big\}. \end{aligned}\tag{5}$$

Using triangle inequality, we get

$$\frac{d(Tx_n, Sx_{n+1})}{2} \leq \max\{d(Sx_n, Sx_{n+1}), d(Sx_{n+1}, Sx_{n+2})\}. \tag{6}$$

Now, if $M(x_n, x_{n+1}) = d(Sx_{n+1}, Sx_{n+2})$, then from (4) we have,

$$\begin{aligned}d(Sx_{n+1}, Sx_{n+2}) >& d(Tx_n, Tx_{n+1})\\ >& d(Sx_{n+1}, Sx_{n+2}),\end{aligned}$$

which is absurd. Therefore $M(x_n, x_{n+1}) = d(Sx_n, Sx_{n+1})$. Employing (4) and the above statement, we obtain

$$\begin{aligned}& d(Tx_n, Tx_{n+1}) < d(Sx_n, Sx_{n+1})\\ \Rightarrow\ & d(Sx_{n+1}, Sx_{n+2}) < d(Sx_n, Sx_{n+1})\end{aligned}\tag{7}$$

for all $n \in \mathbb{N}$. This implies that

$$\{d(Sx_n, Sx_{n+1})\}$$

is a decreasing sequence of positive reals. Thus, there is a real number $r \geq 0$ such that

$$\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = r. \tag{8}$$

We consider $r > 0$. Then, we consider two sequences $\{t_n\}$ and $\{s_n\}$ with same positive limit where

$$t_n = d(Tx_n, Tx_{n+1}) > 0$$

and

$$s_n = d(Sx_n, Sx_{n+1}) > 0$$

for all $n \in \mathbb{N}$ and $s_n > r$ for all $n \in \mathbb{N}$. Finally we obtain from ($\theta 2$),

$$\begin{aligned}C_F \leq& \limsup_{n\to\infty} \theta(d(Tx_n, Tx_{n+1}), d(Sx_n, Sx_{n+1}))\\ <& C_F,\end{aligned}$$

which leads to a contradiction. So we conclude that $r = 0$ and

$$\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = 0. \tag{9}$$

Now we utilize Lemma 2.1 of [24] in this context. We know $\{Sx_n\}$ is a sequence in (X, d) such that (9) holds. Then, if $\{Sx_n\}$ is not a Cauchy sequence in (X, d), then there exist $\epsilon_0 > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of natural numbers with $m_k > n_k > k$,

$$d(Sx_{m_k}, Sx_{n_k}) > \epsilon_0,\ \ d(Sx_{m_k}, Sx_{n_k-1}) \leq \epsilon_0$$

and

$$\lim_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) = \epsilon_0 \tag{10}$$

and

$$\lim_{k\to\infty} d(Sx_{m_k+1}, Sx_{n_k+1}) = \epsilon_0. \tag{11}$$

We have

$$\begin{aligned} M(x_{m_k}, x_{n_k}) = \max\Big\{ & d(Sx_{m_k}, Sx_{n_k}), \\ & d(Sx_{m_k}, Tx_{m_k}), d(Sx_{n_k}, Tx_{n_k}), \\ & \frac{d(Sx_{m_k}, Tx_{n_k}) + d(Tx_{m_k}, Sx_{n_k})}{2}\Big\}. \end{aligned} \tag{12}$$

Passing $k \to \infty$ in (12) and using (10) and (11), we obtain

$$\lim_{k\to\infty} M(x_{m_k}, x_{n_k}) = \epsilon_0. \tag{13}$$

Indeed, we take two sequences $\{t_k\}$ and $\{s_k\}$ with

$$t_k = d(Tx_{m_k}, Tx_{n_k}) = d(Sx_{m_k+1}, Sx_{n_k+1}) > 0$$

and

$$s_k = M(x_{m_k}, x_{n_k}) > 0,$$

for all $k \in \mathbb{N}$. Also we have

$$M(x_{m_k}, x_{n_k}) \geq d(Sx_{m_k}, Sx_{n_k}) > \epsilon_0,$$

for all $k \in \mathbb{N}$. Applying ($\theta 2$) we get,

$$\begin{aligned} C_F \leq & \limsup_{k\to\infty} \theta(d(Sx_{m_k+1}, Sx_{n_k+1}), M(x_{m_k}, x_{n_k})) \\ < & C_F, \end{aligned}$$

which is a contradiction. Hence $\{Sx_n\}$ is a Cauchy sequence. Taking into the completeness of $(S(X), d)$, there exists $w \in X$ such that

$$\lim_{n\to\infty} Sx_n = Sw.$$

We claim that w is a coincidence point of T and S. We consider that

$$d(Tw, Sw) = \ell > 0.$$

Also we have

$$\lim_{n\to\infty} d(Tx_n, Tw) = \ell.$$

Here,

$$M(x_n, w) = \max\Big\{d(Sx_n, Sw), d(Sx_n, Tx_n), d(Sw, Tw), \frac{d(Sx_n, Tw) + d(Tx_n, Sw)}{2}\Big\} \tag{14}$$

and

$$\lim_{n\to\infty} M(x_n, w) = d(Tw, Sw).$$

So, using ($\theta 3$), and (2), we get for all $n \in \mathbb{N}$ with $n \geq n_0$,

$$\begin{aligned} \theta(d(Tx_n, Tu), \ell) =& \theta(d(Tx_n, Tu), M(x_n, w)) \\ >& C_F \\ \Rightarrow \ell =& 0 \\ \Rightarrow d(Sw, Tw) =& 0 \\ Sw =& Tw, \end{aligned} \tag{15}$$

and w is a coincidence point of S and T.

Now, we establish the uniqueness of the coincidence point. Suppose that there exist $w_1, w_2 \in X$ such that $Ss_1 = Ts_1 = w_1$, $Ss_2 = Ts_2 = w_2$ and $w_1 \neq w_2$.

Using (2) and ($\theta 1$) with $x = s_1$ and $y = s_2$, we get

$$C_F \leq \theta(d(Ts_1, Ts_2), M(s_1, s_2)) \tag{16}$$

where

$$\begin{aligned} M(s_1, s_2) =& \max\left\{d(Ss_1, Ss_2), d(Ss_1, Ts_1), d(Ss_2, Ts_2), \frac{d(Ss_1, Ts_2) + d(Ts_1, Ss_2)}{2}\right\} \\ =& d(Ss_1, Ss_2) \\ =& d(w_1, w_2). \end{aligned} \tag{17}$$

Hence from (16), we obtain

$$\begin{aligned} C_F &\leq \theta(d(w_1, w_2), d(w_1, w_2)) \\ &< F(d(w_1, w_2), d(w_1, w_2)) \\ &< C_F, \end{aligned} \tag{18}$$

which is absurd and hence $w_1 = w_2$. So T and S possess a unique coincidence point. Since, these mappings are weakly compatible, employing Theorem 2.11, we can conclude that they have a unique common fixed point. □

The ensuing theorem is another common fixed point result concerning Geraghty functions and extended C_F-simulation functions.

Theorem 3.8. *Let $T, S : X \to X$ be two self-mappings defined on any complete metric space (X, d) such that $T(X) \subseteq S(X)$. Assume that the following conditions hold:*

(i) *there exist an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ and a Geraghty function $\beta : [0, \infty) \to (0, 1)$ such that for each $(x, y) \in X \times X$*

$$\theta(d(Tx, Ty), \beta(M(x, y))M(x, y)) \geq C_F \tag{19}$$

holds with $Sx \neq Sy$, where

$$M(x, y) = \max\left\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), \frac{d(Sx, Ty) + d(Tx, Sy)}{2}\right\},$$

(ii) *$(S(X), d)$ $(or(T(X), d))$ is closed.*

Then T and S possess a unique coincidence point. And if T and S are weakly compatible, then these mappings possess a unique common fixed point.

Proof. We construct the iterative sequence of Picard-Jungck in X such that $Sx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Without loss of generality, we take $Sx_n \neq Sx_{n+1}$ for all $n \in \mathbb{N}$. Because, if $Sx_n = Sx_{n+1}$, for some $n \in \mathbb{N}$, then it implies that x_n is a coincidence point and we are done.

First of all, we claim that

$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = 0. \tag{20}$$

Using (19) and (θ1), with $x = x_n$ and $y = x_{n+1}$, we obtain

$$\begin{aligned} C_F \leq &\theta(d(Tx_n, Tx_{n+1}), \beta(M(x_n, x_{n+1}))M(x_n, x_{n+1})) \\ &< F(\beta(M(x_n, x_{n+1}))M(x_n, x_{n+1}), d(Tx_n, Tx_{n+1})) \\ \Rightarrow \beta(M(x_n, x_{n+1}))M(x_n, x_{n+1}) &> d(Tx_n, Tx_{n+1}) \end{aligned} \tag{21}$$

where

$$\begin{aligned} M(x_n, x_{n+1}) = &\max\Big\{d(Sx_n, Sx_{n+1}), d(Sx_n, Tx_n), d(Sx_{n+1}, Tx_{n+1}), \\ &\frac{d(Tx_n, Sx_{n+1}) + d(Tx_n, Sx_{n+1})}{2}\Big\} \\ = &\max\Big\{d(Sx_n, Sx_{n+1}), d(Sx_n, Sx_{n+1}), d(Sx_{n+1}, Sx_{n+2}), \\ &\frac{d(Sx_{n+1}, Sx_{n+1}) + d(Tx_n, Sx_{n+1})}{2}\Big\} \\ = &\max\Big\{d(Sx_n, Sx_{n+1}), d(Sx_{n+1}, Sx_{n+2}), \frac{d(Tx_n, Sx_{n+1})}{2}\Big\}. \end{aligned} \tag{22}$$

From triangle inequality, we get

$$\frac{d(Tx_n, Sx_{n+1})}{2} \leq \max\{d(Sx_n, Sx_{n+1}), d(Sx_{n+1}, Sx_{n+2})\}. \tag{23}$$

Now, if $M(x_n, x_{n+1}) = d(Sx_{n+1}, Sx_{n+2})$, then from (21) we have,

$$\begin{aligned} \beta(d(Sx_{n+1}, Sx_{n+2}))d(Sx_{n+1}, Sx_{n+2}) &> d(Tx_n, Tx_{n+1}) \\ &> d(Sx_{n+1}, Sx_{n+2}) \\ \Rightarrow \beta(d(Sx_{n+1}, Sx_{n+2})) &> 1, \end{aligned}$$

which is impossible. Hence $M(x_n, x_{n+1}) = d(Sx_n, Sx_{n+1})$. Making use of (21) and the previous arguments, we get

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &< \beta(d(Sx_n, Sx_{n+1}))d(Sx_n, Sx_{n+1}) \\ \Rightarrow d(Sx_{n+1}, Sx_{n+2}) &< \beta(d(Sx_n, Sx_{n+1}))d(Sx_n, Sx_{n+1}) \\ &< d(Sx_n, Sx_{n+1}). \end{aligned} \tag{24}$$

for all $n \in \mathbb{N}$. This implies that

$$\{d(Sx_n, Sx_{n+1})\}$$

is a decreasing sequence of positive real numbers. Thus, there is some $r \geq 0$ such that

$$\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = r. \tag{25}$$

Suppose that $r > 0$. Then it follows from the condition (θ1) with

$$t_n = d(Tx_n, Tx_{n+1}) > 0$$

and

$$s_n = \beta(d(Sx_n, Sx_{n+1}))d(Sx_n, Sx_{n+1}) > 0$$

for some arbitrary $n \in \mathbb{N}$, that

$$\begin{aligned} C_F \leq& \theta(d(Tx_n, Tx_{n+1}), \\ &\beta(d(Sx_n, Sx_{n+1}))d(Sx_n, Sx_{n+1})) \\ <& F(\beta(d(Sx_n, Sx_{n+1}))d(Sx_n, Sx_{n+1}), \\ &d(Sx_{n+1}, Sx_{n+2})) \\ \Rightarrow \beta(d(Sx_n, Sx_{n+1}))d(Sx_n, Sx_{n+1}) >& d(Sx_{n+1}, Sx_{n+2}) \\ \Rightarrow \frac{d(Sx_{n+1}, Sx_{n+2})}{d(Sx_n, Sx_{n+1})} <& \beta(d(Sx_n, Sx_{n+1})) \\ <& 1. \end{aligned}$$

By sandwich theorem,

$$\lim_{n\to\infty} \beta(d(Sx_n, Sx_{n+1})) = 1, \tag{26}$$

which implies that

$$\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = 0, \tag{27}$$

which is a contradiction to (25). Then we conclude that $r = 0$ and from (25), we have

$$\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = 0. \tag{28}$$

Now we make use of Lemma 2.1 of [24] in our context. We know $\{Sx_n\}$ is a sequence in (X, d) such that (28) holds. Then, if $\{Sx_n\}$ is not a Cauchy sequence in (X, d), then there exist $\epsilon > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that $m_k > n_k > k$,

$$d(Sx_{m_k}, Sx_{n_k}) > \epsilon, \;\; d(Sx_{m_k}, Sx_{n_k-1}) \leq \epsilon$$

and

$$\lim_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) = \epsilon \tag{29}$$

and

$$\lim_{k\to\infty} d(Sx_{m_k+1}, Sx_{n_k+1}) = \epsilon. \tag{30}$$

We have

$$\begin{aligned} M(x_{m_k}, x_{n_k}) = \max\Big\{& d(Sx_{m_k}, Sx_{n_k}), \\ & d(Sx_{m_k}, Tx_{m_k}), d(Sx_{n_k}, Tx_{n_k}), \\ & \frac{d(Sx_{m_k}, Tx_{n_k}) + d(Tx_{m_k}, Sx_{n_k})}{2}\Big\}. \end{aligned} \tag{31}$$

Letting $k \to \infty$ in (31) and using (29) and (30), we obtain

$$\lim_{k\to\infty} M(x_{m_k}, x_{n_k}) = \epsilon. \tag{32}$$

Indeed, we consider two sequences $\{t_k\}$ and $\{s_k\}$ with

$$t_k = d(Tx_{m_k}, Tx_{n_k}) = d(Sx_{m_k+1}, Sx_{n_k+1}) > 0$$

and

$$s_k = \beta(M(x_{m_k}, x_{n_k}))M(x_{m_k}, x_{n_k}).$$

Employing we get

$$\begin{aligned} C_F \leq& \theta(d(Sx_{m_k+1}, Sx_{n_k+1}),\\ &\beta(M(x_{m_k}, x_{n_k}))M(x_{m_k}, x_{n_k}))\\ <&F(\beta(M(x_{m_k}, x_{n_k}))M(x_{m_k}, x_{n_k}),\\ &d(Sx_{m_k+1}, Sx_{n_k+1}))\\ \Rightarrow \beta(M(x_{m_k}, x_{n_k}))M(x_{m_k}, x_{n_k}) >& d(Sx_{m_k+1}, Sx_{n_k+1})\\ \Rightarrow d(Sx_{m_k+1}, Sx_{n_k+1}) <& \beta(M(x_{m_k}, x_{n_k}))M(x_{m_k}, x_{n_k})\\ <& M(x_{m_k}, x_{n_k}). \end{aligned}$$

By sandwich theorem,

$$\lim_{k\to\infty} \beta(M(x_{m_k}, x_{n_k}))M(x_{m_k}, x_{n_k}) = \epsilon. \tag{33}$$

Also we have $\beta(M(x_{m_k}, x_{n_k}))M(x_{m_k}, x_{n_k}) \geq d(Sx_{m_k+1}, Sx_{n_k+1}) > \epsilon$. Now, using ($\theta 2$), we obtain that

$$\begin{aligned} C_F \leq& \limsup_{k\to\infty} \theta(t_k, s_k)\\ <& C_F, \end{aligned} \tag{34}$$

which is a contradiction. Hence $\{Sx_n\}$ is a Cauchy sequence. Since $(S(X), d)$ is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} Sx_n = Su.$$

Our claim is that u is a coincidence point of T and S. We consider that

$$d(Tu, Su) = l > 0.$$

Also we have

$$\lim_{n\to\infty} d(Tx_n, Tu) = l.$$

Now,

$$M(x_n, u) = \max\left\{d(Sx_n, Su), d(Sx_n, Tx_n), d(Su, Tu), \frac{d(Sx_n, Tu) + d(Tx_n, Su)}{2}\right\} \tag{35}$$

and

$$\lim_{n\to\infty} M(x_n, u) = d(Tu, Su).$$

Again, choosing two positive sequences $\{t_n\}$ and $\{s_n\}$ with $t_n = d(Tx_n, Tu) > 0$ and $s_n = \beta(M(x_n, u))M(x_n, u)$. Employing in ($\theta 1$), we get

$$\begin{aligned} C_F \leq& \theta(d(Tx_n, Tu), \beta(M(x_n, u))M(x_n, u))\\ <& F(\beta(M(x_n, u))M(x_n, u), d(Tx_n, Tu))\\ \Rightarrow \beta(M(x_n, u))M(x_n, u) >& d(Tx_n, Tu)\\ \Rightarrow d(Tx_n, Tu) <& \beta(M(x_n, u))M(x_n, u)\\ <& M(x_n, u). \end{aligned}$$

By sandwich theorem and using (35),

$$\lim_{k\to\infty} \beta(M(x_n, u))M(x_n, u) = d(Tu, Su). \tag{36}$$

So, using (θ3), and (19), we get for all $n \in \mathbb{N}$ with $n \geq n_1$,

$$\begin{aligned}\theta(d(Tx_n, Tu), l) =& \theta(d(Tx_n, Tu), \beta(M(x_n, u))M(x_n, u)) \\ >& C_F \\ \Rightarrow l =& 0 \\ \Rightarrow d(Su, Tu) =& 0 \\ Su =& Tu,\end{aligned} \tag{37}$$

and u is a coincidence point of S and T.

Now, we establish the uniqueness of the coincidence point. Assume that there exist $w_1, w_2 \in X$ with $Ss_1 = Ts_1 = w_1$, $Ss_2 = Ts_2 = w_2$ and $w_1 \neq w_2$.

Using (19) and (θ1) with $x = s_1$ and $y = s_2$, we get

$$C_F \leq \theta(d(Ts_1, Ts_2), \beta(M(s_1, s_2))M(s_1, s_2)) \tag{38}$$

where

$$\begin{aligned}M(s_1, s_2) =& \max\left\{d(Ss_1, Ss_2), d(Ss_1, Ts_1), d(Ss_2, Ts_2), \frac{d(Ss_1, Ts_2) + d(Ts_1, Ss_2)}{2}\right\} \\ =& d(Ss_1, Ss_2) \\ =& d(w_1, w_2).\end{aligned} \tag{39}$$

Therefore from (38), we obtain

$$\begin{aligned} C_F &\leq \theta(d(w_1, w_2), \beta(d(w_1, w_2))d(w_1, w_2)) \\ &< F(\beta(d(w_1, w_2))d(w_1, w_2), d(w_1, w_2)) \\ \Rightarrow \beta(d(w_1, w_2))d(w_1, w_2) &> d(w_1, w_2)\end{aligned} \tag{40}$$

which is impossible as $\beta(d(w_1, w_2)) < 1$. Hence $w_1 = w_2$ and so T and S have a unique coincidence point. As these mappings are weakly compatible, using Theorem 2.11, we can infer that they possess a unique common fixed point. □

Here we speak briefly of two almost identical results, which can be proved using similar arguments as Theorem 3.8.

Theorem 3.9. *Let $T, S : X \to X$ be two self-mappings defined on any complete metric space (X, d) with $T(X) \subseteq S(X)$. Assume that the following conditions hold:*

(i) *there exist an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ and a Geraghty function $\beta : [0, \infty) \to (0, 1)$ such that for each $(x, y) \in X \times X$*

$$\theta(d(Tx, Ty), \beta(M(x, y))M(x, y)) \geq C_F \tag{41}$$

holds with $Sx \neq Sy$, where

$$M(x, y) = \max\left\{d(Sx, Sy), \frac{d(Sx, Tx) + d(Sy, Ty)}{2}, \frac{d(Sx, Ty) + d(Tx, Sy)}{2}\right\},$$

(ii) *$(S(X), d)$ $(or(T(X), d))$ is closed.*

Then T and S possess a unique coincidence point. And if T and S are weakly compatible, then these mappings possess a unique common fixed point.

Theorem 3.10. *Let $T, S : X \to X$ be two self-mappings defined on any complete metric space (X, d) with $T(X) \subseteq S(X)$. Assume that the following conditions hold:*

(*i*) *there exist an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ and a continuous function $\psi : [0,\infty) \to (0,1)$ with $\psi(t) < t$ such that for each $(x, y) \in X \times X$*

$$\theta(d(Tx, Ty), \psi(M(x, y))) \geq C_F \tag{42}$$

holds with $Sx \neq Sy$, where

$$M(x, y) = \max\left\{d(Sx, Sy), \frac{d(Sx, Tx) + d(Sy, Ty)}{2}, \frac{d(Sx, Ty) + d(Tx, Sy)}{2}\right\},$$

(*ii*) $(S(X), d)$ $(or(T(X), d))$ *is closed.*

Then T and S possess a unique coincidence point. And if T and S are weakly compatible, then these mappings possess a unique common fixed point.

4. Consequences

This section takes care of some corollaries that can be perceived form our derived results. Some of these findings are new and some are existing in the literature.

Corollary 4.1. *[10] Let $T, S : X \to X$ be two self-mappings defined on any complete metric space (X, d) where $T(X) \subseteq S(X)$. Assume that the following conditions hold:*

(*i*) *there exist an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ and a Geraghty function $\beta : [0,\infty) \to (0,1)$ such that for each $(x, y) \in X \times X$*

$$\theta(d(Tx, Ty), \beta(M(x, y))M(x, y)) \geq 0$$

holds with $Sx \neq Sy$, where

$$M(x, y) = \max\left\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), \frac{d(Sx, Ty) + d(Tx, Sy)}{2}\right\},$$

(*ii*) $(S(X), d)$ $(or(T(X), d))$ *is closed.*

Then T and S possess a unique coincidence point. Moreover, if T and S are weakly compatible, then these mappings possess a unique common fixed point.

Proof. Choosing an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ with $C_F = 0$ in Theorem 3.8, we can easily obtain this result. □

Corollary 4.2. *Let $T, S : X \to X$ be two self-mappings defined on any complete metric space (X, d) where $T(X) \subseteq S(X)$. Assume that the following conditions hold:*

(*i*) *there exist an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ and a Geraghty function $\beta : [0,\infty) \to (0,1)$ such that for each $(x, y) \in X \times X$*

$$\theta(d(Tx, Ty), \beta(d(Sx, Sy))d(Sx, Sy)) \geq C_F$$

holds with $Sx \neq Sy$,

(*ii*) $(S(X), d)$ $(or(T(X), d))$ *is closed.*

Then T and S possess a unique coincidence point. Moreover, if T and S are weakly compatible, then these mappings possess a unique common fixed point.

Proof. When $M(x, y) = d(Sx, Sy)$, then for some extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$, we can establish this result employing Theorem 3.8. □

Corollary 4.3. *[10] Let $T, S : X \to X$ be two self-mappings defined on any complete metric space (X, d) with $T(X) \subseteq S(X)$. Assume that the following conditions hold:*

(i) *there exist an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ and a Geraghty function $\beta : [0, \infty) \to (0, 1)$ such that for each $(x, y) \in X \times X$*

$$\theta(d(Tx, Ty), \beta(d(Sx, Sy))d(Sx, Sy)) \geq 0$$

holds with $Sx \neq Sy$,

(ii) *$(S(X), d)$ $(or(T(X), d))$ is closed.*

Then T and S possess a unique coincidence point. Moreover, if T and S are weakly compatible, then these mappings possess a unique common fixed point.

Proof. If $M(x, y) = d(Sx, Sy)$, then picking an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ with $C_F = 0$ in Theorem 3.8, we can easily deduce this statement. □

Corollary 4.4. *Let $T : X \to X$ be a self-map defined on any complete metric space (X, d). Assume that the following condition holds:*

(i) *there exist an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ and a Geraghty function $\beta : [0, \infty) \to (0, 1)$ such that for each $(x, y) \in X \times X$*

$$\theta(d(Tx, Ty), \beta(d(x, y))d(x, y)) \geq C_F.$$

Then T possesses a unique fixed point.

Proof. Putting $Sx = x$ in Corollary 4.2, we get the desired result for some extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$. □

Corollary 4.5. *Let $T : X \to X$ be a self-map defined on any complete metric space (X, d). Assume that the following condition holds:*

(i) *there exist an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ and a Geraghty function $\beta : [0, \infty) \to (0, 1)$ such that for each $(x, y) \in X \times X$*

$$\theta(d(Tx, Ty), \beta(d(x, y))d(x, y)) \geq 0.$$

Then T possesses a unique fixed point.

Proof. Fixing $Sx = x$ in Corollary 4.2, we get the expected corollary for any extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ with $C_F = 0$. □

Corollary 4.6. *Let $T : X \to X$ be a self-map defined on any complete metric space (X, d). Assume that the following condition holds:*

(i) *there exist an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ and a Geraghty function $\beta : [0, \infty) \to (0, 1)$ such that for each $(x, y) \in X \times X$*

$$\theta(d(Tx, Ty), \beta(M(x, y))M(x, y)) \geq C_F$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2}\right\}.$$

Then T possesses a unique fixed point.

Proof. Considering $Sx = x$ in Theorem 3.8, we get this corollary for any extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$. □

Corollary 4.7. *Let $T : X \to X$ be any self-map defined on any complete metric space (X, d). Assume that the following condition holds:*

(*i*) *there exist an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ and a Geraghty function $\beta : [0,\infty) \to (0,1)$ such that for each $(x, y) \in X \times X$*

$$\theta(d(Tx,Ty), \beta(M(x,y))M(x,y)) \geq 0,$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(Tx,y)}{2}\right\}.$$

Then T possesses a unique fixed point.

Proof. Choosing $Sx = x$ and an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ with $C_F = 0$ and putting in Theorem 3.8, we can confirm this result. □

Corollary 4.8. *[10] Assume that $T, S : X \to X$ are two self-maps on a complete metric space (X, d) such that $T(X) \subseteq S(X)$ and the following conditions hold:*

(*i*) *there exists an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ such that for each $(x, y) \in X \times X$*

$$\theta(d(Tx,Ty), M(x,y)) \geq 0$$

holds with $Sx \neq Sy$, where

$$M(x,y) = \max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \frac{d(Sx,Ty)+d(Tx,Sy)}{2}\right\},$$

(*ii*) *$(S(X), d)$ (or$(T(X), d)$) is closed.*

Then T and S have a unique coincidence point. Moreover, if T and S are weakly compatible, then these mappings possess a unique common fixed point.

Proof. Considering an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ with $C_F = 0$ in Theorem 3.7, we can easily affirm the result. □

Corollary 4.9. *[23] Assume that $T, S : X \to X$ are two self-maps on a complete metric space (X, d) such that $T(X) \subseteq S(X)$ and the following conditions hold:*

(*i*) *there exists an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ such that for each $(x, y) \in X \times X$*

$$\theta(d(Tx,Ty), d(Sx,Sy)) \geq C_F$$

holds with $Sx \neq Sy$,

(*ii*) *$(S(X), d)$ (or$(T(X), d)$) is closed.*

Then T and S have a unique coincidence point. Moreover, if T and S are weakly compatible, then these mappings possess a unique common fixed point.

Proof. When $M(x, y) = d(Sx, Sy)$, then for some extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$, we can attain this consequence from Theorem 3.7. □

Corollary 4.10. *[25] Assume that $T, S : X \to X$ are two self-maps on a complete metric space (X, d) such that $T(X) \subseteq S(X)$ and the following conditions hold:*

(i) *there exists an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ such that for each $(x, y) \in X \times X$*

$$\theta(d(Tx, Ty), d(Sx, Sy)) \geq 0$$

holds with $Sx \neq Sy$,

(ii) *$(S(X), d)$ $(or(T(X), d))$ is closed.*

Then T and S have a unique coincidence point. Moreover, if T and S are weakly compatible, then these mappings possess a unique common fixed point.

Proof. If $M(x, y) = d(Sx, Sy)$, then for some extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ with $C_F = 0$, we can conceive this corollary from Theorem 3.7. □

Corollary 4.11. *Assume that $T : X \to X$ is a self-map on a complete metric space (X, d) such that the following condition holds:*

(i) *there exists an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ such that for each $(x, y) \in X \times X$*

$$\theta(d(Tx, Ty), M(x, y)) \geq C_F$$

holds, where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2}\right\}.$$

Then T has a unique fixed point.

Proof. Considering $Sx = x$ in Theorem 3.7, we get this result for any extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$. □

Corollary 4.12. *[4] Assume that $T : X \to X$ is a self-map on a complete metric space (X, d) such that the following condition holds:*

(i) *there exists an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ such that for each $(x, y) \in X \times X$*

$$\theta(d(Tx, Ty), d(x, y)) \geq C_F$$

holds.

Then T has a unique fixed point.

Proof. When $M(x, y) = d(x, y)$, then for some extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$, we obtain this result from Corollary 4.11. □

Corollary 4.13. *[19] Assume that $T : X \to X$ is a self-map on a complete metric space (X, d) such that the following conditions hold:*

(i) *there exists an extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ such that for each $(x, y) \in X \times X$*

$$\theta(d(Tx, Ty), d(x, y)) \geq 0$$

holds.

Then T has a fixed point.

Proof. For some extended C_F-simulation function $\theta \in \mathcal{E}_{(\mathcal{Z},F)}$ with $C_F = 0$, we obtain this result from Corollary 4.12. □

5. An Example

This section deals with a non-trivial example which illustrates one of our obtained result.

Example 5.1. *Consider the metric space l^∞ equipped with the usual metric. Take $X = \{e_0, e_i : i \in \mathbb{N}\}$ where e_0 is the zero sequence and e_i is the sequence whose i-th term is 4 and all the other terms are 0. Then one can easily check that X is complete.*

We define mappings $T : X \to X$ such that $Tx = e_0$ for all $x \in X$ and $S : X \to X$ such that

$$Sx = \begin{cases} e_0, & \text{where } x = e_0; \\ e_{i+1}, & x = e_i. \end{cases}$$

We also consider $\theta(t,s) = \frac{ks}{1+t}$, $t, s \in [0,\infty)$, as the extended C_F-simulation function, where $k = \frac{9}{10}$ and $C_F = 1$. It is easy to check that $d(Tx,Ty) = 0$ and $M(x,y) = 4$ for all $x, y \in X$ with $x \neq y$. Hence,

$$\begin{aligned} \theta(d(Tx,Ty), M(x,y)) =& \frac{kM(x,y)}{1+d(Tx,Ty)} \\ =& \frac{36}{10} \\ \geq& 1 = C_F. \end{aligned}$$

So, T and S satisfy all the hypotheses of Theorem 3.7 and using the theorem, T and S have a unique common fixed point and it is $w = e_0 \in X$.

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