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Subsequences of Triangular Partial Sums of Double Fourier Series on Unbounded Vilenkin Groups

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Abstract. In 1987 Harris proved-among others-that for each $1 \leq p < 2$ there exists a two-dimensional function $f \in L_p$ such that its triangular partial sums $S^{\triangle}_{2^A} f$ of Walsh-Fourier series does not converge almost everywhere. In this paper we prove that subsequences of triangular partial sums $S^{\triangle}_{n_A M_A} f, n_A \in \{1, 2, ..., m_A - 1\}$ on unbounded Vilenkin groups converge almost everywhere to f for each function $f \in L_2$.

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$. Let $m := (m_0, m_1, ...)$ denote a sequence of positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, ..., m_k - 1\}$ the additive group of integers modulo m_k. Define the group G_m as the complete direct product of the groups Z_{m_j}, with the product of the discrete topologies of Z_{m_j}'s. The direct product μ of the measures

$$\mu_k(\{j\}) := \frac{1}{m_k} \quad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$. The elements of G_m can be represented by sequences $x := (x_0, x_1, ..., x_j, ...)$, $(x_j \in Z_{m_j})$. The group operation $+$ in G_m is given by

$$x + y = (x_0 + y_0 \, (\mathrm{mod} m_0), ..., x_k + y_k \, (\mathrm{mod} m_k), ...),$$

where $x = (x_0, ..., x_k, ...)$ and $y = (y_0, ..., y_k, ...) \in G$. The inverse of $+$ will be denoted by $-$.

It is easy to give a base for the neighborhoods of G_m :

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m | y_0 = x_0, ..., y_{n-1} = x_{n-1}\}$$

for $x \in G_m$, $n \in \mathbb{N}$. Define $I_n := I_n(0)$ for $n \in \mathbb{N}$. The sets $I_n(x)$ are called (m-adic) intervals.

Define $\mathcal{A}_{A,B}$ the σ-algebra generated by rectangles $I_A(x^1) \times I_B(x^2)$ as $x = (x^1, x^2)$ rolls over $G_m \times G_m$. Let $E_{A,B}$ be the conditional expectation operator with respect to σ-algebra $\mathcal{A}_{A,B}$. That is,

$$E_{A,B} f(x^1, x^2) = M_A M_B \int_{I_A(x^1) \times I_B(x^2)} f(y^1, y^2) d\mu(y^1, y^2).$$

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If $A = B$, then we simple write $\mathcal{A}_A$ and E_A instead of $\mathcal{A}_{A,A}$ and $E_{A,A}$.

If we define the so-called generalized number system based on m in the following way: $M_0 := 1, M_{k+1} := m_k M_k (k \in \mathbb{N})$, then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N})$ and only a finite number of n_j's differ from zero.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system [1]. At first, define the complex valued functions $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions in this way

$$\rho_k(x) := \exp \frac{2\pi \imath x_k}{m_k} \ (\imath^2 = -1,\ x \in G_m,\ k \in \mathbb{N}).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as follows.

$$\psi_n(x) := \prod_{k=0}^{\infty} \rho_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley one if $m \equiv 2$.

Dirichlet kernels are defined as follows

$$D_n := \sum_{k=0}^{n-1} \psi_k \qquad (n \in \mathbb{N}).$$

Recall that [7]

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \in G_m \backslash I_n. \end{cases} \tag{1}$$

It is well known that (see ([7]))

$$D_n = \psi_n \sum_{j=0}^{\infty} D_{M_j} \sum_{a=m_j-n_j}^{m_j-1} \rho_j^a. \tag{2}$$

The norm of the space $L_p(G_m \times G_m)$ is defined by (μ is the product measue $\mu \times \mu$)

$$\|f\|_p := \left(\int_{G_m \times G_m} \left| f\left(x^1, x^2\right) \right|^p d\mu\left(x^1, x^2\right) \right)^{1/p} < \infty, 1 \leq p < \infty.$$

The rectangular partial sums of the double Vilenkin-Fourier series are defined as follows:

$$S_{n,m}(f; x^1, x^2) := \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \widehat{f}(i,j)\, \psi_i\left(x^1\right) \psi_j\left(x^2\right),$$

where the number

$$\widehat{f}(i,j) = \int_{G_m \times G_m} f\left(x^1, x^2\right) \overline{\psi}_i\left(x^1\right) \overline{\psi}_j\left(x^2\right) d\mu\left(x^1, x^2\right).$$

is said to be the (i, j)th Vilenkin-Fourier coefficient of the function f.

The triangular partial sums are defined as

$$S_k^{\triangle}\left(x^1,x^2;f\right)=\sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1}\hat{f}(i,j)\,\psi_i\left(x^1\right)\psi_j\left(x^2\right).$$

Set

$$S_n^{\square}(x^1,x^2;f):=S_{n,n}(x^1,x^2;f).$$

It is evident that

$$\begin{aligned}S_k^{\triangle}\left(x^1,x^2;f\right) &= \left(f*D_k^{\triangle}\right)\left(x^1,x^2\right)\\ &:= \int_{G_m\times G_m} f\left(y^1,y^2\right)D_k^{\triangle}\left(x^1-y^1,x^2-y^2\right)d\mu\left(y^1,y^2\right),\end{aligned}$$

$$\begin{aligned}S_k^{\square}\left(x^1,x^2;f\right) &= \left(f*D_k^{\square}\right)\left(x^1,x^2\right)\\ &:= \int_{G_m\times G_m} f\left(y^1,y^2\right)D_k^{\square}\left(x^1-y^1,x^2-y^2\right)d\mu\left(y^1,y^2\right),\end{aligned}$$

where

$$D_k^{\triangle}\left(x^1,x^2\right):=\sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1}\psi_i\left(x^1\right)\psi_j\left(x^2\right)$$

and

$$D_k^{\square}\left(x^1,x^2\right):=\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\psi_i\left(x^1\right)\psi_j\left(x^2\right).$$

In 1971 Fefferman proved [3] the following result with respect to the trigonometric system. Let P be an open polygonal region in $\mathbb{R}^2$, containing the origin. Set

$$\lambda P:=\left\{\left(\lambda x^1,\lambda x^2\right):\left(x^1,x^2\right)\in P\right\}$$

for $\lambda>0$. Then for every $p>1$, $f\in L_p\left([-\pi,\pi]^2\right)$ it holds the relation

$$\sum_{(n^1,n^2)\in\lambda P}\widehat{f}\left(n^1,n^2\right)\exp\left(i\left(n^1y^1+n^2y^2\right)\right)\to f\left(y^1,y^2\right)\text{ as }\lambda\to\infty$$

for a. e. $\left(y^1,y^2\right)\in[-\pi,\pi]^2$. That is, $S_{\lambda P}f\to f$ a. e. Sjölin gave [6] a better result in the case when P is a rectangle. He proved the a. e. convergence for the class $f\in L\left(\log^+L\right)^3\log^+\log^+L$ and for functions $f\in L\left(\log^+L\right)^2\log^+\log^+L$ when P is a square. This result for squares is improved by Antonov [2]. There is a sharp constrant between the trigonometric and the Walsh case. In 1987 Harris proved [5] for the Walsh system that if S is a region in $[0,\infty)\times[0,\infty)$ with piecewise C^1 boundary not always paralled to the axes and $1\le p<2$, then there exists an $f\in L_p\left(G_2\times G_2\right)$ such that $S_{\lambda P}f$ does not converges a. e. and in L_p norms as $\lambda\to\infty$. In particular, from theorem of Harris it follows that for any $1\le p<2$ there exists an $f\in L_p\left(G_2\times G_2\right)$ such that $S_{2^A}^{\triangle}f$ does not converges a. e. as $A\to\infty$.

In this paper we prove that the following is true.

Theorem 1. *Let $n_A \in \{1, 2, ..., m_A - 1\}$ and $f \in L_2(G_m \times G_m)$. Then subsequences of triangular partial sums $S^{\triangle}_{n_A M_A} f$ of two-dimensional Fourier series on unbounded Vilenkin group converges almost everywhere to f.*

Proof. We can write

$$\begin{aligned} D^{\triangle}_{n_A M_A}\left(x^1, x^2\right) &= \sum_{i=0}^{n_A M_A - 1} \sum_{j=0}^{n_A M_A - i - 1} \psi_i\left(x^1\right)\psi_j\left(x^2\right) \\ &= \sum_{i=0}^{n_A M_A - 1} \psi_i\left(x^1\right) D_{n_A M_A - i}\left(x^2\right) \\ &= \sum_{r=0}^{n_A - 1} \sum_{i=0}^{M_A - 1} \psi_{i + r M_A}\left(x^1\right) D_{n_A M_A - i - r M_A}\left(x^2\right) \\ &= \sum_{r=0}^{n_A - 1} \rho_A^r\left(x^1\right) \sum_{i=0}^{M_A - 1} \psi_i\left(x^1\right) D_{(n_A - r) M_A - i}\left(x^2\right) \\ &= \sum_{r=0}^{n_A - 2} \rho_A^r\left(x^1\right) \sum_{i=0}^{M_A - 1} \psi_i\left(x^1\right) D_{(n_A - r) M_A - i}\left(x^2\right) \\ &\quad + \rho_A^{n_A - 1}\left(x^1\right) \sum_{i=0}^{M_A - 1} \psi_i\left(x^1\right) D_{M_A - i}\left(x^2\right) \\ &=: T_A^{(1)}\left(x^1, x^2\right) + T_A^{(2)}\left(x^1, x^2\right). \end{aligned} \tag{3}$$

Let $n_A = 1$. Since (see [4])

$$D_{M_A - i}(x) = D_{M_A}(x) - \overline{\psi}_{M_A - 1}(-x) D_i(-x) \tag{4}$$

for $T_A^{(2)}\left(x^1, x^2\right)$ we can write

$$\begin{aligned} T_A^{(2)}\left(x^1, x^2\right) &= \sum_{i=0}^{M_A - 1} \psi_i\left(x^1\right) D_{M_A - i}\left(x^2\right) \\ &= D_{M_A}\left(x^1\right) D_{M_A}\left(x^2\right) \\ &\quad - \overline{\psi}_{M_A - 1}\left(-x^2\right) \sum_{i=0}^{M_A - 1} \psi_i\left(x^1\right) D_i\left(-x^2\right). \end{aligned} \tag{5}$$

Since (see [7])

$$\begin{aligned} D_{r M_{A-1} + i}(x) &= D_{r M_{A-1}}(x) + \psi_{r M_{A-1}}(x) D_i(x), \\ r &= 1, ..., m_{A-1} - 1,\ i = 0, ..., M_{A-1} - 1 \end{aligned}$$

then we have

$$\begin{aligned}
&\sum_{i=0}^{M_A-1}\psi_i\left(x^1\right)D_i\left(-x^2\right)\\
=&\sum_{i=0}^{M_{A-1}-1}\psi_i\left(x^1\right)D_i\left(-x^2\right)\\
&+\sum_{r=1}^{m_{A-1}-1}\sum_{i=0}^{M_{A-1}-1}\psi_{i+rM_{A-1}}\left(x^1\right)D_{i+rM_{A-1}}\left(-x^2\right)\\
=&\sum_{i=0}^{M_{A-1}-1}\psi_i\left(x^1\right)D_i\left(-x^2\right)\\
&+\sum_{r=1}^{m_{A-1}-1}\rho_{A-1}^r\left(x^1\right)D_{M_{A-1}}\left(x^1\right)D_{rM_{A-1}}\left(-x^2\right)\\
&+\sum_{r=1}^{m_{A-1}-1}\rho_{A-1}^r\left(x^1\right)\rho_{A-1}^r\left(-x^2\right)\sum_{i=0}^{M_{A-1}-1}\psi_i\left(x^1\right)D_i\left(-x^2\right)\\
=&\sum_{i=0}^{M_{A-1}-1}\psi_i\left(x^1\right)D_i\left(-x^2\right)\sum_{r=0}^{m_{A-1}-1}\rho_{A-1}^r\left(x^1-x^2\right)\\
&+\left(\sum_{r=1}^{m_{A-1}-1}\rho_{A-1}^r\left(x^1\right)D_{rM_{A-1}}\left(-x^2\right)\right)D_{M_{A-1}}\left(x^1\right).
\end{aligned}$$

Iterating this equality we obtain

$$\begin{aligned}
&\sum_{i=0}^{M_A-1}\psi_i\left(x^1\right)D_i\left(-x^2\right)\\
=&\sum_{j=0}^{A-1}\left(\left(\sum_{r=1}^{m_j-1}\rho_j^r\left(x^1\right)D_{rM_j}\left(-x^2\right)\right)D_{M_j}\left(x^1\right)\right)\\
&\times\prod_{s=j+1}^{A-1}\sum_{l=0}^{m_s-1}\rho_s^l\left(x^1-x^2\right).
\end{aligned}\tag{6}$$

Combining (5) and (6) we have (recall that still $n_A = 1$ is supposed)

$$\begin{aligned}
T_A^{(2)}\left(x^1,x^2\right) =& D_{M_A}\left(x^1\right)D_{M_A}\left(x^2\right)\\
&-\overline{\psi}_{M_A-1}\left(-x^2\right)\sum_{j=0}^{A-1}\left(\sum_{r=1}^{m_j-1}\rho_j^r\left(x^1\right)D_{rM_j}\left(-x^2\right)\right)\\
&\times D_{M_j}\left(x^1\right)\prod_{s=j+1}^{A-1}\sum_{l=0}^{m_s-1}\rho_s^l\left(x^1-x^2\right)\\
=& D_{M_A}\left(x^1\right)D_{M_A}\left(x^2\right)-T_A^{(2,1)}\left(x^1,x^2\right).
\end{aligned}\tag{7}$$

We can write

$$
\begin{aligned}
& T_A^{(2,1)}\left(x^1,x^2\right) \\
= \;& \overline{\psi}_{M_A-1}\left(-x^2\right)\sum_{r=1}^{m_{A-1}-1}\rho_{A-1}^{r}\left(x^1\right)D_{rM_{A-1}}\left(-x^2\right)D_{M_{A-1}}\left(x^1\right) \qquad (8)\\
& +\overline{\psi}_{M_A-1}\left(-x^2\right)\sum_{j=0}^{A-2}\left(\sum_{r=1}^{m_j-1}\rho_j^r\left(x^1\right)D_{rM_j}\left(-x^2\right)\right)\\
& \times D_{M_j}\left(x^1\right)\prod_{s=j+1}^{A-2}\sum_{l=0}^{m_s-1}\rho_s^l\left(x^1-x^2\right)\\
& \times\left(1+\sum_{l=1}^{m_{A-1}-1}\rho_{A-1}^{l}\left(x^1-x^2\right)\right)\\
= \;& : T_A^{(2,1,1)}\left(x^1,x^2\right)+T_A^{(2,1,2)}\left(x^1,x^2\right).
\end{aligned}
$$

The properties of the m-adic number system and the Vilenkin functions give $M_A-1=\sum_{j=0}^{A-1}(m_j-1)M_j$ and then

$$
\begin{aligned}
&\psi_{M_A-1}(x)=\prod_{j=0}^{A-1}\rho_j^{m_j-1}(x)\\
&=\prod_{j=0}^{A-1}\exp(2\pi\imath(m_j-1)x_j/m_j)=\prod_{j=0}^{A-1}\exp(-2\pi\imath x_j/m_j)=\prod_{j=0}^{A-1}\overline{\psi}_{M_j}(x).
\end{aligned}
$$

That is, since

$$
D_{rM_{A-1}}(x)=\left(\sum_{q=0}^{r-1}\rho_{A-1}^{q}(x)\right)D_{M_{A-1}}(x) \tag{9}
$$

and

$$
\psi_{M_A-1}(x)=\overline{\psi}_{M_{A-1}}(x)\,\overline{\psi}_{M_{A-2}}(x)\cdots\overline{\psi}_{M_0}(x) \tag{10}
$$

we get

$$
\begin{aligned}
& T_A^{(2,1,1)}\left(x^1,x^2\right) \qquad (11)\\
= \;& \psi_{M_{A-1}}\left(-x^2\right)\psi_{M_{A-2}}\left(-x^2\right)\cdots\psi_{M_0}\left(-x^2\right)\\
& \times\left(\sum_{r=1}^{m_{A-1}-1}\rho_{A-1}^{r}\left(x^1\right)\sum_{q=0}^{r-1}\rho_{A-1}^{q}\left(-x^2\right)\right)D_{M_{A-1}}\left(x^1\right)D_{M_{A-1}}\left(-x^2\right)\\
= \;& \psi_{M_{A-2}}\left(-x^2\right)\cdots\psi_{M_0}\left(-x^2\right)\\
& \times\left(\sum_{r=1}^{m_{A-1}-1}\rho_{A-1}^{r}\left(x^1\right)\sum_{q=1}^{r}\rho_{A-1}^{q}\left(-x^2\right)\right)D_{M_{A-1}}\left(x^1\right)D_{M_{A-1}}\left(-x^2\right)\\
= \;& \left(\sum_{r=1}^{m_{A-1}-1}\rho_{A-1}^{r}\left(x^1\right)\sum_{q=1}^{r}\rho_{A-1}^{q}\left(-x^2\right)\right)\Phi_{A-1}^{(1)}\left(x^1,x^2\right),
\end{aligned}
$$

where function $\Phi^{(1)}_{A-1}\left(x^1,x^2\right)$ is $\mathcal{A}_{A-1}$ measurable.

From (10) we have

$$
\begin{aligned}
& T_A^{(2,1,2)}\left(x^1,x^2\right) \qquad (12)\\
&= \sum_{j=0}^{A-2}\left(\sum_{r=1}^{m_j-1}\rho_j^r\left(x^1\right)\sum_{q=0}^{r-1}\rho_{A-1}^q\left(-x^2\right)\right)\\
&\times D_{M_j}\left(x^1\right)D_{M_j}\left(-x^2\right)\prod_{s=j+1}^{A-2}\left(1+\sum_{l=1}^{m_s-1}\rho_s^l\left(x^1-x^2\right)\right)\\
&\times\rho_{A-1}\left(-x^2\right)\psi_{M_{A-2}}\left(-x^2\right)\cdots\psi_{M_0}\left(-x^2\right)\\
&+\sum_{j=0}^{A-2}\left(\sum_{r=1}^{m_j-1}\rho_j^r\left(x^1\right)\sum_{q=0}^{r-1}\rho_{A-1}^q\left(-x^2\right)\right)\psi_{M_{A-2}}\left(-x^2\right)\cdots\psi_{M_0}\left(-x^2\right)\\
&\times D_{M_j}\left(x^1\right)D_{M_j}\left(-x^2\right)\prod_{s=j+1}^{A-2}\left(1+\sum_{l=1}^{m_s-1}\rho_s^l\left(x^1-x^2\right)\right)\\
&\times\sum_{l=1}^{m_{A-1}-1}\rho_{A-1}^l\left(x^1\right)\sum_{l=1}^{m_{A-1}-1}\rho_{A-1}^{l+1}\left(-x^2\right)\\
&=:\rho_{A-1}\left(-x^2\right)\Phi^{(2)}_{A-1}\left(x^1,x^2\right)\\
&+\left(\sum_{l=1}^{m_{A-1}-1}\rho_{A-1}^l\left(x^1\right)\sum_{l=1}^{m_{A-1}-1}\rho_{A-1}^{l+1}\left(-x^2\right)\right)\Phi^{(3)}_{A-1}\left(x^1,x^2\right),
\end{aligned}
$$

where functions $\Phi^{(j)}_{A-1}\left(x^1,x^2\right), j=2,3$ are $\mathcal{A}_{A-1}$ measurable.

Combining (8), 11 and (12) we have

$$
\begin{aligned}
& T_A^{(2,1)}\left(x^1,x^2\right) \qquad (13)\\
&= \left(\sum_{r=1}^{m_{A-1}-1}\rho_{A-1}^r\left(x^1\right)\sum_{q=1}^{r}\rho_{A-1}^q\left(-x^2\right)\right)\Phi^{(1)}_{A-1}\left(x^1,x^2\right)\\
&+\rho_{A-1}\left(-x^2\right)\Phi^{(2)}_{A-1}\left(x^1,x^2\right)\\
&+\left(\sum_{l=1}^{m_{A-1}-1}\rho_{A-1}^l\left(x^1\right)\sum_{l=1}^{m_{A-1}-1}\rho_{A-1}^{l+1}\left(-x^2\right)\right)\Phi^{(3)}_{A-1}\left(x^1,x^2\right).
\end{aligned}
$$

Set

$$t_A^{(2,1)}\left(y^1,y^2;f\right):=\left(f*T_A^{(2,1)}\right)\left(y^1,y^2\right).$$

Then it is evident that

$$t_A^{(2,1)}(f)=S^{\square}_{M_A}\left(t_A^{(2,1)}(f)\right)=t_A^{(2,1)}\left(S^{\square}_{M_A}(f)\right).$$

On the other hand, from (13) we conclude that

$$t_A^{(2,1)}\left(S^{\square}_{M_{A-1}}(f)\right)=0.$$

Hence,

$$t_A^{(2,1)}(f) = t_A^{(2,1)}\left(S_{M_A}^{\square}(f) - S_{M_{A-1}}^{\square}(f)\right). \tag{14}$$

Since

$$\left\|t_A^{(2,1)}(f)\right\|_2 \leq c\left\|f\right\|_2,$$

from (14) and Bessel's inequality for two dimensional L_2 functions and the two dimensional Vilenkin system we obtain

$$\begin{aligned}\left\|\sup_A \left|t_A^{(2,1)}(f)\right|\right\|_2^2 &\leq \sum_{A=0}^{\infty}\left\|t_A^{(2,1)}(f)\right\|_2^2 \\ &= \sum_{A=0}^{\infty}\left\|t_A^{(2,1)}\left(S_{M_A}^{\square}(f) - S_{M_{A-1}}^{\square}(f)\right)\right\|_2^2 \\ &\leq c\sum_{A=0}^{\infty}\left\|S_{M_A}^{\square}(f) - S_{M_{A-1}}^{\square}(f)\right\|_2^2 \\ &\leq c\left\|f\right\|_2^2.\end{aligned} \tag{15}$$

Now, we suppose that $n_A > 1$. Then we have

$$\begin{aligned}T_A^{(2)}\left(x^1, x^2\right) &= \rho_A^{n_A-1}\left(x^1\right)\sum_{i=0}^{M_A-1}\psi_i\left(x^1\right)D_{M_A-i}\left(x^2\right) \\ &= \rho_A^{n_A-1}\left(x^1\right)\Phi_A^{(4)}\left(x^1, x^2\right),\end{aligned}$$

where function $\Phi_{A-1}^{(4)}\left(x^1, x^2\right)$ is $\mathcal{A}_A$ measurable. Then we can write

$$t_A^{(2)}(f) := f * T_A^{(2)} = S_{M_{A+1}}^{\square}\left(t_A^{(2)}(f)\right) = t_A^{(2)}\left(S_{M_{A+1}}^{\square}(f)\right),$$

$$t_A^{(2)}\left(S_{M_A}^{\square}(f)\right) = 0,$$

$$t_A^{(2)}(f) = t_A^{(2)}\left(S_{M_{A+1}}^{\square}(f) - S_{M_A}^{\square}(f)\right).$$

Since for any fixed A

$$\left\|t_A^{(2)}(f)\right\|_2 \leq c\left\|f\right\|_2,$$

then as above we can prove that

$$\left\|\sup_A t_A^{(2)}(f)\right\|_2 \leq c\left\|f\right\|_2. \tag{16}$$

Since

$$\left\|\sup_A \left|S_{M_A}^{\square}(f)\right|\right\|_2 \leq c\left\|f\right\|_2,$$

from (5) we obtain that

$$\left\|\sup_A \left|t_A^{(2)}(f)\right|\right\|_2 \leq c\left\|f\right\|_2. \tag{17}$$

Since

$$\begin{aligned} D_{(n_A-r)M_A-i}(x) &= D_{(n_A-r-1)M_A+M_A-i}(x) \\ &= D_{(n_A-r-1)M_A}(x) + \psi_{(n_A-r-1)M_A}(x) D_{M_A-i}(x), \end{aligned}$$

using (9) for $T_A^{(1)}\left(x^1, x^2\right)$ we have

$$\begin{aligned} & T_A^{(1)}\left(x^1, x^2\right) \\ =& \sum_{r=0}^{n_A-2} \rho_A^r\left(x^1\right) D_{(n_A-r-1)M_A}\left(x^2\right) D_{M_A}\left(x^1\right) \\ & + \sum_{r=0}^{n_A-2} \rho_A^r\left(x^1\right) \sum_{i=0}^{M_A-1} \psi_i\left(x^1\right) D_{M_A-i}\left(x^2\right) \psi_{(n_A-r-1)M_A}\left(x^2\right) \\ =& D_{M_A}\left(x^1\right) D_{M_A}\left(x^2\right) \\ & + D_{M_A}\left(x^1\right) D_{M_A}\left(x^2\right) \left\{ \sum_{q=1}^{n_A-2} \rho_A^q\left(x^2\right) + \sum_{r=1}^{n_A-2} \rho_A^r\left(x^1\right) \right. \\ & \left. + \sum_{r=1}^{n_A-2} \sum_{q=1}^{n_A-r-2} \rho_A^q\left(x^2\right) \rho_A^r\left(x^1\right) \right\} \\ & + \sum_{r=1}^{n_A-2} \rho_A^r\left(x^1\right) \rho_A^{n_A-r-1}\left(x^2\right) \sum_{i=0}^{M_A-1} \psi_i\left(x^1\right) D_{M_A-i}\left(x^2\right) \\ & + \sum_{i=0}^{M_A-1} \psi_i\left(x^1\right) D_{M_A-i}\left(x^2\right) \rho_A^{n_A-1}\left(x^2\right) \\ =& D_{M_A}\left(x^1\right) D_{M_A}\left(x^2\right) \\ & + \Phi_A^{(5)}\left(x^1, x^2\right) \left\{ \sum_{q=1}^{n_A-2} \rho_A^q\left(x^2\right) + \sum_{r=1}^{n_A-2} \rho_A^r\left(x^1\right) \right. \\ & \left. + \sum_{r=1}^{n_A-2} \sum_{q=1}^{n_A-r-2} \rho_A^q\left(x^2\right) \rho_A^r\left(x^1\right) \right\} \\ & + \left\{ \sum_{r=1}^{n_A-2} \rho_A^r\left(x^1\right) \rho_A^{n_A-r-1}\left(x^2\right) + \rho_A^{n_A-1}\left(x^2\right) \right\} \Phi_A^{(6)}\left(x^1, x^2\right), \end{aligned}$$

where functions $\Phi_A^{(j)}\left(x^1, x^2\right), j = 5, 6$ are $\mathcal{A}_A$ measurable. Then analogously, as above we can prove that

$$\left\| \sup_A \left| f * T_A^{(1)} \right| \right\|_2 \leq c \left\| f \right\|_2. \tag{18}$$

Combining (3), (17) and (18) we conclude that

$$\left\| \sup_A \left| S_{n_A M_A}^{\triangle}(f) \right| \right\|_2 \leq c \left\| f \right\|_2 \quad \left(f \in L_2\left(G_m \times G_m\right)\right).$$

By the well-known density argument we complete the proof of Theorem 1. □

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