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Some New Inclusion Sets for Eigenvalues of Tensors with Application

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Abstract. In this paper, we are concerned with the eigenvalue inclusion sets for tensors. Some new S-type eigenvalue localization sets for tensors are employed by dividing $N = \{1, 2, \ldots, n\}$ into disjoint subsets S and its complement. Our new sets, are proved to be tighter than that newly derived by Huang et al. (J. Inequal. Appl. 2016 (2016) 254). As applications, we can apply the proposed sets for determining the positive (semi-)definiteness of even-order symmetric tensors. Some examples are given to show the sharpness of our new sets in contrast with the known ones, and verify the effectiveness of those in identifying the positive (semi-)definiteness of tensors.

1. Introduction

Let $\mathbb{R}(\mathbb{C})$ be the real (complex) field. For a positive integer n, N denotes the set $\{1, 2, \ldots, n\}$. An m-order n-dimensional tensor $\mathcal{A}$ consists of n^m entries, which is defined as follows:

$$\mathcal{A} = (a_{i_1 i_2 \ldots i_m}),\ a_{i_1 i_2 \ldots i_m} \in \mathbb{R}(\mathbb{C}),\ 1 \leq i_1, i_2, \ldots, i_m \leq n.$$

$\mathcal{A}$ is called nonnegative (positive) if $a_{i_1 \ldots i_m} \geq 0$ ($a_{i_1 \ldots i_m} > 0$). As usually, we denote the set of all m-order n-dimensional real (complex) tensors by $\mathbb{R}^{[m,n]}$ ($\mathbb{C}^{[m,n]}$). Moreover, a real tensor $\mathcal{A} = (a_{i_1 \ldots i_m})$ is called symmetric [1–6] if

$$a_{i_1 \ldots i_m} = a_{\pi(i_1 \ldots i_m)},\ \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices. And a tensor of m-order n-dimension $\mathcal{I} = (\delta_{i_1 \ldots i_m})$ is called the unit tensor [7], where

$$\delta_{i_1 \ldots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

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Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$, and $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ be an n-dimensional vector, real or complex, where x^{T} denotes the transpose of x. We define the n-dimensional vector

$$\mathcal{A}x^{m-1} = \Big(\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} x_{i_2} \cdots x_{i_m} \Big)_{1\le i\le n}$$

and $x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^{\mathrm{T}}$.

Qi [1] and Lim [8] independently introduced the following definition.

Definition 1.1. *A pair* $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{0\})$ *is called an eigenpair of* $\mathcal{A}$ *if*

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}.$$

In particular, we call (λ, x) *an H-eigenpair if they are both real.*

The eigenvalue problems of tensors have a wide range of practical applications such as automatical control [9], spectral hypergraph theory [10, 11], magnetic resonance imaging [12], high order Markov chains [13], best rank-one approximations in statistical data analysis [14] and so on.

For an m-degree homogeneous polynomial of n variables $f(x)$ denoted as

$$f(x) = \sum_{i_1,\ldots,i_m \in N} a_{i_1\ldots i_m} x_{i_1} \cdots x_{i_m}, \tag{1}$$

where $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$. When m is even, $f(x)$ is called positive definite if

$$f(x) > 0, \text{ for any } x \in \mathbb{R}^n,\ x \neq 0.$$

The homogeneous polynomial $f(x)$ in (1) is equivalent to the tensor product of an m-order n-dimensional symmetric tensor $\mathcal{A}$ and x^m defined by

$$f(x) = \mathcal{A}x^m = \sum_{i_1,\ldots,i_m \in N} a_{i_1\ldots i_m} x_{i_1} \cdots x_{i_m},$$

where $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$. The positive definiteness of multivariate polynomial $f(x)$ plays an important role in the stability study of nonlinear autonomous systems via Lyapunov's direct method in automatic control, such as the multivariate network realizability theory [15], a test for Lyapunov stability in multivariate filters [16], and the output feedback stabilization problems [17].

One of many practical applications of eigenvalues of tensors is that one can identify the positive (semi-)definiteness for an even-order real symmetric tensor by using the smallest H-eigenvalue of a tensor, consequently, can identify the positive (semi-)definiteness of the homogeneous polynomial $f(x)$ determined by this tensor, for details, see [1, 6, 7].

However, as mentioned in [4, 7, 18], it is not easy to compute the smallest H-eigenvalue of tensors when the order and dimension are very large. It is noteworthy that it is difficult to determine a given even-order multivariate polynomial $f(x)$ is positive semi-definite or not because the problem is NP-hard when $n > 3$ and $m \geq 4$ [19]. With this in mind, we shall try to derive a set including all eigenvalues in the complex. In particular, if one of these sets for an even-order real symmetric tensor is in the right-half complex plane, then we can conclude that the smallest H-eigenvalue is positive, consequently, the corresponding tensor is positive definite.

For convenience's sake, throughout this paper, for $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$, $i, j \in N$, $j \neq i$ and a nonempty proper subset S of N, we denote

$$\Delta^N = \{(i_2, i_3, \ldots, i_m) : \text{each } i_j \in N \text{ for } j = 2, 3, \ldots, m\},$$

$$\Delta^S = \{(i_2, i_3, \ldots, i_m) : \text{each } i_j \in S \text{ for } j = 2, 3, \ldots, m\},\ \overline{\Delta^S} = \Delta^N \backslash \Delta^S$$

and

$$r_i(\mathcal{A}) = \sum_{\delta_{ii_2\ldots i_m}=0} |a_{ii_2\ldots i_m}|,\ r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{ii_2\ldots i_m}=0,\\ \delta_{ji_2\ldots i_m}=0}} |a_{ii_2\ldots i_m}| = r_i(\mathcal{A}) - |a_{ij\ldots j}|,$$

$$r_{i,\Delta^S}^j(\mathcal{A}) = \sum_{\substack{(i_2,\ldots,i_m)\in\Delta^S,\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\ldots i_m}| - |a_{ij\ldots j}|,\ r_{i,\overline{\Delta^S}}^j(\mathcal{A}) = \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^S},\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\ldots i_m}| - |a_{ij\ldots j}|.$$

Up till now, many people have focused on locating eigenvalues of tensors and using obtained eigenvalue inclusion theorems to determine the positive (semi-)definiteness of an even-order real symmetric tensor or to derive the lower and upper bounds for the spectral radius of nonnegative tensors and the minimum H-eigenvalue of $\mathcal{M}$-tensors. For details, see [1, 3–7, 18, 20–22]. In this paper, we will continue the study of this topic. Our results, which are related to the order of tensors and the nonempty proper subset of N, improve the set given in [20] in the sense that the eigenvalue inclusion sets are sharper compared with that in [20]. As applications of these new sets, we can utilize them to determine the positive (semi-)definiteness of an even-order real symmetric tensor. Several numerical examples are implemented to illustrate these facts.

In what follows are some existing results that relate to the eigenvalue inclusion sets for tensors are reviewed. In 2005, Qi [1] generalized Geršgorin eigenvalue inclusion theorem from matrices to real supersymmetric tensors, which has been extended to general tensors by Yang and Yang [2]. To get sharper eigenvalue inclusion sets, Li et al. [5] extended the Brauer's eigenvalue localization set of matrices [23, 24] and proposed the following Brauer-type eigenvalue localization set for tensors.

Lemma 1.2. [5] *Let* $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$, $n \geq 2$. *Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) = \bigcup_{i,j\in N, j\neq i} \mathcal{K}_{i,j}(\mathcal{A}),$$

where $\sigma(\mathcal{A})$ *is the set of all the eigenvalues of* $\mathcal{A}$, *and*

$$\mathcal{K}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i\ldots i}| - r_i^j(\mathcal{A}))|z - a_{j\ldots j}| \leq |a_{ij\ldots j}| r_j(\mathcal{A})\}.$$

In addition, in order to reduce computations of determining the set $\mathcal{K}(\mathcal{A})$, Li et al. [5] also presented the following S-type eigenvalue localization set by breaking N into disjoint subsets S and $\bar{S}$, where $\bar{S} = N\backslash S$.

Lemma 1.3. [5] *Let* $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$, $n \geq 2$, *and S be a nonempty proper subset of N. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) = \left(\bigcup_{i\in S, j\in\bar{S}} \mathcal{K}_{i,j}(\mathcal{A})\right) \bigcup \left(\bigcup_{i\in\bar{S}, j\in S} \mathcal{K}_{i,j}(\mathcal{A})\right),$$

where $\mathcal{K}_{i,j}(\mathcal{A})$ ($i \in S, j \in \bar{S}$ or $i \in \bar{S}, j \in S$) *is defined as in Lemma 1.2.*

To further improve the eigenvalue inclusion set in Lemma 1.3, Huang et al. [20] employed a new S-type eigenvalue localization set for tensors as follows.

Lemma 1.4. [20] *Let* $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$ *with* $n \geq 2$. *And let S be a nonempty proper subset of N. Then*

$$\sigma(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A}) = \left(\bigcup_{i\in S, j\in\bar{S}} \Upsilon_i^j(\mathcal{A})\right) \bigcup \left(\bigcup_{i\in\bar{S}, j\in S} \Upsilon_i^j(\mathcal{A})\right),$$

where

$$\Upsilon_i^j(\mathcal{A}) = \left\{z \in \mathbb{C} : |(z - a_{i\ldots i})(z - a_{j\ldots j}) - a_{ij\ldots j}a_{ji\ldots i}| \leq |z - a_{j\ldots j}| r_i^j(\mathcal{A}) + |a_{ij\ldots j}| r_j^i(\mathcal{A})\right\}.$$

Very recently, Zhao and Sang [21] gave a sharper eigenvalue inclusion set for tensors without considering the selection of S compared with that in Lemma 1.4.

Lemma 1.5. [21] *Let $\mathcal{A} = (a_{i_1\dots i_m}) \in \mathbb{C}^{[m,n]}$ with $n \geq 2$. Then*

$$\sigma(\mathcal{A}) \subseteq \Delta^{\cap}(\mathcal{A}) := \bigcup_{i\in N} \bigcap_{j\in N, j\neq i} \Upsilon_i^j(\mathcal{A}),$$

where $\Upsilon_i^j(\mathcal{A})$ is defined as in Lemma 1.4.

2. Some New S-type Eigenvalue Inclusion Sets for Tensors and Their Comparison Theorems

The main focus of this section is to study the eigenvalue inclusion sets for tensor $\mathcal{A}$. Some new S-type eigenvalue sets for tensors as well as the comparisons between the proposed sets with some existing ones are established.

Theorem 2.1. *Let $\mathcal{A} = (a_{i_1\dots i_m}) \in \mathbb{C}^{[m,n]}$ with $n \geq 2$. And let S be a nonempty proper subset of N. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{E}^S(\mathcal{A}) = \left(\bigcup_{i\in S, j\in \bar{S}} \mathcal{E}_{i,j}^S(\mathcal{A})\right) \bigcup \left(\bigcup_{i\in \bar{S}, j\in S} \mathcal{E}_{i,j}^{\bar{S}}(\mathcal{A})\right), \tag{2}$$

where

$$\begin{aligned}\mathcal{E}_{i,j}^S(\mathcal{A}) = \Big\{z \in \mathbb{C} :\ & \Big[|(z - a_{i\dots i})(z - a_{j\dots j}) - a_{ij\dots j}a_{ji\dots i}| - |z - a_{j\dots j}|r^j_{i,\overline{\Delta^S}}(\mathcal{A}) - |a_{ij\dots j}|r^i_{j,\overline{\Delta^S}}(\mathcal{A})\Big] \\ & \cdot|(z - a_{i\dots i})(z - a_{j\dots j}) - a_{ij\dots j}a_{ji\dots i}| \leq \Big[|z - a_{j\dots j}|r^j_{i,\Delta^S}(\mathcal{A}) + |a_{ij\dots j}|r^i_{j,\Delta^S}(\mathcal{A})\Big] \\ & \cdot \Big[|z - a_{i\dots i}|r^i_j(\mathcal{A}) + |a_{ji\dots i}|r^j_i(\mathcal{A})\Big]\Big\},\\ \mathcal{E}_{i,j}^{\bar{S}}(\mathcal{A}) = \Big\{z \in \mathbb{C} :\ & \Big[|(z - a_{i\dots i})(z - a_{j\dots j}) - a_{ij\dots j}a_{ji\dots i}| - |z - a_{j\dots j}|r^j_{i,\overline{\Delta^{\bar{S}}}}(\mathcal{A}) - |a_{ij\dots j}|r^i_{j,\overline{\Delta^{\bar{S}}}}(\mathcal{A})\Big] \\ & \cdot|(z - a_{i\dots i})(z - a_{j\dots j}) - a_{ij\dots j}a_{ji\dots i}| \leq \Big[|z - a_{j\dots j}|r^j_{i,\Delta^{\bar{S}}}(\mathcal{A}) + |a_{ij\dots j}|r^i_{j,\Delta^{\bar{S}}}(\mathcal{A})\Big] \\ & \cdot \Big[|z - a_{i\dots i}|r^i_j(\mathcal{A}) + |a_{ji\dots i}|r^j_i(\mathcal{A})\Big]\Big\}.\end{aligned}$$

Proof. For any $\lambda \in \sigma(\mathcal{A})$, let $x = (x_1, \dots, x_n)^{\mathrm{T}} \in \mathbb{C}^n\backslash\{0\}$ be an eigenvector corresponding to λ, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{3}$$

Let $|x_p| = \max_{i\in S}\{|x_i|\}$ and $|x_q| = \max_{i\in\bar{S}}\{|x_i|\}$. We prove this theorem by distinguishing two cases as follows.

(i) $|x_p| \geq |x_q|$, so $|x_p| = \max_{i\in N}\{|x_i|\}$ and $|x_p| > 0$. It follows from (3) that

$$\begin{cases} \sum\limits_{i_2,\dots,i_m=1}^{n} a_{pi_2\dots i_m}x_{i_2}\cdots x_{i_m} = \lambda x_p^{m-1}, \\ \sum\limits_{i_2,\dots,i_m=1}^{n} a_{qi_2\dots i_m}x_{i_2}\cdots x_{i_m} = \lambda x_q^{m-1}. \end{cases}$$

We can reformulate the above equations in (4) after straightforward derivations

$$\begin{cases} \sum\limits_{\substack{\delta_{pi_2\dots i_m=0},\\ \delta_{qi_2\dots i_m=0}}} a_{pi_2\dots i_m}x_{i_2}\cdots x_{i_m} = (\lambda - a_{p\dots p})x_p^{m-1} - a_{pq\dots q}x_q^{m-1}, \\ \sum\limits_{\substack{\delta_{qi_2\dots i_m=0},\\ \delta_{pi_2\dots i_m=0}}} a_{qi_2\dots i_m}x_{i_2}\cdots x_{i_m} = (\lambda - a_{q\dots q})x_q^{m-1} - a_{qp\dots p}x_p^{m-1}. \end{cases} \tag{4}$$

Premultiplying by $(\lambda - a_{q\dots q})$ in the first equation of (4) results in

$$(\lambda - a_{q\cdots q}) \sum_{\substack{\delta_{pi_2\dots i_m}=0,\\ \delta_{qi_2\dots i_m}=0}} a_{pi_2\dots i_m} x_{i_2}\cdots x_{i_m} = (\lambda - a_{q\dots q})(\lambda - a_{p\dots p})x_p^{m-1} - a_{pq\dots q}(\lambda - a_{q\dots q})x_q^{m-1}. \tag{5}$$

Combining (5) and the second equation of (4) derives

$$\begin{aligned} &(\lambda - a_{q\dots q}) \sum_{\substack{\delta_{pi_2\dots i_m}=0,\\ \delta_{qi_2\dots i_m}=0}} a_{pi_2\dots i_m} x_{i_2}\cdots x_{i_m} + a_{pq\dots q} \sum_{\substack{\delta_{qi_2\dots i_m}=0,\\ \delta_{pi_2\dots i_m}=0}} a_{qi_2\dots i_m} x_{i_2}\cdots x_{i_m} \\ = \ &[(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}]x_p^{m-1}. \end{aligned} \tag{6}$$

It follows from (6) that

$$\begin{aligned} &(\lambda - a_{q\dots q}) \sum_{\substack{\delta_{pi_2\dots i_m}=0,\\ \delta_{qi_2\dots i_m}=0}} a_{pi_2\dots i_m} x_{i_2}\cdots x_{i_m} + a_{pq\dots q} \sum_{\substack{\delta_{qi_2\dots i_m}=0,\\ \delta_{pi_2\dots i_m}=0}} a_{qi_2\dots i_m} x_{i_2}\cdots x_{i_m} \\ = \ &(\lambda - a_{q\dots q}) \left[\sum_{\substack{(i_2,\dots,i_m)\in\overline{\Delta^{\bar S}},\\ \delta_{pi_2\dots i_m}=0}} a_{pi_2\dots i_m} x_{i_2}\cdots x_{i_m} + \sum_{\substack{(i_2,\dots,i_m)\in\Delta^{\bar S},\\ \delta_{qi_2\dots i_m}=0}} a_{pi_2\dots i_m} x_{i_2}\cdots x_{i_m} \right] \\ &+ a_{pq\dots q} \left[\sum_{\substack{(i_2,\dots,i_m)\in\overline{\Delta^{\bar S}},\\ \delta_{pi_2\dots i_m}=0}} a_{qi_2\dots i_m} x_{i_2}\cdots x_{i_m} + \sum_{\substack{(i_2,\dots,i_m)\in\Delta^{\bar S},\\ \delta_{qi_2\dots i_m}=0}} a_{qi_2\dots i_m} x_{i_2}\cdots x_{i_m} \right]. \end{aligned}$$

Taking absolute values and using the triangle inequality in the above equation yield

$$\begin{aligned} &|(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}||x_p|^{m-1} \\ \leq \ &|\lambda - a_{q\dots q}| \left[r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A})|x_p|^{m-1} + r^q_{p,\Delta^{\bar S}}(\mathcal{A})|x_q|^{m-1} \right] + |a_{pq\dots q}| \left[r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A})|x_p|^{m-1} + r^p_{q,\Delta^{\bar S}}(\mathcal{A})|x_q|^{m-1} \right], \end{aligned}$$

which can be rewritten as

$$\begin{aligned} &\left[|(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}| - |\lambda - a_{q\dots q}| r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A}) - |a_{pq\dots q}| r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A}) \right] |x_p|^{m-1} \\ \leq \ &\left[|\lambda - a_{q\dots q}| r^q_{p,\Delta^{\bar S}}(\mathcal{A}) + |a_{pq\dots q}| r^p_{q,\Delta^{\bar S}}(\mathcal{A}) \right] |x_q|^{m-1}. \end{aligned} \tag{7}$$

If $|x_q| = 0$. Note that $|x_p| > 0$, then it follows from (7) that

$$|(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}| - |z - a_{q\dots q}| r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A}) - |a_{pq\dots q}| r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A}) \leq 0.$$

It is evident that $z \in \mathcal{E}^S_{p,q}(\mathcal{A}) \subseteq \bigcup_{i\in S, j\in\bar S} \mathcal{E}^S_{i,j}(\mathcal{A}) \subseteq \mathcal{E}^S(\mathcal{A})$. Otherwise, $|x_q| > 0$. Under this condition, by premultiplying by $(\lambda - a_{p\dots p})$ in the second equation of (4), we obtain

$$(\lambda - a_{p\dots p}) \sum_{\substack{\delta_{qi_2\dots i_m}=0,\\ \delta_{pi_2\dots i_m}=0}} a_{qi_2\dots i_m} x_{i_2}\cdots x_{i_m} = (\lambda - a_{p\dots p})(\lambda - a_{q\dots q})x_q^{m-1} - (\lambda - a_{p\dots p})a_{qp\dots p}x_p^{m-1}. \tag{8}$$

Multiplying the first equation of (4) with $a_{qp\dots p}$ and adding it to (8) give

$$\begin{aligned}&(\lambda - a_{p\dots p})\sum_{\substack{\delta_{pi_2\dots i_m}=0,\\ \delta_{qi_2\dots i_m}=0}} a_{qi_2\dots i_m}x_{i_2}\cdots x_{i_m} + a_{qp\dots p}\sum_{\substack{\delta_{pi_2\dots i_m}=0,\\ \delta_{qi_2\dots i_m}=0}} a_{pi_2\dots i_m}x_{i_2}\cdots x_{i_m}\\ =\ &[(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}]x_q^{m-1}. \end{aligned} \tag{9}$$

Taking absolute values and using the triangle inequality in (9) lead to

$$|(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}||x_q|^{m-1} \le [|\lambda - a_{p\dots p}|r_q^p(\mathcal{A}) + |a_{qp\dots p}|r_p^q(\mathcal{A})]|x_p|^{m-1}. \tag{10}$$

Since $|x_p| \ge |x_q| > 0$, we multiply (7) with (10) and derive

$$\begin{aligned}&\left[|(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}| - |\lambda - a_{q\dots q}|r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A}) - |a_{pq\dots q}|r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A})\right]\\ &\cdot|(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}| \le \left[|\lambda - a_{q\dots q}|r^q_{p,\Delta^{\bar S}}(\mathcal{A}) + |a_{pq\dots q}|r^p_{q,\Delta^{\bar S}}(\mathcal{A})\right]\\ &\cdot\left[|\lambda - a_{p\dots p}|r_q^p(\mathcal{A}) + |a_{qp\dots p}|r_p^q(\mathcal{A})\right],\end{aligned}$$

which means that $\lambda \in \mathcal{E}^S_{p,q}(\mathcal{A}) \subseteq \bigcup_{i\in S, j\in\bar S}\mathcal{E}^S_{i,j}(\mathcal{A}) \subseteq \mathcal{E}^S(\mathcal{A})$.

(ii) $|x_p| \le |x_q|$, so $|x_q| = \max_{i\in N}\{|x_i|\}$ and $|x_q| > 0$. By utilizing the similar method in (i), it holds that

$$\begin{aligned}&\left[|(\lambda - a_{q\dots q})(\lambda - a_{p\dots p}) - a_{qp\dots p}a_{pq\dots q}| - |\lambda - a_{p\dots p}|r^p_{q,\overline{\Delta^S}}(\mathcal{A}) - |a_{qp\dots p}|r^q_{p,\overline{\Delta^S}}(\mathcal{A})\right]|x_q|^{m-1}\\ \le\ &\left[|\lambda - a_{p\dots p}|r^p_{q,\Delta^S}(\mathcal{A}) + |a_{qp\dots p}|r^q_{p,\Delta^S}(\mathcal{A})\right]|x_p|^{m-1}\end{aligned} \tag{11}$$

and

$$|(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}||x_p|^{m-1} \le [|\lambda - a_{q\dots q}|r_p^q(\mathcal{A}) + |a_{pq\dots q}|r_q^p(\mathcal{A})]|x_q|^{m-1}. \tag{12}$$

If $|x_p| = 0$, then it follows from (11) that

$$|(\lambda - a_{q\dots q})(\lambda - a_{p\dots p}) - a_{qp\dots p}a_{pq\dots q}| - |\lambda - a_{p\dots p}|r^p_{q,\overline{\Delta^S}}(\mathcal{A}) - |a_{qp\dots p}|r^q_{p,\overline{\Delta^S}}(\mathcal{A}) \le 0$$

by $|x_q| > 0$. Obviously, $\lambda \in \mathcal{E}^{\bar S}_{q,p}(\mathcal{A}) \subseteq \bigcup_{i\in\bar S, j\in S}\mathcal{E}^{\bar S}_{i,j}(\mathcal{A}) \subseteq \mathcal{E}^S(\mathcal{A})$. If $|x_p| > 0$, then combining (11) and (12) leads to

$$\begin{aligned}&\left[|(\lambda - a_{q\dots q})(\lambda - a_{p\dots p}) - a_{qp\dots p}a_{pq\dots q}| - |\lambda - a_{p\dots p}|r^p_{q,\overline{\Delta^S}}(\mathcal{A}) - |a_{qp\dots p}|r^q_{p,\overline{\Delta^S}}(\mathcal{A})\right]\\ &\cdot|(\lambda - a_{q\dots q})(\lambda - a_{p\dots p}) - a_{qp\dots p}a_{pq\dots q}| \le \left[|\lambda - a_{p\dots p}|r^p_{q,\Delta^S}(\mathcal{A}) + |a_{qp\dots p}|r^q_{p,\Delta^S}(\mathcal{A})\right]\\ &\cdot\left[|\lambda - a_{q\dots q}|r_p^q(\mathcal{A}) + |a_{pq\dots q}|r_q^p(\mathcal{A})\right].\end{aligned}$$

This means that $z \in \mathcal{E}^{\bar S}_{q,p}(\mathcal{A}) \subseteq \bigcup_{i\in\bar S, j\in S}\mathcal{E}^{\bar S}_{i,j}(\mathcal{A}) \subseteq \mathcal{E}^S(\mathcal{A})$. This completes our proof of Theorem 2.1. □

Next, we establish a comparison theorem for the new S-type eigenvalue inclusion set derived in Theorem 2.1 and those in Lemmas 1.2-1.4.

Theorem 2.2. *Let $\mathcal{A} = (a_{i_1\dots i_m}) \in \mathbb{C}^{[m,n]}$ with $n \ge 2$ and S be a nonempty proper subset of N. Then*

$$\mathcal{E}^S(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}).$$

Proof. From Theorem 3.2 of [20], it can be seen that $\Upsilon^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$. Thus, we only need to prove $\mathcal{E}^S(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A})$. Let $z \in \mathcal{E}^S(\mathcal{A})$. Without loss of generality, we assume that $z \in \bigcup_{i\in S, j\in \bar{S}} \mathcal{E}^S_{i,j}(\mathcal{A})$ (we can prove it similarly if $z \in \bigcup_{i\in \bar{S}, j\in S} \mathcal{E}^{\bar{S}}_{i,j}(\mathcal{A})$). Then there exist $p \in S$ and $q \in \bar{S}$ such that $z \in \mathcal{E}^S_{p,q}(\mathcal{A})$, that is

$$\begin{aligned} &\left[|(z-a_{p\ldots p})(z-a_{q\ldots q})-a_{pq\ldots q}a_{qp\ldots p}|-|z-a_{q\ldots q}|r^q_{p,\overline{\Delta^{\bar{S}}}}(\mathcal{A})-|a_{pq\ldots q}|r^p_{q,\overline{\Delta^{\bar{S}}}}(\mathcal{A})\right]\\ &\cdot|(z-a_{p\ldots p})(z-a_{q\ldots q})-a_{pq\ldots q}a_{qp\ldots p}| \le \left[|z-a_{q\ldots q}|r^q_{p,\Delta^{\bar{S}}}(\mathcal{A})+|a_{pq\ldots q}|r^p_{q,\Delta^{\bar{S}}}(\mathcal{A})\right]\\ &\cdot\left[|z-a_{p\ldots p}|r^p_q(\mathcal{A})+|a_{qp\ldots p}|r^q_p(\mathcal{A})\right], \end{aligned}$$

thus

$$\begin{aligned} &|(z-a_{p\ldots p})(z-a_{q\ldots q})-a_{pq\ldots q}a_{qp\ldots p}|-|z-a_{q\ldots q}|r^q_{p,\overline{\Delta^{\bar{S}}}}(\mathcal{A})-|a_{pq\ldots q}|r^p_{q,\overline{\Delta^{\bar{S}}}}(\mathcal{A})\\ \le\ &|z-a_{q\ldots q}|r^q_{p,\Delta^{\bar{S}}}(\mathcal{A})+|a_{pq\ldots q}|r^p_{q,\Delta^{\bar{S}}}(\mathcal{A}) \end{aligned} \tag{13}$$

or

$$|(\lambda-a_{p\ldots p})(\lambda-a_{q\ldots q})-a_{pq\ldots q}a_{qp\ldots p}| \le |\lambda-a_{p\ldots p}|r^p_q(\mathcal{A})+|a_{qp\ldots p}|r^q_p(\mathcal{A}) \tag{14}$$

hold. If Inequality (13) holds, then it has

$$\begin{aligned} &|(z-a_{p\ldots p})(z-a_{q\ldots q})-a_{pq\ldots q}a_{qp\ldots p}|\\ \le\ &|z-a_{q\ldots q}|\left(r^q_{p,\overline{\Delta^{\bar{S}}}}(\mathcal{A})+r^q_{p,\Delta^{\bar{S}}}(\mathcal{A})\right)+|a_{pq\ldots q}|\left(r^p_{q,\overline{\Delta^{\bar{S}}}}(\mathcal{A})+r^p_{q,\Delta^{\bar{S}}}(\mathcal{A})\right)\\ =\ &|z-a_{q\ldots q}|r^q_p(\mathcal{A})+|a_{pq\ldots q}|r^p_q(\mathcal{A}). \end{aligned}$$

This means that $z \in \Upsilon^q_p(\mathcal{A}) \subseteq \bigcup_{i\in S, j\in\bar{S}} \Upsilon^j_i(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A})$. In addition, if Inequality (14) holds, then it is evident that $z \in \Upsilon^p_q(\mathcal{A}) \subseteq \bigcup_{i\in\bar{S}, j\in S} \Upsilon^j_i(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A})$.

We summarize the above discussions and infer that $\mathcal{E}^S(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A})$. This proof is completed. □

In the following theorem, another S-type eigenvalue localization set for tensors is established, which is sharper than that in Theorem 2.1.

Theorem 2.3. *Let $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$ with $n \ge 2$. And let S be a nonempty proper subset of N. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{W}^S(\mathcal{A}) = \left(\mathcal{W}^S_{i,j}(\mathcal{A})\right)\bigcup\left(\mathcal{W}^{\bar{S}}_{i,j}(\mathcal{A})\right), \tag{15}$$

where

$$\begin{aligned} \mathcal{W}^S_{i,j}(\mathcal{A}) &= \left(\bigcup_{i\in S, j\in\bar{S}} \hat{\mathcal{W}}^1_{i,j}(\mathcal{A})\right)\bigcup\left(\bigcup_{i\in S, j\in\bar{S}}\left(\tilde{\mathcal{W}}^1_{i,j}(\mathcal{A})\bigcap\Theta^j_i(\mathcal{A})\right)\right),\\ \mathcal{W}^{\bar{S}}_{i,j}(\mathcal{A}) &= \left(\bigcup_{i\in\bar{S}, j\in S} \hat{\mathcal{W}}^2_{i,j}(\mathcal{A})\right)\bigcup\left(\bigcup_{i\in\bar{S}, j\in S}\left(\tilde{\mathcal{W}}^2_{i,j}(\mathcal{A})\bigcap\Theta^j_i(\mathcal{A})\right)\right), \end{aligned}$$

with

$$\begin{aligned}
\hat{\mathcal{W}}^1_{i,j}(\mathcal{A}) &= \{z \in \mathbb{C} \;:\; |(z-a_{i\ldots i})(z-a_{j\ldots j}) - a_{ij\ldots j}a_{ji\ldots i}| \le |z-a_{j\ldots j}| r^j_{i,\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + |a_{ij\ldots j}| r^i_{j,\overline{\Delta^{\bar{S}}}}(\mathcal{A})\},\\
\hat{\mathcal{W}}^2_{i,j}(\mathcal{A}) &= \{z \in \mathbb{C} \;:\; |(z-a_{i\ldots i})(z-a_{j\ldots j}) - a_{ij\ldots j}a_{ji\ldots i}| \le |z-a_{j\ldots j}| r^j_{i,\overline{\Delta^{S}}}(\mathcal{A}) + |a_{ij\ldots j}| r^i_{j,\overline{\Delta^{S}}}(\mathcal{A})\},\\
\tilde{\mathcal{W}}^1_{i,j}(\mathcal{A}) &= \Big\{z \in \mathbb{C} \;:\; \Big[|(z-a_{i\ldots i})(z-a_{j\ldots j}) - a_{ij\ldots j}a_{ji\ldots i}| - |z-a_{j\ldots j}| r^j_{i,\overline{\Delta^{\bar{S}}}}(\mathcal{A}) - |a_{ij\ldots j}| r^i_{j,\overline{\Delta^{\bar{S}}}}(\mathcal{A})\Big]\\
&\quad \cdot \Big[|(z-a_{j\ldots j})(z-a_{i\ldots i}) - a_{ij\ldots j}a_{ji\ldots i}| - |z-a_{i\ldots i}| r^i_{j,\Delta^{\bar{S}}}(\mathcal{A}) - |a_{ji\ldots i}| r^j_{i,\Delta^{\bar{S}}}(\mathcal{A})\Big]\\
&\quad \le \Big[|z-a_{j\ldots j}| r^j_{i,\Delta^{\bar{S}}}(\mathcal{A}) + |a_{ij\ldots j}| r^i_{j,\Delta^{\bar{S}}}(\mathcal{A})\Big]\Big[|z-a_{i\ldots i}| r^i_{j,\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + |a_{ji\ldots i}| r^j_{i,\overline{\Delta^{\bar{S}}}}(\mathcal{A})\Big]\Big\},\\
\tilde{\mathcal{W}}^2_{i,j}(\mathcal{A}) &= \Big\{z \in \mathbb{C} \;:\; \Big[|(z-a_{i\ldots i})(z-a_{j\ldots j}) - a_{ij\ldots j}a_{ji\ldots i}| - |z-a_{j\ldots j}| r^j_{i,\overline{\Delta^{S}}}(\mathcal{A}) - |a_{ij\ldots j}| r^i_{j,\overline{\Delta^{S}}}(\mathcal{A})\Big]\\
&\quad \cdot \Big[|(z-a_{j\ldots j})(z-a_{i\ldots i}) - a_{ij\ldots j}a_{ji\ldots i}| - |z-a_{i\ldots i}| r^i_{j,\Delta^{S}}(\mathcal{A}) - |a_{ji\ldots i}| r^j_{i,\Delta^{S}}(\mathcal{A})\Big]\\
&\quad \le \Big[|z-a_{j\ldots j}| r^j_{i,\Delta^{S}}(\mathcal{A}) + |a_{ij\ldots j}| r^i_{j,\Delta^{S}}(\mathcal{A})\Big]\Big[|z-a_{i\ldots i}| r^i_{j,\overline{\Delta^{S}}}(\mathcal{A}) + |a_{ji\ldots i}| r^j_{i,\overline{\Delta^{S}}}(\mathcal{A})\Big]\Big\},\\
\Theta^j_i(\mathcal{A}) &= \{z \in \mathbb{C} \;:\; |(z-a_{i\ldots i})(z-a_{j\ldots j}) - a_{ij\ldots j}a_{ji\ldots i}| \le |z-a_{j\ldots j}| r^j_i(\mathcal{A}) + |a_{ij\ldots j}| r^i_j(\mathcal{A})\}.
\end{aligned}$$

Proof. For any $\lambda \in \sigma(\mathcal{A})$, let $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n\backslash\{0\}$ be an eigenvector corresponding to λ, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{16}$$

Let $|x_p| = \max_{i\in S}\{|x_i|\}$ and $|x_q| = \max_{i\in \bar{S}}\{|x_i|\}$. There two situations stated as follows.

(i) $|x_p| \ge |x_q|$, so $|x_p| = \max_{i\in N}\{|x_i|\}$ and $|x_p| > 0$. It follows from (16) that

$$\begin{cases}\displaystyle\sum_{\substack{\delta_{pi_2\ldots i_m}=0,\\ \delta_{qi_2\ldots i_m}=0}} a_{pi_2\ldots i_m} x_{i_2}\cdots x_{i_m} = (\lambda - a_{p\ldots p}) x_p^{m-1} - a_{pq\ldots q} x_q^{m-1},\\ \displaystyle\sum_{\substack{\delta_{qi_2\ldots i_m}=0,\\ \delta_{pi_2\ldots i_m}=0}} a_{qi_2\ldots i_m} x_{i_2}\cdots x_{i_m} = (\lambda - a_{q\ldots q}) x_q^{m-1} - a_{qp\ldots p} x_p^{m-1}.\end{cases} \tag{17}$$

Similar to the proof of Theorem 2.1, we can derive

$$\begin{aligned}&\Big[|(\lambda-a_{p\ldots p})(\lambda-a_{q\ldots q}) - a_{pq\ldots q}a_{qp\ldots p}| - |\lambda - a_{q\ldots q}| r^q_{p,\overline{\Delta^{\bar{S}}}}(\mathcal{A}) - |a_{pq\ldots q}| r^p_{q,\overline{\Delta^{\bar{S}}}}(\mathcal{A})\Big] |x_p|^{m-1}\\ \le\ &\Big[|\lambda - a_{q\ldots q}| r^q_{p,\Delta^{\bar{S}}}(\mathcal{A}) + |a_{pq\ldots q}| r^p_{q,\Delta^{\bar{S}}}(\mathcal{A})\Big] |x_q|^{m-1}.\end{aligned} \tag{18}$$

If $|x_q| = 0$, then from (18), it holds that

$$|(\lambda-a_{p\ldots p})(\lambda-a_{q\ldots q}) - a_{pq\ldots q}a_{qp\ldots p}| \le |\lambda - a_{q\ldots q}| r^q_{p,\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + |a_{pq\ldots q}| r^p_{q,\overline{\Delta^{\bar{S}}}}(\mathcal{A}).$$

Evidently, $\lambda \in \hat{\mathcal{W}}^1_{p,q}(\mathcal{A}) \subseteq \bigcup_{i\in S, j\in\bar{S}} \hat{\mathcal{W}}^1_{i,j}(\mathcal{A}) \subseteq \mathcal{W}^S(\mathcal{A})$. Otherwise, $|x_q| > 0$. If $\lambda \notin \bigcup_{i\in S, j\in\bar{S}} \hat{\mathcal{W}}^1_{i,j}(\mathcal{A})$, then for any $i \in S$ and $j \in \bar{S}$, we have

$$|(\lambda-a_{i\ldots i})(\lambda-a_{j\ldots j}) - a_{ij\ldots j}a_{ji\ldots i}| > |\lambda - a_{j\ldots j}| r^j_{i,\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + |a_{ij\ldots j}| r^i_{j,\overline{\Delta^{\bar{S}}}}(\mathcal{A}).$$

It follows from (18) that

$$|(\lambda-a_{p\ldots p})(\lambda-a_{q\ldots q}) - a_{pq\ldots q}a_{qp\ldots p}| \le |\lambda - a_{q\ldots q}| r^q_p(\mathcal{A}) + |a_{pq\ldots q}| r^p_q(\mathcal{A}),$$

i.e., $\lambda \in \Theta_p^q(\mathcal{A})$. Besides, it follows from (9) that

$$\begin{aligned}
&(\lambda - a_{p\dots p}) \sum_{\substack{\delta_{pi_2\dots i_m}=0,\\ \delta_{qi_2\dots i_m}=0}} a_{qi_2\dots i_m} x_{i_2}\cdots x_{i_m} + a_{qp\dots p} \sum_{\substack{\delta_{pi_2\dots i_m}=0,\\ \delta_{qi_2\dots i_m}=0}} a_{pi_2\dots i_m} x_{i_2}\cdots x_{i_m} \\
= \; &[(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}]x_q^{m-1}. \qquad (19)
\end{aligned}$$

It can be seen that

$$\begin{aligned}
&(\lambda - a_{p\dots p}) \sum_{\substack{\delta_{pi_2\dots i_m}=0,\\ \delta_{qi_2\dots i_m}=0}} a_{qi_2\dots i_m} x_{i_2}\cdots x_{i_m} + a_{qp\dots p} \sum_{\substack{\delta_{qi_2\dots i_m}=0,\\ \delta_{pi_2\dots i_m}=0}} a_{pi_2\dots i_m} x_{i_2}\cdots x_{i_m} \\
= \; &(\lambda - a_{p\dots p}) \left[\sum_{\substack{(i_2,\dots,i_m)\in\overline{\Delta^{\bar S}},\\ \delta_{pi_2\dots i_m}=0}} a_{qi_2\dots i_m} x_{i_2}\cdots x_{i_m} + \sum_{\substack{(i_2,\dots,i_m)\in\Delta^{\bar S},\\ \delta_{qi_2\dots i_m}=0}} a_{qi_2\dots i_m} x_{i_2}\cdots x_{i_m} \right] \\
&+ a_{qp\dots p} \left[\sum_{\substack{(i_2,\dots,i_m)\in\overline{\Delta^{\bar S}},\\ \delta_{pi_2\dots i_m}=0}} a_{pi_2\dots i_m} x_{i_2}\cdots x_{i_m} + \sum_{\substack{(i_2,\dots,i_m)\in\Delta^{\bar S},\\ \delta_{qi_2\dots i_m}=0}} a_{pi_2\dots i_m} x_{i_2}\cdots x_{i_m} \right],
\end{aligned}$$

then we apply the triangle inequality in (19) and obtain

$$\begin{aligned}
&|(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}||x_q|^{m-1} \\
\le \; &|\lambda - a_{p\dots p}| \left[r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A})|x_p|^{m-1} + r^p_{q,\Delta^{\bar S}}(\mathcal{A})|x_q|^{m-1} \right] + |a_{qp\dots p}| \left[r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A})|x_p|^{m-1} + r^q_{p,\Delta^{\bar S}}(\mathcal{A})|x_q|^{m-1} \right],
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\left[|(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}| - |\lambda - a_{p\dots p}|r^p_{q,\Delta^{\bar S}}(\mathcal{A}) - |a_{qp\dots p}|r^q_{p,\Delta^{\bar S}}(\mathcal{A}) \right] |x_q|^{m-1} \\
\le \; &\left[|\lambda - a_{p\dots p}|r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A}) + |a_{qp\dots p}|r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A}) \right] |x_p|^{m-1}. \qquad (20)
\end{aligned}$$

Under the condition $|x_p| \ge |x_q| > 0$, we combine (18) with (20) and obtain

$$\begin{aligned}
&\left[|(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}| - |\lambda - a_{q\dots q}|r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A}) - |a_{pq\dots q}|r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A}) \right] \\
\cdot &\left[|(\lambda - a_{q\dots q})(\lambda - a_{p\dots p}) - a_{pq\dots q}a_{qp\dots p}| - |\lambda - a_{p\dots p}|r^p_{q,\Delta^{\bar S}}(\mathcal{A}) - |a_{qp\dots p}|r^q_{p,\Delta^{\bar S}}(\mathcal{A}) \right] \\
\le \; &\left[|\lambda - a_{q\dots q}|r^q_{p,\Delta^{\bar S}}(\mathcal{A}) + |a_{pq\dots q}|r^p_{q,\Delta^{\bar S}}(\mathcal{A}) \right] \left[|\lambda - a_{p\dots p}|r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A}) + |a_{qp\dots p}|r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A}) \right],
\end{aligned}$$

which implies that

$$\lambda \in \left(\tilde{\mathcal{W}}^1_{p,q}(\mathcal{A}) \bigcap \Theta_p^q(\mathcal{A}) \right) \subseteq \bigcup_{i\in S, j\in\bar S} \left(\tilde{\mathcal{W}}^1_{i,j}(\mathcal{A}) \bigcap \Theta_i^j(\mathcal{A}) \right) \subseteq \mathcal{W}^S(\mathcal{A}).$$

(ii) $|x_p| \le |x_q|$, so $|x_q| = \max_{i\in N}\{|x_i|\}$ and $|x_q| > 0$. Similar to the proof of (i), we can derive

$$\begin{aligned}
&\left[|(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}| - |\lambda - a_{p\dots p}|r^p_{q,\overline{\Delta^S}}(\mathcal{A}) - |a_{qp\dots p}|r^q_{p,\overline{\Delta^S}}(\mathcal{A}) \right] |x_q|^{m-1} \\
\le \; &\left[|\lambda - a_{p\dots p}|r^p_{q,\Delta^S}(\mathcal{A}) + |a_{qp\dots p}|r^q_{p,\Delta^S}(\mathcal{A}) \right] |x_p|^{m-1} \qquad (21)
\end{aligned}$$

and

$$\begin{aligned}&\left[|(\lambda-a_{p\ldots p})(\lambda-a_{q\ldots q})-a_{pq\ldots q}a_{qp\ldots p}|-|\lambda-a_{q\ldots q}|r^{q}_{p,\Delta^{S}}(\mathcal{A})-|a_{pq\ldots q}|r^{p}_{q,\Delta^{S}}\right]|x_p|^{m-1}\\ \leq&\left[|\lambda-a_{q\ldots q}|r^{q}_{p,\overline{\Delta^{S}}}(\mathcal{A})+|a_{pq\ldots q}|r^{p}_{q,\overline{\Delta^{S}}}(\mathcal{A})\right]|x_q|^{m-1}.\end{aligned}\tag{22}$$

If $|x_p|=0$, then it follows from (21) that

$$|(\lambda-a_{p\ldots p})(\lambda-a_{q\ldots q})-a_{pq\ldots q}a_{qp\ldots p}|\leq|\lambda-a_{p\ldots p}|r^{p}_{q,\overline{\Delta^{S}}}(\mathcal{A})+|a_{qp\ldots p}|r^{q}_{p,\overline{\Delta^{S}}}(\mathcal{A}),$$

then $\lambda\in\hat{\mathcal{W}}^{2}_{q,p}(\mathcal{A})\subseteq\bigcup_{i\in\bar{S},j\in S}\hat{\mathcal{W}}^{2}_{i,j}(\mathcal{A})\subseteq\mathcal{W}^{S}(\mathcal{A})$. Otherwise, $|x_p|>0$. If $\lambda\notin\bigcup_{i\in\bar{S},j\in S}\hat{\mathcal{W}}^{2}_{i,j}(\mathcal{A})$, then for any $i\in\bar{S}$ and $j\in S$, we have

$$|(z-a_{i\ldots i})(z-a_{j\ldots j})-a_{ij\ldots j}a_{ji\ldots i}|>|z-a_{j\ldots j}|r^{j}_{i,\overline{\Delta^{S}}}(\mathcal{A})+|a_{ij\ldots j}|r^{i}_{j,\overline{\Delta^{S}}}(\mathcal{A}).$$

It follows from (21) that

$$|(\lambda-a_{p\ldots p})(\lambda-a_{q\ldots q})-a_{pq\ldots q}a_{qp\ldots p}|\leq|\lambda-a_{p\ldots p}|r^{p}_{q}(\mathcal{A})+|a_{qp\ldots p}|r^{q}_{p}(\mathcal{A}),$$

i.e., $\lambda\in\Theta^{p}_{q}(\mathcal{A})$. Based on $|x_q|\geq|x_p|>0$, multiplying (21) with (22) leads to

$$\begin{aligned}&\left[|(z-a_{q\ldots q})(z-a_{p\ldots p})-a_{qp\ldots p}a_{pq\ldots q}|-|z-a_{p\ldots p}|r^{p}_{q,\overline{\Delta^{S}}}(\mathcal{A})-|a_{qp\ldots p}|r^{q}_{p,\overline{\Delta^{S}}}(\mathcal{A})\right]\\ &\cdot\left[|(z-a_{p\ldots p})(z-a_{q\ldots q})-a_{qp\ldots p}a_{pq\ldots q}|-|z-a_{q\ldots q}|r^{q}_{p,\Delta^{S}}(\mathcal{A})-|a_{pq\ldots q}|r^{p}_{q,\Delta^{S}}(\mathcal{A})\right]\\ \leq&\left[|z-a_{p\ldots p}|r^{p}_{q,\Delta^{S}}(\mathcal{A})+|a_{qp\ldots p}|r^{q}_{p,\Delta^{S}}(\mathcal{A})\right]\left[|z-a_{q\ldots q}|r^{q}_{p,\overline{\Delta^{S}}}(\mathcal{A})+|a_{pq\ldots q}|r^{p}_{q,\overline{\Delta^{S}}}(\mathcal{A})\right],\end{aligned}$$

which means that

$$\lambda\in\left(\tilde{\mathcal{W}}^{2}_{q,p}(\mathcal{A})\bigcap\Theta^{p}_{q}(\mathcal{A})\right)\subseteq\bigcup_{i\in\bar{S},j\in S}\left(\tilde{\mathcal{W}}^{2}_{i,j}(\mathcal{A})\bigcap\Theta^{j}_{i}(\mathcal{A})\right)\subseteq\mathcal{W}^{S}(\mathcal{A}).$$

This completes our proof in this theorem. □

Next, we turn to the relations between $\mathcal{W}^{S}(\mathcal{A})$ and $\mathcal{E}^{S}(\mathcal{A})$ in the following theorem. To this end, we start with a useful lemma as follows.

Lemma 2.4. [4] *Let $a,b,c\geq 0$ and $d>0$.*
(I) If $\frac{a}{b+c+d}\leq 1$, then

$$\frac{a-(b+c)}{d}\leq\frac{a-b}{c+d}\leq\frac{a}{b+c+d}.$$

(II) If $\frac{a}{b+c+d}\geq 1$, then

$$\frac{a-(b+c)}{d}\geq\frac{a-b}{c+d}\geq\frac{a}{b+c+d}.$$

Theorem 2.5. *Let $\mathcal{A}=(a_{i_1\ldots i_m})\in\mathbb{C}^{[m,n]}$ with $n\geq 2$ and S be a nonempty proper subset of N. Then*

$$\mathcal{W}^{S}(\mathcal{A})\subseteq\mathcal{E}^{S}(\mathcal{A}).$$

Proof. Let $z \in \mathcal{W}^S(\mathcal{A})$, then

$$z \in \left(\mathcal{W}^S_{i,j}(\mathcal{A})\right) \bigcup \left(\mathcal{W}^{\bar{S}}_{i,j}(\mathcal{A})\right).$$

Without loss of generality, we assume that $z \in \mathcal{W}^S_{i,j}(\mathcal{A})$ (we can prove it similarly if $z \in \mathcal{W}^{\bar{S}}_{i,j}(\mathcal{A})$). That is, $z \in \bigcup_{i\in S, j\in \bar{S}} \hat{\mathcal{W}}^1_{i,j}(\mathcal{A})$ or $z \in \bigcup_{i\in S, j\in \bar{S}} \left(\tilde{\mathcal{W}}^1_{i,j}(\mathcal{A}) \bigcap \Theta^j_i(\mathcal{A})\right)$. If $z \in \bigcup_{i\in S, j\in \bar{S}} \hat{\mathcal{W}}^1_{i,j}(\mathcal{A})$, then there exist $p \in S$ and $q \in \bar{S}$ such that

$$|(z - a_{p\dots p})(z - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}| - |z - a_{q\dots q}| r^q_{p,\overline{\Delta^{\bar{S}}}}(\mathcal{A}) - |a_{pq\dots q}| r^p_{q,\overline{\Delta^{\bar{S}}}}(\mathcal{A}) \le 0,$$

then it can be easily obtained that

$$\begin{aligned} &\left[|(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}| - |\lambda - a_{q\dots q}| r^q_{p,\overline{\Delta^{\bar{S}}}}(\mathcal{A}) - |a_{pq\dots q}| r^p_{q,\overline{\Delta^{\bar{S}}}}(\mathcal{A})\right] \\ &\cdot |(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}| \le \left[|\lambda - a_{q\dots q}| r^q_{p,\Delta^{\bar{S}}}(\mathcal{A}) + |a_{pq\dots q}| r^p_{q,\Delta^{\bar{S}}}(\mathcal{A})\right] \\ &\cdot \left[|\lambda - a_{p\dots p}| r^p_q(\mathcal{A}) + |a_{qp\dots p}| r^q_p(\mathcal{A})\right], \end{aligned}$$

which means that $\lambda \in \mathcal{E}^S_{p,q}(\mathcal{A}) \subseteq \bigcup\limits_{i\in S, j\in \bar{S}} \mathcal{E}^S_{i,j}(\mathcal{A}) \subseteq \mathcal{E}^S(\mathcal{A})$. Otherwise, $z \notin \bigcup_{i\in S, j\in \bar{S}} \hat{\mathcal{W}}^1_{i,j}(\mathcal{A})$, thus we have that

$$|(z - a_{i\dots i})(z - a_{j\dots j}) - a_{ij\dots j}a_{ji\dots i}| - |z - a_{j\dots j}| r^j_{i,\overline{\Delta^{\bar{S}}}}(\mathcal{A}) - |a_{ij\dots j}| r^i_{j,\overline{\Delta^{\bar{S}}}}(\mathcal{A}) > 0 \tag{23}$$

holds for any $i \in S$ and $j \in \bar{S}$, and $z \in \bigcup_{i\in S, j\in \bar{S}} \left(\tilde{\mathcal{W}}^1_{i,j}(\mathcal{A}) \bigcap \Theta^j_i(\mathcal{A})\right)$. Thus we can infer that there exist $p \in S$ and $q \in \bar{S}$ such that

$$\begin{aligned} &\left[|(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}| - |\lambda - a_{q\dots q}| r^q_{p,\overline{\Delta^{\bar{S}}}}(\mathcal{A}) - |a_{pq\dots q}| r^p_{q,\overline{\Delta^{\bar{S}}}}(\mathcal{A})\right] \\ &\cdot \left[|(\lambda - a_{q\dots q})(\lambda - a_{p\dots p}) - a_{pq\dots q}a_{qp\dots p}| - |\lambda - a_{p\dots p}| r^p_{q,\Delta^{\bar{S}}}(\mathcal{A}) - |a_{qp\dots p}| r^q_{p,\Delta^{\bar{S}}}(\mathcal{A})\right] \\ &\le \left[|\lambda - a_{q\dots q}| r^q_{p,\Delta^{\bar{S}}}(\mathcal{A}) + |a_{pq\dots q}| r^p_{q,\Delta^{\bar{S}}}(\mathcal{A})\right]\left[|\lambda - a_{p\dots p}| r^p_{q,\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + |a_{qp\dots p}| r^q_{p,\overline{\Delta^{\bar{S}}}}(\mathcal{A})\right] \end{aligned} \tag{24}$$

and

$$|(z - a_{p\dots p})(z - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}| \le |z - a_{q\dots q}| r^q_p(\mathcal{A}) + |a_{pq\dots q}| r^p_q(\mathcal{A}). \tag{25}$$

The following analysis will be based on two cases:

(1) $\left[|\lambda - a_{q\dots q}| r^q_{p,\Delta^{\bar{S}}}(\mathcal{A}) + |a_{pq\dots q}| r^p_{q,\Delta^{\bar{S}}}(\mathcal{A})\right]\left[|\lambda - a_{p\dots p}| r^p_{q,\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + |a_{qp\dots p}| r^q_{p,\overline{\Delta^{\bar{S}}}}(\mathcal{A})\right] = 0$, then combining (23) and (24) leads to

$$|(\lambda - a_{q\dots q})(\lambda - a_{p\dots p}) - a_{pq\dots q}a_{qp\dots p}| - |\lambda - a_{p\dots p}| r^p_{q,\Delta^{\bar{S}}}(\mathcal{A}) - |a_{qp\dots p}| r^q_{p,\Delta^{\bar{S}}}(\mathcal{A}) \le 0.$$

Then it holds that

$$|(\lambda - a_{q\dots q})(\lambda - a_{p\dots p}) - a_{pq\dots q}a_{qp\dots p}| \le |\lambda - a_{p\dots p}| r^p_q(\mathcal{A}) + |a_{qp\dots p}| r^q_p(\mathcal{A}),$$

it follows that

$$\begin{aligned} &|(\lambda - a_{p\dots p})(\lambda - a_{q\dots q}) - a_{pq\dots q}a_{qp\dots p}| - |\lambda - a_{p\dots p}| r^p_{q,\overline{\Delta^S}}(\mathcal{A}) - |a_{qp\dots p}| r^q_{p,\overline{\Delta^S}}(\mathcal{A}) \\ \le \;& |\lambda - a_{p\dots p}| r^p_{q,\Delta^S}(\mathcal{A}) + |a_{qp\dots p}| r^q_{p,\Delta^S}(\mathcal{A}). \end{aligned} \tag{26}$$

Multiplying (25) with (26) yields

$$\left[|(z-a_{p\ldots p})(z-a_{q\ldots q})-a_{qp\ldots p}a_{pq\ldots q}|-|z-a_{p\ldots p}|r^p_{q,\overline{\Delta^S}}(\mathcal{A})-|a_{qp\ldots p}|r^q_{p,\overline{\Delta^S}}(\mathcal{A})\right]$$
$$\cdot|(z-a_{q\ldots q})(z-a_{p\ldots p})-a_{qp\ldots p}a_{pq\ldots q}|\leq\left[|z-a_{p\ldots p}|r^p_{q,\Delta^S}(\mathcal{A})+|a_{qp\ldots p}|r^q_{p,\Delta^S}(\mathcal{A})\right]$$
$$\cdot\left[|z-a_{q\ldots q}|r^q_p(\mathcal{A})+|a_{pq\ldots q}|r^p_q(\mathcal{A})\right],$$

this follows that $z\in\mathcal{E}^{\bar S}_{q,p}(\mathcal{A})\subseteq\bigcup_{i\in\bar S,j\in S}\mathcal{E}^{\bar S}_{i,j}(\mathcal{A})\subseteq\mathcal{E}^S(\mathcal{A})$.

(ii) $\left[|\lambda-a_{q\ldots q}|r^q_{p,\Delta^{\bar S}}(\mathcal{A})+|a_{pq\ldots q}|r^p_{q,\Delta^{\bar S}}(\mathcal{A})\right]\left[|\lambda-a_{p\ldots p}|r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A})+|a_{qp\ldots p}|r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A})\right]>0$. Dividing (24) by $\left[|\lambda-a_{q\ldots q}|r^q_{p,\Delta^{\bar S}}(\mathcal{A})+|a_{pq\ldots q}|r^p_{q,\Delta^{\bar S}}(\mathcal{A})\right]\left[|\lambda-a_{p\ldots p}|r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A})+|a_{qp\ldots p}|r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A})\right]$ yields

$$\frac{|(\lambda-a_{p\ldots p})(\lambda-a_{q\ldots q})-a_{pq\ldots q}a_{qp\ldots p}|-|\lambda-a_{q\ldots q}|r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A})-|a_{pq\ldots q}|r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A})}{|\lambda-a_{q\ldots q}|r^q_{p,\Delta^{\bar S}}(\mathcal{A})+|a_{pq\ldots q}|r^p_{q,\Delta^{\bar S}}(\mathcal{A})}$$
$$\cdot\frac{|(\lambda-a_{q\ldots q})(\lambda-a_{p\ldots p})-a_{pq\ldots q}a_{qp\ldots p}|-|\lambda-a_{p\ldots p}|r^p_{q,\Delta^{\bar S}}(\mathcal{A})-|a_{qp\ldots p}|r^q_{p,\Delta^{\bar S}}(\mathcal{A})}{|\lambda-a_{p\ldots p}|r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A})+|a_{qp\ldots p}|r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A})}\leq 0. \tag{27}$$

Let $a=|(\lambda-a_{q\ldots q})(\lambda-a_{p\ldots p})-a_{pq\ldots q}a_{qp\ldots p}|\geq 0$, $b=|\lambda-a_{p\ldots p}|r^p_{q,\Delta^{\bar S}}(\mathcal{A})\geq 0$, $c=|a_{qp\ldots p}|r^q_{p,\Delta^{\bar S}}(\mathcal{A})\geq 0$ and $d=|\lambda-a_{p\ldots p}|r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A})+|a_{qp\ldots p}|r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A})>0$. If

$$\frac{|(\lambda-a_{q\ldots q})(\lambda-a_{p\ldots p})-a_{pq\ldots q}a_{qp\ldots p}|-|\lambda-a_{p\ldots p}|r^p_{q,\Delta^{\bar S}}(\mathcal{A})-|a_{qp\ldots p}|r^q_{p,\Delta^{\bar S}}(\mathcal{A})}{|\lambda-a_{p\ldots p}|r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A})+|a_{qp\ldots p}|r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A})}\geq 1,$$

then by (II) of Lemma 2.4, it follows from (27) that

$$\frac{|(\lambda-a_{p\ldots p})(\lambda-a_{q\ldots q})-a_{pq\ldots q}a_{qp\ldots p}|-|\lambda-a_{q\ldots q}|r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A})-|a_{pq\ldots q}|r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A})}{|\lambda-a_{q\ldots q}|r^q_{p,\Delta^{\bar S}}(\mathcal{A})+|a_{pq\ldots q}|r^p_{q,\Delta^{\bar S}}(\mathcal{A})}$$
$$\cdot\frac{|(\lambda-a_{q\ldots q})(\lambda-a_{p\ldots p})-a_{pq\ldots q}a_{qp\ldots p}|}{|\lambda-a_{p\ldots p}|r^p_q(\mathcal{A})+|a_{qp\ldots p}|r^q_p(\mathcal{A})}$$
$$\leq\frac{|(\lambda-a_{p\ldots p})(\lambda-a_{q\ldots q})-a_{pq\ldots q}a_{qp\ldots p}|-|\lambda-a_{q\ldots q}|r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A})-|a_{pq\ldots q}|r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A})}{|\lambda-a_{q\ldots q}|r^q_{p,\Delta^{\bar S}}(\mathcal{A})+|a_{pq\ldots q}|r^p_{q,\Delta^{\bar S}}(\mathcal{A})}$$
$$\cdot\frac{|(\lambda-a_{q\ldots q})(\lambda-a_{p\ldots p})-a_{pq\ldots q}a_{qp\ldots p}|-|\lambda-a_{p\ldots p}|r^p_{q,\Delta^{\bar S}}(\mathcal{A})-|a_{qp\ldots p}|r^q_{p,\Delta^{\bar S}}(\mathcal{A})}{|\lambda-a_{p\ldots p}|r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A})+|a_{qp\ldots p}|r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A})}\leq 0,$$

straightforwardly leads to

$$\left[|(\lambda-a_{p\ldots p})(\lambda-a_{q\ldots q})-a_{pq\ldots q}a_{qp\ldots p}|-|\lambda-a_{q\ldots q}|r^q_{p,\overline{\Delta^{\bar S}}}(\mathcal{A})-|a_{pq\ldots q}|r^p_{q,\overline{\Delta^{\bar S}}}(\mathcal{A})\right]$$
$$\cdot|(\lambda-a_{q\ldots q})(\lambda-a_{p\ldots p})-a_{pq\ldots q}a_{qp\ldots p}|\leq\left[|\lambda-a_{q\ldots q}|r^q_{p,\Delta^{\bar S}}(\mathcal{A})+|a_{pq\ldots q}|r^p_{q,\Delta^{\bar S}}(\mathcal{A})\right]$$
$$\cdot\left[|\lambda-a_{p\ldots p}|r^p_q(\mathcal{A})+|a_{qp\ldots p}|r^q_p(\mathcal{A})\right],$$

which implies that $z \in \mathcal{E}^S_{p,q}(\mathcal{A}) \subseteq \bigcup_{i\in S, j\in \bar{S}} \mathcal{E}^S_{i,j}(\mathcal{A}) \subseteq \mathcal{E}^S(\mathcal{A})$. Additionally, if

$$\frac{|(\lambda - a_{q\ldots q})(\lambda - a_{p\ldots p}) - a_{pq\ldots q}a_{qp\ldots p}| - |\lambda - a_{p\ldots p}|r^p_{q,\Delta^{\bar{S}}}(\mathcal{A}) - |a_{qp\ldots p}|r^q_{p,\Delta^{\bar{S}}}(\mathcal{A})}{|\lambda - a_{p\ldots p}|r^p_{q,\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + |a_{qp\ldots p}|r^q_{p,\overline{\Delta^{\bar{S}}}}(\mathcal{A})} \leq 1,$$

then it is not difficult to verify that (26) holds. Multiplying (25) with (26) leads to

$$\begin{aligned}&\left[|(z - a_{q\ldots q})(z - a_{p\ldots p}) - a_{qp\ldots p}a_{pq\ldots q}| - |z - a_{p\ldots p}|r^p_{q,\overline{\Delta^S}}(\mathcal{A}) - |a_{qp\ldots p}|r^q_{p,\overline{\Delta^S}}(\mathcal{A})\right]\\ &\cdot|(z - a_{q\ldots q})(z - a_{p\ldots p}) - a_{qp\ldots p}a_{pq\ldots q}| \leq \left[|z - a_{p\ldots p}|r^p_{q,\Delta^S}(\mathcal{A}) + |a_{qp\ldots p}|r^q_{p,\Delta^S}(\mathcal{A})\right]\\ &\cdot\left[|z - a_{q\ldots q}|r^q_p(\mathcal{A}) + |a_{pq\ldots q}|r^p_q(\mathcal{A})\right],\end{aligned}$$

which means that $z \in \mathcal{E}^{\bar{S}}_{q,p}(\mathcal{A}) \subseteq \bigcup_{i\in \bar{S}, j\in S} \mathcal{E}^{\bar{S}}_{i,j}(\mathcal{A}) \subseteq \mathcal{E}^S(\mathcal{A})$. This completes the proof of this theorem. □

According to Theorems 2.2 and 2.5, the comparisons among $\mathcal{W}^S(\mathcal{A})$, $\mathcal{E}^S(\mathcal{A})$, $\Upsilon^S(\mathcal{A})$ $\mathcal{K}^S(\mathcal{A})$ and $\mathcal{K}(\mathcal{A})$ are given as follows.

Theorem 2.6. *Let $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$ with $n \geq 2$ and S be a nonempty proper subset of N. Then*

$$\mathcal{W}^S(\mathcal{A}) \subseteq \mathcal{E}^S(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}).$$

In the sequel, we construct an example to compare the regions of Theorems 2.1 and 2.3 with those in Theorem 3.1 of [20] and Theorem 4 of [21].

Example 2.7. *Let the tensor $\mathcal{A}$ be defined as follows:*

$$\mathcal{A} = [A(1,:,:), A(2,:,:), A(3,:,:), A(4,:,:)] \in \mathbb{R}^{[3,4]},$$

where

$$A(1,:,:) = \begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}, \; A(2,:,:) = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2} & 0\\ 0 & \frac{1}{2} & 0 & \frac{1}{2}\\ 0 & 0 & \frac{1}{2} & 0\end{pmatrix},$$

$$A(3,:,:) = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}, \; A(4,:,:) = \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 0 & 0\end{pmatrix}.$$

By computations, we get that $\sigma(\mathcal{A}) \subseteq \Delta^{\cap}(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq 5\}$ (by Lemma 1.5), $\sigma(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq 5\}$ for all $S \subseteq N$ and $S \neq N$ (by Lemma 1.4), $\sigma(\mathcal{A}) \subseteq \mathcal{E}^S(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq 4.7200\}$ (by Theorem 2.1) and $\sigma(\mathcal{A}) \subseteq \mathcal{W}^S(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq 4.6458\}$ (by Theorem 2.3) as $S = \{3, 4\}$ and $\bar{S} = \{1, 2\}$. It noteworthy that the localization sets $\mathcal{E}^S(\mathcal{A})$ and $\mathcal{W}^S(\mathcal{A})$ are the sharpest as $S = \{3, 4\}$ and $\bar{S} = \{1, 2\}$. It can clearly be seen that $\mathcal{W}^S(\mathcal{A}) \subseteq \mathcal{E}^S(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A}) = \Delta^{\cap}(\mathcal{A})$ (see Figure 1).

Remark 2.8. *Example 2.7 shows that the eigenvalue localization sets in Theorems 2.1 and 2.3 are sharper than that in Theorem 3.1 of [20], and $\mathcal{W}^S(\mathcal{A})$ outperforms $\mathcal{E}^S(\mathcal{A})$. What we want to point out here is that: Although $\mathcal{W}^S(\mathcal{A})$ can capture all eigenvalues of $\mathcal{A}$ more precisely than $\mathcal{E}^S(\mathcal{A})$, there are more computations to determine $\mathcal{W}^S(\mathcal{A})$ and its form is more complicated compared with $\mathcal{E}^S(\mathcal{A})$.*

Compare the eigenvalue localization sets exhibited in Theorems 2.1 and 2.3 with that in Theorem 3.1 of [20], it is observed that the forms of those are different. In the following, based on Theorem 3.1 of [20], we develop an S-type eigenvalue localization set, whose form is similar to that in Theorem 3.1 of [20].

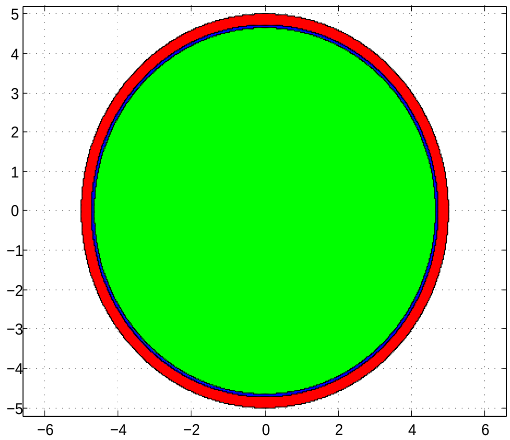

Figure 1: $\Delta^{\cap}(\mathcal{A})$ and $\Upsilon^S(\mathcal{A})$ are filled by red color, $\mathcal{E}^S(\mathcal{A})$ and $\mathcal{W}^S(\mathcal{A})$ are filled by blue and green colors, respectively.

Theorem 2.9. *Let $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$ with $n \geq 2$. And let S be a nonempty proper subset of N. Then*

$$\sigma(\mathcal{A}) \subseteq \hat{\Upsilon}^S(\mathcal{A}) = \left(\bigcup_{i\in S}\bigcap_{j\in\bar{S}} \Upsilon_i^j(\mathcal{A})\right) \bigcup \left(\bigcup_{i\in\bar{S}}\bigcap_{j\in S} \Upsilon_i^j(\mathcal{A})\right), \tag{28}$$

where $\Upsilon_i^j(\mathcal{A})$ is defined as in Lemma 1.4.

Proof. For any $\lambda \in \sigma(\mathcal{A})$, let $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n\backslash\{0\}$ be an eigenvector corresponding to λ, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}.$$

Let $|x_p| = \max_{i\in S}\{|x_i|\}$ and $|x_q| = \max_{i\in\bar{S}}\{|x_i|\}$. Now, let us distinguish two cases to prove.

(i) $|x_p| \geq |x_q|$, so $|x_p| = \max_{i\in N}\{|x_i|\}$ and $|x_p| > 0$. For any $j \in \bar{S}$. In the same manner applied in the proof of Theorem 3.1 in [20], it holds that

$$\begin{aligned} & |(\lambda - a_{j\ldots j})(\lambda - a_{p\ldots p}) - a_{pj\ldots j}a_{jp\ldots p}||x_p|^{m-1} \\ \leq\ & |\lambda - a_{j\ldots j}|r_p^j(\mathcal{A})|x_p|^{m-1} + |a_{pj\ldots j}|r_j^p(\mathcal{A})|x_p|^{m-1}. \end{aligned}$$

Note that $|x_p| > 0$, thus

$$|(\lambda - a_{j\ldots j})(\lambda - a_{p\ldots p}) - a_{pj\ldots j}a_{jp\ldots p}| \leq |\lambda - a_{j\ldots j}|r_p^j(\mathcal{A}) + |a_{pj\ldots j}|r_j^p(\mathcal{A}),$$

which implies that $\lambda \in \Upsilon_p^j(\mathcal{A})$. From the arbitrariness of $j \in \bar{S}$, it has $\lambda \in \bigcap_{j\in\bar{S}} \Upsilon_p^j(\mathcal{A}) \subseteq \bigcup_{i\in S}\bigcap_{j\in\bar{S}} \Upsilon_i^j(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A})$.

(ii) $|x_p| \leq |x_q|$, so $|x_q| = \max_{i\in N}\{|x_i|\}$ and $|x_q| > 0$. For any $i \in S$, with a quite similar strategy utilized in Theorem 3.1 of [20], we can obtain

$$\begin{aligned} & |(\lambda - a_{q\ldots q})(\lambda - a_{i\ldots i}) - a_{iq\ldots q}a_{qi\ldots i}||x_q|^{m-1} \\ \leq\ & |\lambda - a_{i\ldots i}|r_q^i(\mathcal{A})|x_q|^{m-1} + |a_{qi\ldots i}|r_i^q(\mathcal{A})|x_q|^{m-1}. \end{aligned}$$

Note that $|x_q| > 0$, thus

$$|(\lambda - a_{q\ldots q})(\lambda - a_{i\ldots i}) - a_{iq\ldots q}a_{qi\ldots i}| \leq |\lambda - a_{i\ldots i}|r_q^i(\mathcal{A}) + |a_{qi\ldots i}|r_i^q(\mathcal{A}).$$

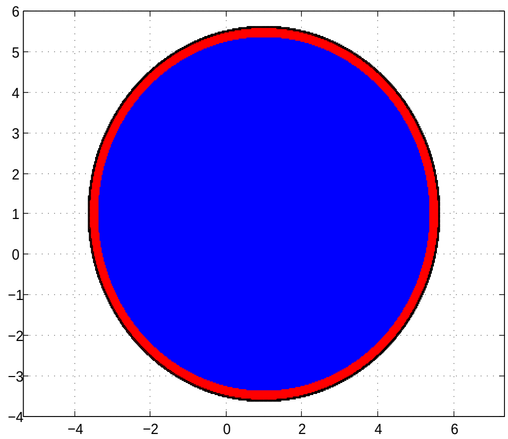

Figure 2: $\mathcal{B}(\mathcal{A})$, $\Upsilon^S(\mathcal{A})$ and $\hat{\Upsilon}^S(\mathcal{A})$ are filled by black, red and blue colors, respectively.

This means that $\lambda \in \Upsilon_q^i(\mathcal{A})$. It follows from the arbitrariness of $i \in S$ that $\lambda \in \bigcap_{j\in S} \Upsilon_q^j(\mathcal{A}) \subseteq \bigcup_{i\in\bar{S}} \bigcap_{j\in S} \Upsilon_i^j(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A})$. Then the result of this theorem is immediately obtained. □

The following theorem concentrates on comparing Theorem 2.9 with Theorem 3.1 of [20].

Theorem 2.10. *Let $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$ with $n \geq 2$ and S be a nonempty proper subset of N. Then*

$$\hat{\Upsilon}^S(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A}).$$

Proof. It is not difficult to verify that

$$\bigcup_{i\in S} \bigcap_{j\in\bar{S}} \Upsilon_i^j(\mathcal{A}) \subseteq \bigcup_{i\in S, j\in\bar{S}} \Upsilon_i^j(\mathcal{A}) \text{ and } \bigcup_{i\in\bar{S}} \bigcap_{j\in S} \Upsilon_i^j(\mathcal{A}) \subseteq \bigcup_{i\in\bar{S}, j\in S} \Upsilon_i^j(\mathcal{A}),$$

then we have $\hat{\Upsilon}^S(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A})$. This proof is completed. □

An example is given to compare the region of Theorem 2.9 with those of Lemma 1.4 and Theorem 3.1 of [22].

Example 2.11. *Let*

$$\mathcal{A} = \left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{112} & a_{113} & a_{211} & a_{212} & a_{213} & a_{311} & a_{312} & a_{313} \\ a_{121} & a_{122} & a_{123} & a_{221} & a_{222} & a_{223} & a_{321} & a_{322} & a_{323} \\ a_{131} & a_{132} & a_{133} & a_{231} & a_{232} & a_{233} & a_{331} & a_{332} & a_{333} \end{array}\right) = \left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 0.5 & 1 & 1.5 & 0 & 2 & 0 & 0 \\ 0.5 & 0.5 & 0 & 1 & 1 & 0 & 0 & 3 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 1 \end{array}\right).$$

By Theorem 3.1 of [22], we obtain

$$\sigma(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{A}) = \{z \in \mathbb{C} : |z - 1| \leq 4.6416\}.$$

From Lemma 1.4, we obtain

$$\sigma(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A}) = \{z \in \mathbb{C} : |z - 1| \leq 4.6056\}.$$

According to Theorem 2.9, we derive

$$\sigma(\mathcal{A}) \subseteq \hat{\Upsilon}^S(\mathcal{A}) = \{z \in \mathbb{C} : |z - 1| \leq 4.3723\}.$$

The comparative results are given in Figure 2. Clearly, $\hat{\Upsilon}^S(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{A})$.

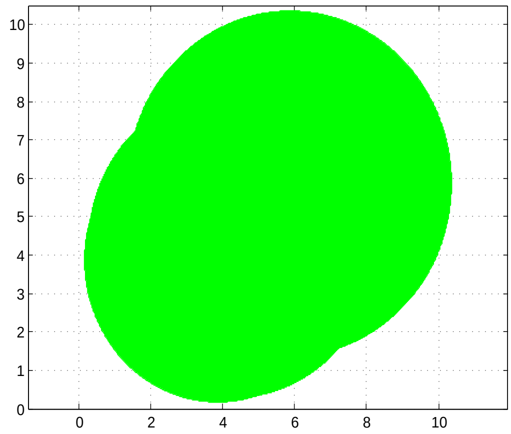

Figure 3: $\mathcal{E}^S(\mathcal{A})$ is fulled by green color.

3. Applications of the New *S*-type Eigenvalue Inclusion Sets for Tensors

The eigenvalue inclusion sets of Theorems 2.1 and 2.3 can be used to judge the positive definiteness of the tensor $\mathcal{A}$. The applications of those will be stressed by Examples 3.1 and 3.2.

Example 3.1. [4] *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,3]}$ be a real symmetric tensor with elements defined as follows:*

$$a_{1111} = 5,\ a_{2222} = 6,\ a_{3333} = 3.3,\ a_{1112} = -0.1,\ a_{1113} = 0.1,$$
$$a_{1122} = -0.2,\ a_{1123} = -0.2,\ a_{1133} = 0,\ a_{1222} = -0.1,\ a_{1223} = 0.3,$$
$$a_{1233} = 0.1,\ a_{1333} = -0.1,\ a_{2223} = 0.1,\ a_{2233} = -0.1,\ a_{2333} = 0.2.$$

After some calculations, we conclude that the tensor $\mathcal{A}$ can not meet the conditions of Theorem 3.2 in [7] and Theorem 4.1 of [5], and for any nonempty proper subset S of N, Theorem 4.2 of [5] can not be applied to determine the positive definiteness of $\mathcal{A}$, while we choose $S = \{1, 2\}$, $\bar{S} = \{3\}$, and utilize Theorem 2.1, we can obtain the eigenvalue localization set $\mathcal{E}^S(\mathcal{A})$ in Figure 3.

As observed in Figure 3, all eigenvalues of $\mathcal{A}$ are located in right hand side of the complex plane, which means that the smallest H-eigenvalue of $\mathcal{A}$ is positive, hence $\mathcal{A}$ is positive definite.

Example 3.2. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,3]}$ be a real symmetric tensor with elements defined as follows:

$$a_{1111} = 5.2,\ a_{2222} = 6.05,\ a_{3333} = 3.3,\ a_{1112} = -0.1,\ a_{1113} = 0.1,$$
$$a_{1122} = -0.2,\ a_{1123} = -0.2,\ a_{1133} = 0,\ a_{1222} = -0.1,\ a_{1223} = 0.3,$$
$$a_{1233} = 0.1,\ a_{1333} = -0.2,\ a_{2223} = 0.1,\ a_{2233} = -0.1,\ a_{2333} = 0.2.$$

By proper calculations, we confirm that the tensor $\mathcal{A}$ can not meet the conditions of Theorem 3.2 in [7] and Theorem 4.1 of [5]. Theorem 4.2 of [5] and Theorem 7 of [4] can not be utilized to determine the positive definiteness of $\mathcal{A}$ for any nonempty proper subset S of N. Besides, by using the methods in Theorems 1 and 2 of [25], we also can not determine the positive definiteness of $\mathcal{A}$. Let $S = \{1, 2\}$, $\bar{S} = \{3\}$. According to Theorem 2.3, the eigenvalue localization set $\mathcal{W}^S(\mathcal{A})$ is depicted in Figure 4.

From Figure 4, it can be seen that all eigenvalues of $\mathcal{A}$ are located in right hand side of the complex plane, and therefore the smallest H-eigenvalue of $\mathcal{A}$ is positive, which implies that $\mathcal{A}$ is positive definite.

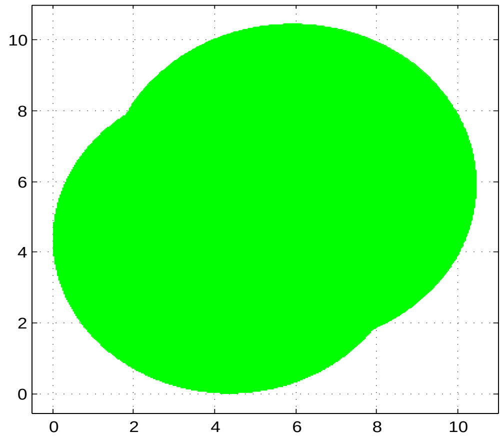

Figure 4: $\mathcal{W}^S(\mathcal{A})$ is fulled by green color.

4. Concluding Remarks

In this paper, some new S-type eigenvalue inclusion sets for tensors as well as their comparison theorems are derived, which show that the proposed sets are sharper than the one in [20]. As applications, these eigenvalue inclusion sets can be applied to determine the positive definiteness or positive semi-definiteness of the even-order symmetric tensors.

However, the new S-type eigenvalue inclusion sets depend on the set S. How to choose S to make $\mathcal{E}^S(\mathcal{A})$, $\mathcal{W}^S(\mathcal{A})$ and $\hat{\Upsilon}^S(\mathcal{A})$ established in this paper as tight as possible is very important and interesting, while if the dimension of the tensor $\mathcal{A}$ is large, this work is still underdeveloped and very difficult. Therefore, future work will include numerical or theoretical studies for finding the best choice for S.

Competing Interests

The authors declare that they have no competing interests.

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