Filomat 32:11 (2018), 4089–4098
https://doi.org/10.2298/FIL1811089L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Jordan Derivations of Generalized One Point Extensions

Yanbo Li[a], Feng Wei[b]

[a]*School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Qinhuangdao, 066004, P.R. China*
[b]*School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, 100081, P. R. China*

Abstract. For a generalized one-point extension algebra, it is proved that under certain conditions, each Jordan derivation is the sum of a derivation and an anti-derivation. Moreover, we prove that every Jordan derivation of a dual extension algebra is a derivation.

1. Introduction

Let us begin with some basic definitions. Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{A}$ a unital algebra over $\mathcal{R}$ and $\mathcal{Z}(\mathcal{A})$ the center of $\mathcal{A}$. We denote the *Jordan product* by

$$a \circ b = ab + ba$$

for all $a, b \in \mathcal{A}$. Recall that an $\mathcal{R}$-linear mapping Θ from $\mathcal{A}$ into itself is called a *derivation* if

$$\Theta(ab) = \Theta(a)b + a\Theta(b)$$

for all $a, b \in \mathcal{A}$, an *anti-derivation* if

$$\Theta(ab) = \Theta(b)a + b\Theta(a)$$

for all $a, b \in \mathcal{A}$, and a *Jordan derivation* if

$$\Theta(a \circ b) = \Theta(a) \circ b + a \circ \Theta(b).$$

Every derivation is obviously a Jordan derivation. The converse statement is in general not true. Moreover, in the 2-torsion free case the definition of a Jordan derivation is equivalent to that for all $x \in \mathcal{A}$,

$$\Theta(x^2) = \Theta(x)x + x\Theta(x).$$

Those Jordan derivations which are not derivations are said to be *proper*.

There has been an increasing interest in the study of Jordan derivations of various algebras since last decades. The standard problem is to find out whether a Jordan derivation degenerates to a derivation.

2010 *Mathematics Subject Classification*. Primary 16W25; Secondary 47B47
Keywords. generalized one-point extension, dual extension, Jordan derivation
Received: 18 October 2016; Accepted: 15 November 2018
Communicated by Dragan S. Djordjević
The work of the first author is supported by the Natural Science Foundation of Hebei Province, China (Grant No. A2017501003) and the Science and Technology support program of Northeastern University at Qinhuangdao (No. XNK201601).
Email addresses: liyanbo707@163.com (Yanbo Li), daoshuo@hotmail.com (Feng Wei)

Jacobson and Rickart [10] proved that every Jordan derivation of the full matrix algebra over a 2-torsion free unital ring is a derivation by relating the problem to the decomposition of Jordan homomorphisms. In [9], Herstein showed that every Jordan derivation from a 2-torsion free prime ring into itself is also a derivation. Zhang and Yu [24] obtained that every Jordan derivation on a triangular algebra with faithful assumption is a derivation. This result was extended to the higher case by Xiao and Wei [22]. They obtained that any Jordan higher derivation on a triangular algebra is a higher derivation. The aforementioned results have been extended to various algebras in different directions, see [4, 8, 17, 18, 23, 24] and the references therein.

Note that each associative algebra with non trivial idempotents is isomorphic to a generalized matrix algebra. The form of Jordan derivations on generalized matrix algebras has been characterized by current authors in [15]. We proved that under certain conditions, each Jordan derivation is the sum of a derivation and an anti-derivation. An example of proper Jordan derivations was also given there. To find a proper Jordan derivation is not an easy task in general. Fortunately, the so-called generalized one-point extension algebras introduced by the current authors in [14] just provide us another class of examples of proper Jordan derivations. We prove that under certain conditions, each Jordan derivation on a generalized one-point extension algebra is the sum of a derivation and an anti-derivation. This result implies that the faithful condition in [15] is not necessary.

More recently, Bencovič and Širovnik [3] introduced the so-called singular Jordan derivations which are usually anti-derivations. They gave a sufficient condition for a Jordan derivation on a unital algebra with a nontrivial idempotent to be the sum of a derivation and a singular Jordan derivation. It is natural to ask whether the conditions in [3] are necessary. We will give a negative answer in this paper by studying dual extension algebras, which was introduced by Xi in [19].

The paper is organized as follows. After a quick review of some preliminaries on path algebras and generalized matrix algebras in Section 2, we investigate Jordan derivations of generalized one-point extension algebras in Section 3. Then in Section 4, we study Jordan derivations of dual extension algebras.

2. Path algebras and generalized matrix algebras

In this section, we recall some basic facts concerning path algebras of quivers and generalized matrix algebras. For more details, we refer the reader to [1, 2, 22].

2.1. Path algebras

Recall that a *finite quiver* $\Gamma = (\Gamma_0, \Gamma_1)$ is an oriented graph with the set of vertices Γ_0 and the set of arrows between vertices Γ_1 being both finite. For an arrow α, we write $s(\alpha) = i$ and $e(\alpha) = j$ if it is from the vertex i to the vertex j. A *sink* is a vertex without arrows beginning at it and a *source* is a vertex without arrows ending at it. A *nontrivial path* in Γ is an ordered sequence of arrows $p = \alpha_n \cdots \alpha_1$ such that $e(\alpha_m) = s(\alpha_{m+1})$ for each $1 \leq m < n$. Define $s(p) = s(\alpha_1)$ and $e(p) = e(\alpha_n)$. The length of p is defined to be n. A *trivial path* is the symbol e_i for each $i \in \Gamma_0$ and its length is defined to be zero. A nontrivial path p is called an *oriented cycle* if $s(p) = e(p)$. Denote the set of all paths by $\mathscr{P}$.

Let K be a field and Γ a quiver. Then the path algebra $K\Gamma$ is the K-algebra generated by the paths in Γ and the product of two paths $x = \alpha_n \cdots \alpha_1$ and $y = \beta_t \cdots \beta_1$ is defined by

$$xy = \begin{cases} \alpha_n \cdots \alpha_1 \beta_t \cdots \beta_1, & e(y) = s(x) \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $K\Gamma$ is an associative algebra with the identity $1 = \sum_{i \in \Gamma_0} e_i$.

A *relation* σ on a quiver Γ over a field K is a K-linear combination of paths $\sigma = \sum_{i=1}^{n} k_i p_i$, where $k_i \in K$ and $e(p_1) = \cdots = e(p_n)$, $s(p_1) = \cdots = s(p_n)$. Moreover, the length of each path is assumed to be at least 2. Let ρ be a set of relations on Γ over K. The pair (Γ, ρ) is called a *quiver* with relations over K. Denote by $< \rho >$ the ideal of $K\Gamma$ generated by ρ. The K-algebra $K(\Gamma, \rho) = K\Gamma / < \rho >$ is always associated with (Γ, ρ). For arbitrary element $x \in K\Gamma$, write by $\overline{x}$ the corresponding element in $K(\Gamma, \rho)$. We often *write $\overline{x}$ as x if there is no confusion caused.*

2.2. Generalized matrix algebras

Let $\mathcal{R}$ be a commutative ring with identity. A Morita context consists of two $\mathcal{R}$-algebras A and B, two bimodules ${}_AM_B$ and ${}_BN_A$, and two bimodule homomorphisms called the pairings $\Phi_{MN} : M \underset{B}{\otimes} N \longrightarrow A$ and $\Psi_{NM} : N \underset{A}{\otimes} M \longrightarrow B$ satisfying the following commutative diagrams:

$$\begin{array}{ccc} M \underset{B}{\otimes} N \underset{A}{\otimes} M & \xrightarrow{\Phi_{MN}\otimes I_M} & A \underset{A}{\otimes} M \\ \Big\downarrow {\scriptstyle I_M\otimes\Psi_{NM}} & & \Big\downarrow {\scriptstyle \cong} \\ M \underset{B}{\otimes} B & \xrightarrow{\cong} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} N \underset{A}{\otimes} M \underset{B}{\otimes} N & \xrightarrow{\Psi_{NM}\otimes I_N} & B \underset{B}{\otimes} N \\ \Big\downarrow {\scriptstyle I_N\otimes\Phi_{MN}} & & \Big\downarrow {\scriptstyle \cong} \\ N \underset{A}{\otimes} A & \xrightarrow{\cong} & N\,. \end{array}$$

If $(A, B, {}_AM_B, {}_BN_A, \Phi_{MN}, \Psi_{NM})$ is a Morita context, then the set

$$\begin{bmatrix} A & M \\ N & B \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} \middle| \, a \in A, m \in M, n \in N, b \in B \right\}$$

forms an $\mathcal{R}$-algebra under matrix-like addition and matrix-like multiplication. Such an $\mathcal{R}$-algebra is called a *generalized matrix algebra* and is usually denoted by $\mathcal{G} = (A, M, N, B)$. The structure and properties of linear mappings on generalized matrix algebras have been investigated in our systemic works [13, 15, 16, 22].

From now on, we always assume, without specially mentioned, that every algebra and every bimodule considered is 2-torsion free. We end this section by recalling some indispensable descriptions about derivations and Jordan derivations of generalized matrix algebras.

Lemma 2.1. *[13, Proposition 4.2] An additive map Θ from $\mathcal{G}$ into itself is a derivation if and only if it has the form*

$$\Theta\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) - mn_0 - m_0n & am_0 - m_0b + \tau_2(m) \\ n_0a - bn_0 + \nu_3(n) & n_0m + nm_0 + \mu_4(b) \end{bmatrix}, (\bigstar 1)$$
$$\forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G},$$

where $m_0 \in M, n_0 \in N$ and

$$\delta_1 : A \longrightarrow A, \quad \tau_2 : M \longrightarrow M, \quad \nu_3 : N \longrightarrow N, \quad \mu_4 : B \longrightarrow B$$

are all $\mathcal{R}$-linear mappings satisfying the following conditions:

(1) δ_1 is a derivation of A with $\delta_1(mn) = \tau_2(m)n + m\nu_3(n)$;
(2) μ_4 is a derivation of B with $\mu_4(nm) = n\tau_2(m) + \nu_3(n)m$;
(3) $\tau_2(am) = a\tau_2(m) + \delta_1(a)m$ and $\tau_2(mb) = \tau_2(m)b + m\mu_4(b)$;
(4) $\nu_3(na) = \nu_3(n)a + n\delta_1(a)$ and $\nu_3(bn) = b\nu_3(n) + \mu_4(b)n$.

Lemma 2.2. *[15, Proposition 4.2] An additive map Θ from $\mathcal{G}$ into itself is a Jordan derivation if and only if it is of the form*

$$\Theta\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) - mn_0 - m_0n & am_0 - m_0b + \tau_2(m) + \tau_3(n) \\ n_0a - bn_0 + \nu_2(m) + \nu_3(n) & n_0m + nm_0 + \mu_4(b) \end{bmatrix}, (\bigstar 2)$$
$$\forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G},$$

where $m_0 \in M, n_0 \in N$ and

$$\delta_1 : A \longrightarrow A, \quad \tau_2 : M \longrightarrow M, \quad \tau_3 : N \longrightarrow M,$$
$$\nu_2 : M \longrightarrow N, \quad \nu_3 : N \longrightarrow N, \quad \mu_4 : B \longrightarrow B$$

are all $\mathcal{R}$-linear mappings satisfying the following conditions:

(1) δ_1 *is a Jordan derivation on* A *and* $\delta_1(mn) = \tau_2(m)n + m\nu_3(n)$;
(2) μ_4 *is a Jordan derivation on* B *and* $\mu_4(nm) = n\tau_2(m) + \nu_3(n)m$;
(3) $\tau_2(am) = a\tau_2(m) + \delta_1(a)m$ *and* $\tau_2(mb) = \tau_2(m)b + m\mu_4(b)$;
(4) $\nu_3(bn) = b\nu_3(n) + \mu_4(b)n$ *and* $\nu_3(na) = \nu_3(n)a + n\delta_1(a)$;
(5) $\tau_3(na) = a\tau_3(n)$, $\tau_3(bn) = \tau_3(n)b$, $n\tau_3(n) = 0$, $\tau_3(n)n = 0$;
(6) $\nu_2(am) = \nu_2(m)a$, $\nu_2(mb) = b\nu_2(m)$, $m\nu_2(m) = 0$, $\nu_2(m)m = 0$.

3. Jordan derivations of generalized one-point extensions

We introduced generalized one-point extension algebras in [14]. In this section, we prove that under certain conditions, each Jordan derivation of a generalized one-point extension algebra is the sum of a derivation and an anti-derivation.

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a finite quiver without oriented cycles and $|\Gamma_0| \geq 2$. Let Γ^* be the quiver whose vertex set is Γ_0 and

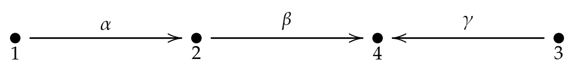
$$\Gamma_1^* = \{\alpha^* : i \to j \mid \alpha : j \to i \text{ is an arrow in } \Gamma_1\}.$$

For a path $p = \alpha_n \cdots \alpha_1$ in Γ, write the path $\alpha_1^* \cdots \alpha_n^*$ in Γ^* by p^*. Given a set ρ of relations, denote by $\Lambda = K(\Gamma, \rho)$. Define the generalized one-point extension algebra $E(\Lambda)$ to be the path algebra of the quiver $(\Gamma_0, \Gamma_1 \cup \Gamma_1^*)$ with relations

$$\rho \ \cup \ \rho^* \ \cup \ \{\alpha\beta^* \mid \alpha, \beta \in \Gamma_1\} \ \cup \ \{\alpha^*\beta \mid \alpha, \beta \in \Gamma_1\}.$$

If we choose a suitable idempotent, then neither M nor N need to be faithful. Let us illustrate an example here.

Example 3.1. *Let* K *be a field. Let* Γ *be a quiver as follows*



and let $\Lambda = K\Gamma$. *The generalized one-point extension algebra* $E(\Lambda)$ *has a basis*

$$\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \alpha^*, \beta^*, \gamma^*, \beta\alpha, \alpha^*\beta^*\}.$$

Taking the idempotent $e_1 + e_2$, *then* $E(\Lambda)$ *is isomorphic to a generalized matrix algebra* $\mathcal{G} = (A, M, N, B)$, *where* A *has a basis* $\{e_1, e_2, \alpha, \alpha^*\}$, B *has a basis* $\{e_3, e_4, \gamma, \gamma^*\}$, M *has a basis* $\{\alpha^*\beta^*, \beta^*\}$ *and* N *has a basis* $\{\beta, \beta\alpha\}$. *It is easy to check that* $\alpha \in \mathrm{Ann}({}_AM)$ *and* $\gamma \in \mathrm{Ann}(M_B)$, *that is,* M *is neither faithful as a left* A*-module nor as a right* B*-module. Similarly, we obtain* $\gamma \in \mathrm{Ann}({}_BN)$ *and* $\alpha \in \mathrm{Ann}(N_A)$, *that is,* N *is neither faithful as a left* B*-module nor as a right* A*-module.*

In [3] Benkovič proved that for $\mathcal{G} = (A, M, N, B)$, if

(1) $aM = 0$ and $Na = 0$ imply that $a = 0$;
(2) $Mb = 0$ and $bN = 0$ imply that $b = 0$,

then every Jordan derivation on $\mathcal{G}$ is the sum of a derivation and an anti-derivation. Clearly, our example does not satisfy Benkovič's conditions.

Let us characterize anti-derivations of generalized one-point extension algebras.

Lemma 3.2. *Let* Γ *be a finite quiver without oriented cycles and* $\Lambda = K(\Gamma, \rho)$. *Let* θ *be an anti-derivation on* $E(\Lambda)$ *and* $\alpha \in \Gamma_1 \cup \Gamma_1^*$ *with* $s(\alpha) = r$ *and* $e(\alpha) = t$. *Then*

(1) $\Theta(e_i) = \sum_{s(p)=i, \text{ or } e(p)=i} k_p^i p$;

(2) $\Theta(\alpha) = \sum_{s(p)=t, e(p)=r} k_p^\alpha p.$

Moreover, $\Theta(p) = 0$ *for all path* p *with length more than one. If there exists a nontrivial path* β *such that* $\beta\alpha \neq 0$ *or* $\alpha\beta \neq 0$*, then* $\Theta(\alpha) = 0$*.*

Proof. (1) Suppose that

$$\Theta(e_r) = \sum_{i\in\Gamma_0} k_i e_i + \sum_{s(p)\neq e(p)} k_p^r p. \tag{3.1}$$

It follows from the fact $e_r^2 = e_i$ that

$$\Theta(e_r) = \Theta(e_r)e_r + e_r\Theta(e_r). \tag{3.2}$$

Combining (3.1) with (3.2) gives that $k_r = 0$. If there exists $j \in \Gamma_0$ with $i \neq j$ such that $k_j \neq 0$, then the coefficient of e_j in the expansion of $\Theta(e_r)e_j$ is k_j. On the other hand, since e_j does not appear in the expansion of $\Theta(e_j)$, we conclude that e_j does not appear in the expansion of $e_r\Theta(e_j)$ too. This implies that $\Theta(e_je_r) \neq 0$, which is impossible.

(2) Let Θ be an anti-derivation on $E(\Lambda)$ and let $\alpha \in \Gamma_1$ with $s(\alpha) = r$ and $e(\alpha) = t$. Suppose that

$$\Theta(\alpha) = \sum_{i\in\Gamma_0} k_i e_i + \sum_{s(p)\neq e(p)} k_p^\alpha p. \tag{3.3}$$

Then

$$\Theta(\alpha) = \Theta(e_t\alpha) = \Theta(\alpha)e_t + \alpha\Theta(e_t). \tag{3.4}$$

Taking (3.3) into (3.4) gives that

$$\begin{aligned}\sum_{i\in\Gamma_0} k_i e_i + \sum_{s(p)\neq e(p)} k_p^\alpha p &= (\sum_{i\in\Gamma_0} k_i e_i + \sum_{s(p)\neq e(p)} k_p^\alpha p)e_t + \alpha\Theta(e_t)\\ &= k_t e_t + \sum_{s(p)=t} k_p^\alpha p + \alpha\Theta(e_t).\end{aligned} \tag{3.5}$$

On the other hand,

$$\Theta(\alpha) = \Theta(\alpha e_r) = e_r\Theta(\alpha) + \Theta(e_r)\alpha. \tag{3.6}$$

Substituting (3.3) into (3.6) yields that

$$\begin{aligned}\sum_{i\in\Gamma_0} k_i e_i + \sum_{s(p)\neq e(p)} k_p^\alpha p &= e_r(\sum_{i\in\Gamma_0} k_i e_i + \sum_{s(p)\neq e(p)} k_p^\alpha p) + \Theta(e_r)\alpha\\ &= k_r e_r + \sum_{e(p)=r} k_p^\alpha p + \Theta(e_r)\alpha.\end{aligned} \tag{3.7}$$

Combining (3.5) with (3.7) leads to

$$\begin{aligned}\sum_{i\in\Gamma_0} k_i e_i + \sum_{s(p)\neq e(p)} k_p^\alpha p &= k_t e_t + \sum_{s(p)=t} k_p^\alpha p + \alpha\Theta(e_t)\\ &= k_r e_r + \sum_{e(p)=r} k_p^\alpha p + \Theta(e_r)\alpha.\end{aligned}$$

This implies that $k_i = 0$ for all $i \in \Gamma_0$ and the coefficients of all paths p with $s(p) \neq t$ or $e(p) \neq r$ in the expansion of $\Theta(\alpha)$ are zero, that is,

$$\Theta(\alpha) = \sum_{s(p)=t, e(p)=r} k_p^\alpha p.$$

If there exists a non trivial path β such that $\beta\alpha \neq 0$, then $\alpha\beta^* = 0$. However,

$$\Theta(\alpha\beta^*) = \Theta(\beta^*)\alpha + \beta^*\Theta(\alpha).$$

If $\Theta(\alpha) \neq 0$, then by (3.5) we know that

$$\beta^*\Theta(\alpha) = \sum_{s(p)=t,e(p)=r} k_p^\alpha \beta p \neq 0,$$

and hence $\Theta(\alpha\beta^*) \neq 0$, which is a contradiction. This forces that $\Theta(\alpha) = 0$. Similarly, one can show that if $\alpha\beta \neq 0$, then $\Theta(\alpha) = 0$. □

Since Γ is a quiver without oriented cycles, we can take a source i in Γ. Let e_i be the corresponding idempotent in $E(\Lambda)$. Then $E(\Lambda) \simeq \mathcal{G} = (A, M, N, B)$ with $A \simeq E(\Lambda')$, where the quiver Γ' of Λ' is obtained by removing the vertex i and the relations starting at i. Moreover, we have from the construction of $E(\Lambda)$ that the bilinear pairings are both zero. In this case, the form of an arbitrary Jordan derivation of $E(\Lambda)$ is as follows:

Lemma 3.3. *Let $\Lambda = K\Gamma$ and $E(\Lambda)$ be the generalized one-point extension. Then an arbitrary Jordan derivation Θ on $E(\Lambda)$ is of the form*

$$\Theta\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) & am_0 - m_0b + \tau_2(m) + \tau_3(n) \\ n_0a - bn_0 + \nu_2(m) + \nu_3(n) & 0 \end{bmatrix}, (\blacklozenge 1)$$
$$\forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G},$$

where $m_0 \in M, n_0 \in N$ and

$$\delta_1 : A \longrightarrow A, \quad \tau_2 : M \longrightarrow M, \quad \tau_3 : N \longrightarrow M, \quad \nu_2 : M \longrightarrow N, \quad \nu_3 : N \longrightarrow N$$

are all $\mathcal{R}$-linear mappings satisfying the following conditions:

(1) δ_1 is a Jordan derivation on A;
(2) $\tau_2(am) = a\tau_2(m) + \delta_1(a)m$ and $\tau_2(mb) = \tau_2(m)b$;
(3) $\nu_3(bn) = b\nu_3(n)$ and $\nu_3(na) = \nu_3(n)a + n\delta_1(a)$;
(4) $\tau_3(na) = a\tau_3(n)$, $\tau_3(bn) = \tau_3(n)b$;
(5) $\nu_2(am) = \nu_2(m)a$, $\nu_2(mb) = b\nu_2(m)$.

Proof. We have from Lemma 2.2 that it is sufficient to prove that $\mu_4 = 0$. But, this is clear because μ_4 is a Jordan derivation on $B = K$. □

In [15], the form of an arbitrary anti-derivation on a generalized matrix algebra $\mathcal{G} = (A, M, N, B)$ has been characterized under the condition that M being faithful as left A-module and also as right B-module. If we remove the faithful assumption on M, the form of an anti-derivation on $\mathcal{G}$ is as follows:

Lemma 3.4. *An additive mapping Θ is an anti-derivation of $\mathcal{G}$ if and only if Θ has the form*

$$\Theta\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) & am_0 - m_0b + \tau_3(n) \\ n_0a - bn_0 + \nu_2(m) & \mu_4(b) \end{bmatrix}, (\blacklozenge 2)$$
$$\forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G},$$

where $m_0 \in M, n_0 \in N$ are two elements such that for all $a, a' \in A$, $b, b' \in B$, $m \in M$ and $n \in N$

(1) $[a, a']m_0 = 0$, $m_0[b, b'] = 0$, $n_0[a, a'] = 0$, $[b, b']n_0 = 0$;

(2) $m_0n = 0$, $nm_0 = 0$, $mn_0 = 0$, $n_0m = 0$

and

$$\delta_1 : A \longrightarrow A, \quad \tau_3 : N \longrightarrow M, \quad \nu_2 : M \longrightarrow N, \quad \mu_4 : B \longrightarrow B$$

are $\mathcal{R}$-linear mappings satisfying the following condition: for all $a \in A$, $b \in B$, $m, m' \in M$ and $n, n' \in N$

(3) δ_1 *is an anti-derivation on A and* $\delta_1(mn) = 0$, $\delta_1(a)m = 0$, $n\delta_1(a) = 0$;

(4) μ_4 *is an anti-derivation on B and* $\mu_4(nm) = 0$, $m\mu_4(b) = 0$, $\mu_4(b)n = 0$;

(5) $\tau_3(na) = a\tau_3(n)$, $\tau_3(bn) = \tau_3(n)b$, $n\tau_3(n') = 0$, $\tau_3(n)n' = 0$;

(6) $\nu_2(am) = \nu_2(m)a$, $\nu_2(mb) = b\nu_2(m)$, $m\nu_2(m') = 0$, $\nu_2(m)m' = 0$.

Proof. It can be proved as that of [15, Proposition 3.6]. □

As a consequence of Lemma 3.3 and Lemma 3.4 we have

Proposition 3.5. *Let Θ be a Jordan derivation on a generalized one-point extension algebra $E(\Lambda) \simeq \mathcal{G} = (A, M, N, B)$. If there exists an anti-derivation f on A with $\mathrm{Im}(f) \subset \mathrm{Ann}({}_AM)$ such that $\delta_1 - f$ is a derivation of A, then Θ is the sum of a derivation and an anti-derivation.*

We are now in a position to state the main result of this section.

Theorem 3.6. *Let Γ be a finite quiver without oriented cycles and $\Lambda = K(\Gamma, \rho)$. If there is no path p with length more than one, then every Jordan derivation on the generalized one point extension algebra $E(\Lambda)$ is the sum of a derivation and an anti-derivation.*

Proof. Let Θ be a Jordan derivation on $E(\Lambda)$. Then by Lemma 3.2 it is of the form (♦1). We claim that if each Jordan derivation on A is the sum of a derivation and an anti-derivation, then so is $E(\Lambda)$. In fact, assume $\delta_1 = d + f$, where d is a derivation of A and f is an anti-derivation of A. By Lemma 3.2 we know that all e_i do not appear in $f(a)$ for $a \in A$. Note that the length of each path is not more than one. This implies that $f(a)m = 0$ for all $a \in A$ and $m \in M$. Similarly, we can show that $nf(a) = 0$ for all $a \in A$ and $n \in N$. Define a linear mapping f' on $E(\Lambda)$ by

$$\Delta\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} f(a) & \tau_3(n) \\ \nu_2(m) & 0 \end{bmatrix}, \quad \forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in E(\Lambda).$$

Then Lemma 3.2 and Lemma 3.4 give that Δ is a anti-derivation of $E(\Lambda)$. Furthermore, the linear mapping

$$D\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} d(a) & am_0 - m_0b + \tau_2(m) \\ n_0a - bn_0 + \nu_3(n) & 0 \end{bmatrix}$$

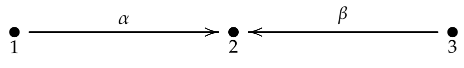
is a derivation of $E(\Lambda)$. This completes the proof of our claim. Repeating this process, we arrive at the algebra K, on which every Jordan derivation is zero. This completes the proof. □

By Lemma 3.2 and Theorem 3.6 we immediately get

Corollary 3.7. *Let Γ be a finite quiver without oriented cycles and ρ a relations set containing all paths of length 2. Then each Jordan derivation of $E(\Lambda)$ is the sum of a derivation and an anti-derivation.*

Finally, we illustrate an example which satisfies the condition of Theorem 3.6.

Example 3.8. *Let Γ be a quiver as follows*



and let $\Lambda = K\Gamma$. Then $E(\Lambda)$ has a basis $\{e_1, e_2, e_3, \alpha, \beta, \alpha^*, \beta^*\}$. Define a linear mapping on $E(\Lambda)$ by

$$\Theta(e_1) = \Theta(e_2) = \Theta(e_3) = 0, \quad \Theta(\alpha) = \alpha^*, \quad \Theta(\alpha^*) = \alpha,$$
$$\Theta(\beta) = \beta + \beta^*, \quad \Theta(\beta^*) = \beta - \beta^*.$$

Then a direct computation shows that Θ is a proper Jordan derivation on $E(\Lambda)$.

On the other hand, we can also define two linear mappings Θ_1 and Θ_2 by

$$\Theta_1(e_1) = \Theta_1(e_2) = \Theta_1(e_3) = 0, \quad \Theta_1(\alpha) = 0, \quad \Theta_1(\alpha^*) = 0,$$
$$\Theta_1(\beta) = \beta, \quad \Theta_1(\beta^*) = -\beta^*$$

and

$$\Theta_2(e_1) = \Theta_2(e_2) = \Theta_2(e_3) = 0, \quad \Theta_2(\alpha) = \alpha^*, \quad \Theta_2(\alpha^*) = \alpha,$$
$$\Theta_2(\beta) = \beta^*, \quad \Theta_2(\beta^*) = \beta.$$

It is easy to see that Θ_1 is a derivation on $E(\Lambda)$ and Θ_2 is an anti-derivation on $E(\Lambda)$. Therefore, Θ is the sum of the derivation Θ_1 and the anti-derivation Θ_2.

4. Jordan Derivations of Dual Extensions of Algebras

For path algebras of finite quivers without oriented cycles, Xi [19] constructed their dual extension algebras to study quasi-hereditary algebras. This construction were further refined by Deng and Xi in [5, 7, 20]. A more general construction, the twisted doubles, were studied in [6, 11, 21]. In this section, we prove that every Jordan derivation of a dual extension algebra is a derivation.

Let $\Lambda = K(\Gamma, \rho)$, where Γ is a finite quiver. Define $\mathscr{D}(\Lambda)$ to be the path algebra of the quiver $(\Gamma_0, \Gamma_1 \cup \Gamma_1^*)$ with relations

$$\rho \ \cup \ \rho^* \ \cup \ \{\alpha\beta^* \mid \alpha, \beta \in \Gamma_1\}.$$

If Γ has no oriented cycles, then $\mathscr{D}(\Lambda)$ is called the *dual extension* of Λ. Assume that $|\Gamma_0| \geq 2$. Then $\mathscr{D}(\Lambda)$ is isomorphic to a generalized matrix algebra $\mathcal{G} = (A, M, N, B)$. According to the construction of dual extension, it is easy to verify that the pairings $\Phi_{MN} = 0$ and $\Psi_{NM} \neq 0$. If $M \neq 0$, then $N \neq 0$. Moreover, it is helpful to point out that M need not to be faithful as left A-module or as right B-module. Some examples were given in [14].

Let $\Lambda = K(\Gamma, \rho)$, where Γ is a finite connected quiver without oriented cycles, and let $\mathscr{D}(\Lambda)$ be the dual extension algebra. Assume that $i \in \Gamma_0$ is a source and $\mathscr{D}(\Lambda) \simeq \mathcal{G} = (A, M, N, B)$, where $B \simeq e_i\mathscr{D}(\Lambda)e_i$. Thus a Jordan derivation on $\mathscr{D}(\Lambda)$ can be characterized by the methods of generalized matrix algebras as follows.

Lemma 4.1. *Let Θ be a Jordan derivation of $\mathscr{D}(\Lambda)$. Then Θ is of the form*

$$\Theta\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) & am_0 - m_0b + \tau_2(m) \\ n_0a - bn_0 + \nu_3(n) & n_0m + nm_0 + \mu_4(b) \end{bmatrix}, \ \forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G},$$

where $m_0 \in M, n_0 \in N$ and

$$\delta_1 : A \longrightarrow A, \quad \tau_2 : M \longrightarrow M, \quad \nu_3 : N \longrightarrow N, \quad \mu_4 : B \longrightarrow B$$

are all $\mathcal{R}$-linear mappings satisfying the following conditions:

(1) δ_1 is a Jordan derivation on A;
(2) μ_4 is a derivation on B and $\mu_4(nm) = n\tau_2(m) + \nu_3(n)m$;
(3) $\tau_2(am) = a\tau_2(m) + \delta_1(a)m$ and $\tau_2(mb) = \tau_2(m)b + m\mu_4(b)$;
(4) $\nu_3(bn) = b\nu_3(n) + \mu_4(b)n$ and $\nu_3(na) = \nu_3(n)a + n\delta_1(a)$.

Proof. Let Θ be a Jordan derivation of $\mathscr{D}(\Lambda)$ with the form ($\bigstar$2). We first prove that $\tau_3 = 0$, $\nu_2 = 0$. Let $\alpha \in N$ be an arbitrary arrow. Then $e(\alpha) = i$. Assume that $s(\alpha) = j$, where $j \in \Gamma_0$. In view of condition (5) of Lemma 2.2 we know that $\tau_3(\alpha) = \tau_3(\alpha e_j) = e_j \tau_3(\alpha)$. This implies that if $\tau_3(\alpha) \neq 0$, then $\alpha\tau_3(\alpha) \neq 0$. However, $\alpha\tau_3(\alpha) \neq 0$ is impossible by condition (5) of Lemma 2.2. Thus $\tau_3(\alpha) = 0$ for all $\alpha \in N$. Note that all path $p \in N$ with length more than 1 is of the form $\alpha p'$, where α is an arrow ending at i. Then $\tau_3(p) = \tau_3(\alpha p') = p'\tau_3(\alpha) = 0$. This shows that $\tau_3 = 0$. Similarly, one can prove that $\nu_2 = 0$. Furthermore, we have from the commutativity of B that every Jordan derivation of B is a derivation. Finally, the fact $\Phi_{MN} = 0$ leads to $mn_0 = m_0 n = 0$ for all $m \in M$ and $n \in N$. $\square$

Now we can describe Jordan derivations of a dual extension algebra.

Theorem 4.2. *Let Γ be a finite connected quiver without oriented cycles and $\Lambda = K(\Gamma, \rho)$. Let $\mathscr{D}(\Lambda)$ be the dual extension algebra of Λ. Then each Jordan derivation of $\mathscr{D}(\Lambda)$ is a derivation.*

Proof. If the algebra $\mathscr{D}(\Lambda)$ is trivial, then the theorem clearly holds. Suppose that $\Gamma_0 \geq 2$. Let Θ be a Jordan derivation on $\mathscr{D}(\Lambda)$. Let us denote by (Γ', ρ') the quiver obtained by removing the vertex i and the relations starting at i and write $\Lambda' = K(\Gamma', \rho')$. It follows from Lemma 2.1 and Lemma 4.1 that each Jordan derivation on $\mathscr{D}(\Lambda)$ is a derivation if each Jordan derivation on $\mathscr{D}(\Lambda')$ is a derivation. Thus it is sufficient to determine whether every Jordan derivation on $\mathscr{D}(\Lambda')$ is a derivation. We continuously repeat this process and ultimately arrive at the algebra K after finite times, since Γ_0 is a finite set. Clearly, every Jordan derivation on K is a derivation. This completes the proof. $\square$

Remarks 4.3. *(1) Our result on Jordan derivations of dual extension algebras implies that neither the conditions in [15] nor those in [3] are necessary.*

(2) As applications of Theorem 4.2, we can prove that every Jordan generalized derivation (see [12] for the definition) and every generalized Jordan derivation of dual extension algebras are both generalized derivations. We omit the details here and left it to the reader.

Acknowledgement We thanks the reviewers for the valuable suggestions which helped to clarify the whole paper. The first author would like to express his sincere thanks to Chern Institute of Mathematics of Nankai University for the hospitality during his visit.

References

[1] I. Assem, D. Simson and A. Skowroński, Elements of the Representation Theory of Associative Algebras, Volume 1. Techniques of Representation Theory, London Mathematical Society Student Texts, 65, Cambridge University Press, Cambridge, 2006.

[2] M. Auslander, I. Reiten, S. Smalø, Representation Theory of Artin Algebras, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1995.

[3] D. Benkovič, N. Širovnik Jordan derivations of unital algebras with idempotents, Linear Algebra Appl., 437 (2012) 2271–2284.

[4] J. M. Cusack, Jordan derivations on rings, Proc. Amer. Math. Soc., 53 (1975) 321–324.

[5] B.-M. Deng, C.-C. Xi, Quasi-hereditary algebras which are dual extensions of algebras, Comm. algebra, 22 (1994) 4717–4735.

[6] B.-M. Deng, C.-C. Xi, Quasi-hereditary algebras which are twisted double incidence algebras of posets, Beiträge Algebra Geom., 36 (1995) 37–71.

[7] B.-M. Deng, C.-C. Xi, Ringel duals of quasi-hereditary algebras, Comm. Algebra, 24 (1996) 2825–2838.

[8] D. Han, F. Wei, Jordan (α, β)-derivation on triangular algebras and related mappings, Linear Algebra Appl., 434 (2011) 259–284.

[9] I. Herstein, Jordan derivations of prime rings, Proc. Amer. Math. Soc., 8 (1957) 1104–1110.

[10] N. Jacobson, C. Rickart, Jordan homomorphisms of rings, Trans. Amer. Math. Soc., 69 (1950) 479–502.

[11] S. Koenig, C.-C. Xi, Strong symmetry defined by twisting modules, applied to quasi-hereditary algebras with triangular decomposition and vanishing radical cube, Comm. Math. Phys., 197 (1998) 427–441.

[12] Y.-B. Li, D. D. Benkovič, Jordan generalized derivations on triangular algebras, Linear Multilinear Algebra, 59 (2011), 841–849.

[13] Y.-B. Li, F. Wei, Semi-centralizing maps of generalized matrix algebras, Linear Algebra Appl., 436 (2012) 1122–1153.

[14] Y.-B. Li, F. Wei, Lie derivations of dual extensions of algebras, Colloquium Mathematicum, 141 (2015) 65–82.

[15] Y.-B. Li, L. V. Wyk, Feng Wei, Jordan derivations and antiderivations of generalized matrix algebras, Operators and Matrices, 7 (2013) 399–415.

[16] Y.-B. Li, Z.-K. Xiao, Additivity of maps on generalized matrix algebras, Electronic Journal of linear alegbra, 22 (2011) 743–757.

[17] F.-Y. Lu, The Jordan structure of CSL algebras, Studia Math., 190 (2009) 283–299.

[18] F.-Y. Lu, Jordan derivations of reflexive algebras, Integr. Equ. Oper. Theory, 67 (2010) 51–56.

[19] C.-C. Xi, Quasi-hereditary algebras with a duality, J. Reine Angew. Math., 449 (1994) 201–215.

[20] C.-C. Xi, Global dimensions of dual extension algebras, Manuscripta Math., 88 (1995) 25–31.
[21] C.-C. Xi, Twisted doubles of algebras I: Deformations and the Jones index, CMS Conf. Proc., 24 (1998) 513–523.
[22] Z.-K. Xiao, F. Wei, Jordan higher derivations on triangular algebras, Linear Algebra Appl., 433 (2010) 2615–2622.
[23] Z.-K. Xiao, F. Wei, Jordan higher derivations on some operator algebras, Houston J. Math., 38 (2012) 275–293.
[24] J.-H. Zhang, W.-Y. Yu, Jordan derivations of triangular algebras, Linear Algebra Appl., 419 (2006) 251–255.