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On a Conjecture of the Harmonic Index and the Minimum Degree of Graphs

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Abstract. The harmonic index of a graph G is defined as the sum of the weights $\frac{2}{d(u)+d(v)}$ of all edges uv of G, where $d(u)$ denotes the degree of the vertex u in G. Cheng and Wang [4] proposed a conjecture: For all connected graphs G with $n \geq 4$ vertices and minimum degree $\delta(G) \geq k$, where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$, then $H(G) \geq H(K^*_{k,n-k})$ with equality if and only if $G \cong K^*_{k,n-k}$. $K^*_{k,n-k}$ is a complete split graph which has only two degrees, i.e. degree k and degree $n-1$, and the number of vertices of degree k is $n-k$, while the number of vertices of degree $n-1$ is k. In this work, we prove that this conjecture is true when $k \leq \frac{n}{2}$, and give a counterexample to show that the conjecture is not correct when $k = \lfloor \frac{n}{2} \rfloor + 1$, n is even, that is $k = \frac{n}{2} + 1$.

1. Introduction

Throughout this paper we consider only simple connected graphs. Such a graph will be denoted by $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ are the vertex set and edge set of G, respectively. For a vertex v of a graph G, we denote the degree of v by $d_G(v)$ ($d(v)$ for short). The minimum degree of G is denoted by $\delta(G)$. We use $G - uv$ to denote the graph that arises from G by deleting the edge $uv \in E(G)$. Let $G(n, k)$ be the set of connected simple graphs of order n with minimum degree k. We use $K^*_{k,n-k}$ to denote the graph that arises from the complete bipartite graph $K_{k,n-k}$ by joining every pair of vertices in the partite set with k vertices by a new edge. Our other notations are standard and taken mainly from [2].

The Randić index $R(G)$, proposed by Randić [11] in 1975, is defined as the sum of the weights $\frac{1}{\sqrt{d(u)d(v)}}$ over all edges uv of G, that is,

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}}.$$

The Randić index is one of the most successful molecular descriptors in structure-property and structure-activity relationship studies. Mathematical properties of this descriptor have been studied extensively (see [7-9] and the references cited therein).

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Another variant of the Randić index is called the harmonic index. For a graph G, the harmonic index $H(G)$ is defined in [6] as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)}.$$

Zhong [13] determined the minimum and maximum values of the harmonic index for simple connected graphs and trees. Wu et al. [12] determined the graph with minimum harmonic index among all the graphs (or all triangle-free graphs) with minimum degree at least two. Chang and Zhu [3] gave the minimum value of the harmonic index for the graphs with minimum degree at least two and obtained a lower bound on the harmonic index of a triangle-free graph with arbitrary minimum degree. Cheng and Wang [4] determined the graph which attains the minimum harmonic index of connected graphs with minimum degree at least three, and posed the following Conjecture 1.1.

Conjecture 1.1. [4] *Let G be a graph with $n \geq 4$ vertices and $\delta(G) \geq k$, where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$. Then $H(G) \geq H(K^*_{k,n-k})$ with equality if and only if $G \cong K^*_{k,n-k}$.*

Ali [1] gave some counterexamples to show that the Conjecture 1.1 is not always true. In this work, using the methods in [5,10], we prove that Conjecture 1.1 is true when $k \leq \frac{n}{2}$, and give a counterexample to show that the conjecture is not correct when $k = \frac{n}{2} + 1$ (n is even).

For an edge $e = uv$ of a graph G with harmonic index $H(G)$, its weight is defined to be $\frac{2}{d(u)+d(v)}$. The harmonic index of G is the sum of weights over all its edges.

Lemma 1.2. [12] *If e is an edge with maximal weight in G, then $H(G - e) < H(G)$.*

Let G be a graph with $n \geq 4$ vertices and $\delta(G) \geq k$. If $\delta(G) > k$, then by Lemma 1.2, the deletion of an edge with maximal weight yields a graph G' of minimal degree at least k such that $H(G') < H(G)$. So we only need prove the Conjecture 1.1 is true for G with $\delta(G) = k$. We can modify the Conjecture 1.1 to obtain the following Conjecture 1.3.

Conjecture 1.3. *Let $G \in G(n,k)$, where $n \geq 4$, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$. Then $H(G) \geq H(K^*_{k,n-k})$ with equality if and only if $G \cong K^*_{k,n-k}$.*

2. Preliminaries

First, we will give some linear equalities and nonlinear inequalities which must be satisfied in any graph from the class $G(n,k)$. Denote by n_i the number of vertices of degree i , and by $x_{i,j}$ the number of edges joining the vertices of degrees i and j in $G(n,k)$. The mathematical description of the problem P to determining minimum of $H(G) = \sum_{k \leq i \leq j \leq n-1} \frac{2x_{i,j}}{i+j}$ is:

$$\min \sum_{k \leq i \leq j \leq n-1} \frac{2x_{i,j}}{i+j}$$

subject to:

$$\begin{aligned}
&2x_{k,k} + x_{k,k+1} + x_{k,k+2} + \cdots + x_{k,n-1} = kn_k,\\
&x_{k,k+1} + 2x_{k+1,k+1} + x_{k+1,k+2} + \cdots + x_{k+1,n-1} = (k+1)n_{k+1},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (1)\\
&x_{k,n-1} + x_{k+1,n-1} + x_{k+2,n-1} + \cdots + 2x_{n-1,n-1} = (n-1)n_{n-1},
\end{aligned}$$

$$n_k + n_{k+1} + n_{k+2} + \cdots + n_{n-1} = n, \tag{2}$$

$$x_{i,j} \leq n_i n_j, \text{ for } k \leq i \leq n-1,\ i < j \leq n-1, \tag{3}$$

$$x_{i,i} \leq \binom{n_i}{2}, \text{ for } k \leq i \leq n-1, \tag{4}$$

$$x_{i,j}, n_i \text{ are nonnegative integers, for } k \leq i \leq j \leq n-1. \tag{5}$$

Constraints (1) − (5) define a nonlinearly optimization problem.

If the first equality from (1) divide by k, second by $k+1$, third by $k+2$ and so on, the last by $n-1$ and sum them all, we get

$$\sum_{k\le i\le j\le n-1}\left(\frac{1}{i}+\frac{1}{j}\right)x_{i,j}=n_k+n_{k+1}+n_{k+2}+\cdots+n_{n-1}=n$$

because of (2). Then

$$\begin{aligned}H(G)=&\sum_{k\le i\le j\le n-1}\frac{2x_{i,j}}{i+j}=\frac{1}{2}\sum_{k\le i\le j\le n-1}\left(\frac{1}{i}+\frac{1}{j}+\frac{4}{i+j}-\frac{1}{i}-\frac{1}{j}\right)x_{i,j}\\ =&\frac{1}{2}\sum_{k\le i\le j\le n-1}\left(\frac{1}{i}+\frac{1}{j}\right)x_{i,j}-\frac{1}{2}\sum_{k\le i\le j\le n-1}\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)x_{i,j}\\ =&\frac{n}{2}-\frac{1}{2}\sum_{k\le i\le j\le n-1}\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)x_{i,j}=\frac{n}{2}-\frac{1}{2}\sum_{k\le i\le j\le n-1}\frac{(i-j)^2}{ij(i+j)}x_{i,j}\\ =&\frac{n}{2}-\frac{1}{2}\sum_{k\le i<j\le n-1}\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)x_{i,j}.\end{aligned}$$

Define

$$\gamma=\sum_{k\le i<j\le n-1}\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)x_{i,j}.$$

Henceforth we will consider the problem of maximizing γ instead of minimizing $H(G)$.

3. Main result

Theorem 3.1. *Let $G\in G(n,k)$, where $n\ge 4$, $1\le k\le\frac{n}{2}$. Then $H(G)\ge H(K^*_{k,n-k})$ with equality if and only if $G\cong K^*_{k,n-k}$.*

Proof. Since the minimum degree of G is k, it is obvious that $n_{n-1}\le k$. Let m be the index such that $n_m+n_{m+1}+\cdots+n_{n-2}+n_{n-1}\ge k$ and $n_{m+1}+\cdots+n_{n-2}+n_{n-1}<k$. We distinguish two cases: (1) $n_m+\cdots+n_{n-2}+n_{n-1}=k$, (2) $n_m+\cdots+n_{n-2}+n_{n-1}>k$.

Case 1. $n_m+\cdots+n_{n-2}+n_{n-1}=k$.

Note that

$$\begin{aligned}\gamma=&\sum_{k\le i<j\le n-1}\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)x_{i,j}=\sum_{j=k+1}^{n-1}\left(\frac{1}{k}+\frac{1}{j}-\frac{4}{k+j}\right)x_{k,j}+\sum_{j=k+2}^{n-1}\left(\frac{1}{k+1}+\frac{1}{j}-\frac{4}{k+1+j}\right)x_{k+1,j}+\cdots\\ &+\sum_{j=m}^{n-1}\left(\frac{1}{m-1}+\frac{1}{j}-\frac{4}{m-1+j}\right)x_{m-1,j}+\sum_{m\le i<j\le n-1}\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)x_{i,j}.\end{aligned}$$

Weights of all edges which join vertices of degree i, with vertices of degree j, $i+1\le j\le n-1$ are represented by $\sum_{j=i+1}^{n-1}(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j})x_{i,j}$. We give the maximum possible weights to these edges. Since $n_m+\cdots+n_{n-2}+n_{n-1}=k$ and $\sum_{j=i+1}^{n-1}x_{i,j}\le in_i$, first we join a vertex of degree i to all k vertices of degrees $n-1,\cdots,m$ (maximum weights) and with $i-k$ vertices of other degrees j, $i+1\le j\le m-1$. Furthermore, $h(x,y)=\frac{1}{x}+\frac{1}{y}-\frac{4}{x+y}$ is increasing for y, where $k\le x<y\le m-1$ (since $\frac{\partial h(x,y)}{\partial y}=\frac{(y-x)(3y+x)}{y^2(x+y)^2}>0$ for $k\le x<y\le m-1$). We will

maximize the weights of these last $i-k$ edges joining a vertex of degree i to $i-k$ vertices of degree $m-1$. Thus,

$$\begin{aligned}
&\sum_{j=i+1}^{n-1}\Big(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\Big)x_{i,j}\\
=&\sum_{j=m}^{n-1}\Big(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\Big)x_{i,j}+\sum_{j=i+1}^{m-1}\Big(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\Big)x_{i,j}\\
\leq&\sum_{j=m}^{n-1}\Big(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\Big)n_in_j+\Big(\frac{1}{i}+\frac{1}{m-1}-\frac{4}{i+m-1}\Big)(i-k)n_i\\
=&n_i\Bigg(\sum_{j=m}^{n-1}\Big(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\Big)n_j+\Big(\frac{1}{i}+\frac{1}{m-1}-\frac{4}{i+m-1}\Big)(i-k)\Bigg).
\end{aligned}$$

Then

$$\begin{aligned}
\gamma\leq& n_k\Bigg(\Big(\frac{1}{k}+\frac{1}{n-1}-\frac{4}{k+n-1}\Big)n_{n-1}+\Big(\frac{1}{k}+\frac{1}{n-2}-\frac{4}{k+n-2}\Big)n_{n-2}+\cdots+\Big(\frac{1}{k}+\frac{1}{m}-\frac{4}{k+m}\Big)n_m\Bigg)\\
&+n_{k+1}\Bigg(\Big(\frac{1}{k+1}+\frac{1}{n-1}-\frac{4}{k+n}\Big)n_{n-1}+\Big(\frac{1}{k+1}+\frac{1}{n-2}-\frac{4}{k+n-1}\Big)n_{n-2}+\cdots\\
&+\Big(\frac{1}{k+1}+\frac{1}{m}-\frac{4}{k+1+m}\Big)n_m+\Big(\frac{1}{k+1}+\frac{1}{m-1}-\frac{4}{k+m}\Big)\Bigg)+\cdots\\
&+n_{m-1}\Bigg(\Big(\frac{1}{m-1}+\frac{1}{n-1}-\frac{4}{m+n-2}\Big)n_{n-1}+\Big(\frac{1}{m-1}+\frac{1}{n-2}-\frac{4}{m+n-3}\Big)n_{n-2}+\cdots\\
&+\Big(\frac{1}{m-1}+\frac{1}{m}-\frac{4}{2m-1}\Big)n_m+\Big(\frac{1}{m-1}+\frac{1}{m-1}-\frac{4}{2m-2}\Big)(m-1-k)\Bigg)\\
&+\sum_{m\leq i<j\leq n-1}\Big(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\Big)x_{i,j}\\
=&\sum_{k\leq i\leq m-1}f(i)n_i+\sum_{m\leq i<j\leq n-1}\Big(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\Big)x_{i,j}\\
=&\gamma_1+\gamma_2
\end{aligned}$$

where $f(i)=\sum_{j=m}^{n-1}(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j})n_j+(\frac{1}{i}+\frac{1}{m-1}-\frac{4}{i+m-1})(i-k)$. Let $g(x)=x(\frac{1}{x}+\frac{1}{y}-\frac{4}{x+y})$, where $0<x<y$. Note that $g'(x)=\frac{(x-y)(x+3y)}{y(x+y)^2}<0$ for $0<x<y$, then $g(x)$ is monotonous decreasing for $0<x<y$. Thus we have for $k+1\leq i\leq m-1$, $m\leq j\leq n-1$:

$$k\Big(\frac{1}{k}+\frac{1}{j}-\frac{4}{k+j}\Big)\geq i\Big(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\Big).$$

Therefore

$$
\begin{aligned}
f(i) \leq & \frac{k}{i}\left(\sum_{j=m}^{n-1}\left(\frac{1}{k}+\frac{1}{j}-\frac{4}{k+j}\right)n_j+\left(\frac{1}{k}+\frac{1}{m-1}-\frac{4}{k+m-1}\right)(i-k)\right)\\
= & \left(1-\frac{i-k}{i}\right)\left(\sum_{j=m}^{n-1}\left(\frac{1}{k}+\frac{1}{j}-\frac{4}{k+j}\right)n_j\right)+\frac{k(i-k)}{i}\left(\frac{1}{k}+\frac{1}{m-1}-\frac{4}{k+m-1}\right)\\
= & \sum_{j=m}^{n-1}\left(\frac{1}{k}+\frac{1}{j}-\frac{4}{k+j}\right)n_j+\frac{i-k}{i}\left(\left(\frac{1}{k}+\frac{1}{m-1}-\frac{4}{k+m-1}\right)k-\sum_{j=m}^{n-1}\left(\frac{1}{k}+\frac{1}{j}-\frac{4}{k+j}\right)n_j\right)\\
\leq & \sum_{j=m}^{n-1}\left(\frac{1}{k}+\frac{1}{j}-\frac{4}{k+j}\right)n_j,
\end{aligned}
$$

because $\sum_{j=m}^{n-1} n_j = k$ and $m \leq j \leq n-1$. Since $n_k + \cdots + n_{m-1} = n-k$, we have

$$
\begin{aligned}
\gamma_1 = & \sum_{k\leq i\leq m-1} f(i)n_i \leq \left(\sum_{j=m}^{n-1}\left(\frac{1}{k}+\frac{1}{j}-\frac{4}{k+j}\right)n_j\right)\sum_{i=k}^{m-1} n_i\\
= & (n-k)\sum_{j=m}^{n-1}\left(\frac{1}{k}+\frac{1}{j}-\frac{4}{k+j}\right)n_j = \sum_{j=m}^{n-1}\left(\frac{1}{k}+\frac{1}{n-1}-\frac{4}{k+n-1}\right)(n-k)n_j\\
& -\sum_{j=m}^{n-2}\left(\frac{1}{k}+\frac{1}{n-1}-\frac{4}{k+n-1}\right)(n-k)n_j+\sum_{j=m}^{n-2}\left(\frac{1}{k}+\frac{1}{j}-\frac{4}{k+j}\right)(n-k)n_j\\
= & \left(\frac{1}{k}+\frac{1}{n-1}-\frac{4}{k+n-1}\right)(n-k)k-\sum_{j=m}^{n-2}\left(\frac{1}{n-1}+\frac{4}{k+j}-\frac{4}{k+n-1}-\frac{1}{j}\right)(n-k)n_j.
\end{aligned}
$$

Since $x_{i,j} \leq n_i n_j$, $m \leq i < j \leq n-1$, and $n_{n-1} = k - \sum_{j=m}^{n-2} n_j$, we have

$$
\begin{aligned}
\gamma_2 = & \sum_{m\leq i<j\leq n-1}\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)x_{i,j} \leq \sum_{m\leq i<j\leq n-1}\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)n_i n_j\\
= & \sum_{i=m}^{n-2}\left(\frac{1}{i}+\frac{1}{n-1}-\frac{4}{i+n-1}\right)n_i\left(k-\sum_{j=m}^{n-2} n_j\right)+\sum_{m\leq i<j\leq n-2}\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)n_i n_j\\
= & k\sum_{i=m}^{n-2}\left(\frac{1}{i}+\frac{1}{n-1}-\frac{4}{i+n-1}\right)n_i-\sum_{i=m}^{n-2}\left(\frac{1}{i}+\frac{1}{n-1}-\frac{4}{i+n-1}\right)n_i^2\\
& +\sum_{m\leq i<j\leq n-2}\left(\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)-\left(\frac{1}{i}+\frac{1}{n-1}-\frac{4}{i+n-1}\right)-\left(\frac{1}{j}+\frac{1}{n-1}-\frac{4}{j+n-1}\right)\right)n_i n_j\\
= & k\sum_{i=m}^{n-2}\left(\frac{1}{i}+\frac{1}{n-1}-\frac{4}{i+n-1}\right)n_i-\sum_{i=m}^{n-2}\left(\frac{1}{i}+\frac{1}{n-1}-\frac{4}{i+n-1}\right)n_i^2\\
& -2\sum_{m\leq i<j\leq n-2}\left(\frac{2}{i+j}+\frac{1}{n-1}-\frac{2}{i+n-1}-\frac{2}{j+n-1}\right)n_i n_j.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\gamma \leq & \gamma_1+\gamma_2 \leq \Big(\frac{1}{k}+\frac{1}{n-1}-\frac{4}{k+n-1}\Big)(n-k)k-\sum_{i=m}^{n-2}\Big(\frac{1}{n-1}+\frac{4}{k+i}-\frac{4}{k+n-1}-\frac{1}{i}\Big)(n-k)n_i \\
& +k\sum_{i=m}^{n-2}\Big(\frac{1}{i}+\frac{1}{n-1}-\frac{4}{i+n-1}\Big)n_i-\sum_{i=m}^{n-2}\Big(\frac{1}{i}+\frac{1}{n-1}-\frac{4}{i+n-1}\Big)n_i^2 \\
& -2\sum_{m\leq i<j\leq n-2}\Big(\frac{2}{i+j}+\frac{1}{n-1}-\frac{2}{i+n-1}-\frac{2}{j+n-1}\Big)n_in_j \\
= & \Big(\frac{1}{k}+\frac{1}{n-1}-\frac{4}{k+n-1}\Big)(n-k)k-\sum_{i=m}^{n-2}\Big(\frac{1}{i}+\frac{1}{n-1}-\frac{4}{i+n-1}\Big)n_i^2 \\
& -2\sum_{m\leq i<j\leq n-2}\Big(\frac{2}{i+j}+\frac{1}{n-1}-\frac{2}{i+n-1}-\frac{2}{j+n-1}\Big)n_in_j \\
& +\sum_{i=m}^{n-2}\Bigg(\Big(\frac{1}{i}+\frac{1}{n-1}-\frac{4}{i+n-1}\Big)k-\Big(\frac{1}{n-1}+\frac{4}{k+i}-\frac{4}{k+n-1}-\frac{1}{i}\Big)(n-k)\Bigg)n_i \\
\leq & \Big(\frac{1}{k}+\frac{1}{n-1}-\frac{4}{k+n-1}\Big)(n-k)k-\sum_{i=m}^{n-2}\Big(\frac{1}{i}+\frac{1}{n-1}-\frac{4}{i+n-1}\Big)n_i^2 \\
& -2\sum_{m\leq i<j\leq n-2}\Big(\frac{2}{i+j}+\frac{1}{n-1}-\frac{2}{i+n-1}-\frac{2}{j+n-1}\Big)n_in_j \\
& +(n-k)\sum_{i=m}^{n-2}\Bigg(\Big(\frac{1}{i}+\frac{1}{n-1}-\frac{4}{i+n-1}\Big)-\Big(\frac{1}{n-1}+\frac{4}{k+i}-\frac{4}{k+n-1}-\frac{1}{i}\Big)\Bigg)n_i \\
= & \Big(\frac{1}{k}+\frac{1}{n-1}-\frac{4}{k+n-1}\Big)(n-k)k-\Bigg(\sum_{i=m}^{n-2}\Big(\frac{1}{i}+\frac{1}{n-1}-\frac{4}{i+n-1}\Big)n_i^2 \\
& +2\sum_{m\leq i<j\leq n-2}\frac{(n-j-1)\big(2(n-1)(n+j-1)-(i+j)(n+i-1)\big)}{(i+j)(n-1)(n+i-1)(n+j-1)}n_in_j\Bigg) \\
& -2(n-k)\sum_{i=m}^{n-2}\frac{(i-k)\big((i+n-1)(k+n-1)-(i+i)(k+i)\big)}{i(k+i)(i+n-1)(k+n-1)}n_i \\
\leq & \Big(\frac{1}{k}+\frac{1}{n-1}-\frac{4}{k+n-1}\Big)(n-k)k.
\end{aligned}
$$

The second inequality holds because $k\leq n-k$. So $H(G)=\frac{n}{2}-\frac{\gamma}{2}\geq\frac{n}{2}-\frac{1}{2}(\frac{1}{k}+\frac{1}{n-1}-\frac{4}{k+n-1})(n-k)k=\frac{k(k-1)}{2(n-1)}+\frac{2k(n-k)}{k+n-1}=H(K^*_{k,n-k})$. Equality holds when $n_i=0$ for $k+1\leq i\leq n-2$, $n_k=n-k$, $n_{n-1}=k$, $x_{k,n-1}=(n-k)k$, $x_{n-1,n-1}=\binom{k}{2}$, and all other $x_{i,j}$ are equal to zero. Thus, graphs for which the harmonic index attains its minimum value are $K^*_{k,n-k}$.

Case 2. $n_m+\cdots+n_{n-2}+n_{n-1}>k$.

We put $n_m=n_{m'}+n_{m''}$, such that $n_{m''}+n_{m+1}+\cdots+n_{n-1}=k$. Then $n_k+\cdots+n_{m-1}+n_{m'}=n-k$. We will color the vertices of degree m with red and white, such that the number of red vertices is $n_{m''}$. Denote by $x_{i,m'}$ (resp. $x_{i,m''}$) for $i\neq m$, the number of edges between vertices of degree i and the white (resp. red) vertices of degree m, by $x_{m',m'}$ (resp. $x_{m'',m''}$) the number of edges between white (resp. red) vertices of degree m, and by $x_{m',m''}$ the number of edges between white and red vertices of degree m. Then $x_{i,m}=x_{i,m'}+x_{i,m''}$ for

$i \neq m$, and $x_{m,m} = x_{m',m'} + x_{m',m''} + x_{m'',m''}$. We will replace system (1) by:

$$\begin{aligned} &x_{k,i} + \cdots + x_{i,m-1} + x_{i,m'} + x_{i,m''} + x_{i,m+1} + \cdots + x_{i,n-1} = in_i, k \leq i \leq n-1, i \neq m, \\ &x_{k,m'} + \cdots + x_{m-1,m'} + 2x_{m',m'} + x_{m',m''} + x_{m',m+1} + \cdots + x_{m',n-1} = mn_{m'}, \\ &x_{k,m''} + \cdots + x_{m-1,m''} + x_{m',m''} + 2x_{m'',m''} + x_{m'',m+1} + \cdots + x_{m'',n-1} = mn_{m''}. \end{aligned} \tag{1'}$$

We will proceed similarly as in the Case 1. The rest of the proof is omitted, because it is similar to the one of Case 1. □

Remark When $k = \frac{n}{2} + 1$, $n \equiv 0 \pmod 4$ and $n \geq 8$, we give a counterexample to show that the Conjecture 1.1 is not correct. Let G' be an n-vertex graph obtained from K_n by deleting the edges of an $(\frac{n}{2} - 2)$-regular graph on $\frac{n}{2}$ vertices. Then $G' \in G(n, \frac{n}{2} + 1)$. By some elementary calculations, we have $H(K^*_{\frac{n}{2}+1,\frac{n}{2}-1}) = \frac{n^2+2n}{8(n-1)} + \frac{n^2-4}{3n}$, $H(G') = \frac{n^2-2n}{8(n-1)} + \frac{n}{2(n+2)} + \frac{n}{3}$. Thus, $H(K^*_{\frac{n}{2}+1,\frac{n}{2}-1}) - H(G') = \frac{(n-4)^2}{6n(n-1)(n+2)} > 0$ for $n \geq 8$. This implies $H(K^*_{\frac{n}{2}+1,\frac{n}{2}-1}) > H(G')$.

When $k = \lfloor \frac{n}{2} \rfloor + 1$, n is odd, that is $k = \frac{n+1}{2}$, the graph G_1 in [1] is a counterexample which can show that the Conjecture 1.1 is also not correct.

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