Filomat 32:12 (2018), 4113–4130
https://doi.org/10.2298/FIL1812113M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence of Solutions of a Non-variational Bi-harmonic System via Fixed Point Theory

Idir Mechai[a], Metib Alghamdi[a], Habib Yazidi[a,b]

[a]*Department of Mathematics, Faculty of Science, Jazan University, P.O.Box 277, Jazan 45142, Saudi Arabia*
[b] *University of Tunis, National School of Engineers of Tunis, 5 Street Taha Hssine, Bab Mnar 1008 Tunis, Tunisia*

Abstract. We prove existence of a positive solution for a system of non-variational bi-harmonic equations. Furthermore, we give some a priori estimates of solutions and a non-existence result. In addition we compute numerical solutions to illustrate the theoretical results.

1. Introduction

We consider the following 2^k, $k \geq 1$ strongly coupled elliptic system

$$\begin{cases} \Delta^2 u_i = f_i(u_{i+1}), \quad u_i > 0 \text{ in } B, \; i = 1, 2, \ldots, 2^k - 1, \\ \Delta^2 u_{2^k} = f_{2^k}(u_1), \quad u_{2^k} > 0 \text{ in } B, \end{cases} \tag{1}$$

with the boundary conditions

$$\begin{cases} u_i = 0, \quad \frac{\partial u_i}{\partial \nu} = 0, \text{on } \partial B, \; i = 1, 2, \ldots, 2^k - 1, \\ u_{2^k} = 0, \; \frac{\partial u_{2^k}}{\partial \nu} = 0, \text{ on } \partial B \end{cases} \tag{2}$$

where B is the unit ball in $\mathbb{R}^N$ $(N > 4)$, the functions f_i: $[0, \infty) \to [0, \infty)$ are continuous, verifying $f_i(0) = 0$ for $i = 1, 2, 3, \ldots, 2^k$.

The system described by (1)-(2) is ubiquitous in physics and chemistry where steady-states are answers to problematic questions in a great variety of systems of reaction-diffusion equations. These equations interact everywhere in nature. This interaction takes place in such disparate phenomena as the proliferation of virile mutants over a substantially wide habitat, the dispersion of fire flames in spacious forests, in combustion chambers, or in nuclear reactors where neutron populations evolve and develop. Hence, the reaction-diffusion equations represent a significant research area in mathematics see [6] and the references therein.

2010 *Mathematics Subject Classification*. Primary 35B45; Secondary 35P30, 35J65, 35J55
Keywords. Bi-harmonic equation; non-variational system; a priori estimates; topological methods.
Received: 30 November 2017; Accepted: 22 February 2018
Communicated by Calogero Vetro
Corresponding author: Habib Yazidi
Email addresses: imechai@jazanu.edu (Idir Mechai), malghamdi@jazanu.edu.sa (Metib Alghamdi), hyazidi@jazanu.edu, habib.yazidi@gmail.com (Habib Yazidi)

The non-variational Laplacian systems are extensively studied in several research papers. Existence, non existence, and a priori estimates for solutions are addressed in many papers [2], [4], [5] and [15]. Similar results are obtained for the bi-Laplacian systems, fractional differential equations and nonlinear elastic beam equations using topological methods, namely fixed point theorem and degree theory [1], [7], [10], [11], [13], [18].

The particular case of the system (1)-(2), corresponding to $k = 1$ was treated in ([17]). The authors established the existence of a non-trivial solution provided that a priori estimates on the L^∞-norm of solutions holds true. In the present work, we propose to study the general strongly coupled elliptic system (1)-(2). We carry out a detailed analysis of the expected solutions for our problem, and we extract suitable conditions on the source terms f_i for $i = 1, 2, 3, \dots, 2^k$, which allow us to prove existence and non-existence results.

This paper is organized as follows. In Section 2 we recall some preliminary results related to the bi-laplacian problem. Furthermore, we study the eigenvalue problem associate to the system (1)-(2) and prove some properties of its solutions. The main results are presented and proved in Section 3. We end the paper, Section 4, by giving examples and computing numerical solutions related to the system (1)-(2).

2. Preliminary Results

In this work, we seek a positive radial summetric solution to system (1)-(2). Then, let $r = |x| \in [0, 1)$, $u_i = u_i(r)$ for $i = 1, 2, 3, \dots, 2^k - 1$, and $u_{2^k} = u_{2^k}(r)$

$$\begin{cases} u_i^{(4)} + \frac{2(N-1)}{r} u_i^{(3)} + \frac{(N-1)(N-3)}{r^2} u_i'' - \frac{(N-1)(N-3)}{r^3} u_i' = f_i(u_{i+1}), \quad u_i > 0, \\ u_{2^k}^{(4)} + \frac{2(N-1)}{r} u_{2^k}^{(3)} + \frac{(N-1)(N-3)}{r^2} u_{2^k}'' - \frac{(N-1)(N-3)}{r^3} u_{2^k}' = f_{2^k}(u_1), \quad u_{2^k} > 0, \end{cases} \tag{3}$$

with the following boundary conditions

$$\begin{cases} u_i'(0) = 0, \quad u_i^{(3)}(0) = 0, \quad u_i(1) = 0, \quad u_i'(1) = 0, \\ u_{2^k}'(0) = 0, \quad u_{2^k}^{(3)}(0) = 0, \quad u_{2^k}(1) = 0, \quad u_{2^k}'(1) = 0. \end{cases} \tag{4}$$

It's well known that any solution $(u(r), v(r)) \in C^4(0, 1) \times C^4(0, 1)$ of (3)-(4) is a radial symmetric solution of (1)-(2).

The eigenvalue problem for the operator Δ^2 plays a crucial a role in studying our problem, we cite the following result from [18, Lemma 2].

Lemma 2.1. *There is a $\mu_1 > 0$ such that the problem*

$$\Delta^2 v = \mu_1 v \quad in\ B, \quad v = \frac{\partial v}{\partial \nu} = 0 \quad on\ \partial B$$

possesses a positive, radial symmetric solution $\varphi_1(x)$ which satisfies, for some positive constants C_1 and C_2,

$$C_1(1 - |x|)^2 \le \varphi_1(x) \le C_2(1 - |x|)^2, \quad x \in \overline{B}. \tag{5}$$

We recall the Green function $G(r, s)$ for the operator Δ^2, $N > 4$, see [11] and [18],

$$G(r, s) = \begin{cases} a_N(s) + r^2 b_N(s), & \text{for } 0 \le r \le s \le 1 \\ (\frac{s}{r})^{N-1}(a_N(r) + s^2 b_N(r)), & \text{for } 0 \le s \le r \le 1, \end{cases} \tag{6}$$

where

$$a_N(t) = \frac{t^3}{4(N-2)(N-4)}\left[2 + (N-4)t^{N-2} - (N-2)t^{N-4}\right],$$

and

$$b_N(t) = \frac{t}{4N(N-2)}\left[Nt^{N-2} - (N-2)t^N - 2\right].$$

The following proprieties of the kernel $G(r,s)$ are in [18]. There exists a positive constant C such that

$$0 \le G(r,s) \le Cs^{N-1}(1-s)^2 \ (\max(r,s))^{4-N}, \tag{7}$$

$$\frac{\partial}{\partial r}G(r,s)(r,s) \le 0, \tag{8}$$

and

$$\frac{\partial^2}{\partial r^2}G(r,s)\Big|_{r=1} = \frac{1}{2}s^{N-1}(1-s^2). \tag{9}$$

Hence, the problem (3)-(4) is transformed into the integral equations

$$\begin{cases} u_i(r) &= \displaystyle\int_0^1 G(r,s)f_i(u_{i+1}(s))\mathrm{d}s, \quad \text{for } i = 1,2,\dots,2^k-1 \\ u_{2^k}(r) &= \displaystyle\int_0^1 G(r,s)f_{2^k}(u_1(s))\mathrm{d}s. \end{cases} \tag{10}$$

It's natural that problem (3)-(4) and problem (10) are equivalent.
Consider the following eigenvalue problem,

$$\begin{cases} \Delta^2\phi_i = \lambda_{i+1}\ \phi_{i+1}, \quad i = 1,2,\dots,2^k-1 \qquad \text{in } B, \\ \Delta^2\phi_{2^k} = \lambda_1\ \phi_1 \\ \phi_i = 0,\ \dfrac{\partial\phi_i}{\partial\nu} = 0, i = 1,2,\dots,2^k-1 \\ \phi_{2^k} = 0,\ \dfrac{\partial\phi_{2^k}}{\partial\nu} = 0. \end{cases} \tag{11}$$

where $\lambda_i > 0,\ \ i = 1,2,3,\dots,2^k$.
Note φ_1 the corresponding eigenfunction of μ_1 the first eigenvalue of Δ^2 on the unit ball B, we prove the following result.

Lemma 2.2. *Assume that* $\prod_{i=1}^{2^k}\lambda_i = \mu_1^{2^k}$, *then the problem (11) has a positive solution* $(\phi_1,\phi_2,\phi_3,\dots,\phi_{2^k})$ *verifying (modulo a constant)* $\phi_1 = \varphi_1$, $\phi_i = \dfrac{\lambda_1\lambda_{i+1}\dots\lambda_{2^k}}{\mu_1^{2^k-(i-1)}}\varphi_1$ *for* $i = 2,3,\dots,2^k-1$, $\phi_{2^k} = \dfrac{\lambda_1}{\mu}\varphi_1$.

Proof. We define

$$w_1 = \phi_1, \quad w_i = \frac{\mu_1^{2^k-(i-1)}}{\prod\limits_{\substack{l=1,\\ l\neq,2,3,\dots,i}}^{2^k}\lambda_l}\ \phi_i, \quad \text{for } i = 2,\dots,2^k-1, \text{and } w_{2^k} = \frac{\mu_1}{\lambda_1}\,\phi_{2^k}. \tag{12}$$

We put (12) in the problem (11), after some simplifications, we obtain

$$\begin{cases} \Delta^2 w_i = \mu_1 \, w_{i+1}, \; \Delta^2 w_{2^k} = \mu_1 \, w_1 & \text{in } B, \\ w_i = \dfrac{\partial w_i}{\partial \nu} = 0, \; w_{2^k} = \dfrac{\partial w_{2^k}}{\partial \nu} = 0 & \text{on } \partial B. \end{cases} \tag{13}$$

for $i = 1, 2, 3, \ldots, 2^k - 1$.
Adding all the equations, we get

$$\begin{cases} \Delta^2 \left(\displaystyle\sum_{i=1}^{2^k} w_i \right) = \mu_1 \displaystyle\sum_{i=1}^{2^k} w_i & \text{in } B, \\ \displaystyle\sum_{i=1}^{2^k} w_i = 0, \; \dfrac{\partial}{\partial \nu} \left(\displaystyle\sum_{i=1}^{2^k} w_i \right) = 0 & \text{on } \partial B. \end{cases} \tag{14}$$

Applying $(\Delta^2)^{2^{k-1}-1}$ on the i^{th} and $(i+k)^{th}$ equations of the system (13) for $i = 1, 2, 3, \ldots, 2^{k-1}$, yields

$$\begin{cases} (\Delta^2)^{2^{k-1}-1} w_i = \mu_1^{2^{k-1}-1} w_{i+2^{k-1}} & \text{in } B, \\ w_i = 0, \; \dfrac{\partial w_i}{\partial \nu} = 0 & \text{on } \partial B, \end{cases} \tag{15}$$

and

$$\begin{cases} (\Delta^2)^{2^{k-1}-1} w_{i+2^{k-1}} = \mu_1^{2^{k-1}-1} w_i & \text{in } B, \\ w_{i+2^{k-1}} = 0, \; \dfrac{\partial w_{i+2^{k-1}}}{\partial \nu} = 0 & \text{on } \partial B. \end{cases} \tag{16}$$

Next, subtracting the equation (16) from (15), gives

$$\begin{cases} (\Delta^2)^{2^{k-1}-1} (w_i - w_{i+2^{k-1}}) = \mu_1^{2^{k-1}-1} (w_{i+2^{k-1}} - w_i) & \text{in } B, \\ w_i - w_{i+2^{k-1}} = 0, \; \dfrac{\partial}{\partial \nu} (w_i - w_{i+2^{k-1}}) = 0 & \text{on } \partial B. \end{cases} \tag{17}$$

We multiply (17) by $w_i - w_{i+2^{k-1}}$ and we make a 2^k integration by parts, we obtain

$$\int_B |\Delta(w_i - w_{i+2^{k-1}})|^{2^k} \mathrm{d}x = -\mu_1^{2^{k-1}-1} \int_B |w_i - w_{i+2^{k-1}}|^2 \mathrm{d}x,$$

this proves that $w_i = w_{i+2^{k-1}}$ for $i = 1, 2, 3, \ldots, 2^{k-1}$ in $\overline{B}$, which reduce the system (13) to 2^{k-1} equations. Repeating the same argument $k-1$ times, where at each j^{th} iteration we apply the operator $(\Delta^2)^{2^{k-j}-1}$ to the reduced system with 2^{k-j} equations and following the same steps as the previous iteration. Finally, we obtain $w_1 = w_2 = w_3 = \ldots = w_{2^k}$.

The properties of the eigenvalue problem for the bi-Laplacian, imply that the only solution of the system (14) is the first eigenfunction φ_1. Looking at (14), we have, modulo a positive constant, $w_1 = \ldots = w_{2^k} = \varphi_1$. Then we deduce directly the desired result. □

Let us, now, give the following identity which is important in studying our problem. Let F_i be the primitive of f_i such that $F_i(0) = 0$, for $i = 1, \ldots, 2^k$.

Lemma 2.3. *Let $(u_1, u_2, u_3, \ldots, u_{2^k})$ a solution of the system (1)-(2) and α_i for $i = 1, 2, \ldots, 2^k$ are some positive constants. We have the following*

$$\begin{aligned}\sum_{i=1}^{2^k-1}\int_{\partial B}(\Delta u_i, \Delta u_{i+1})(x.\nu)d\sigma_x &= \sum_{i=1}^{2^k-1}\int_B N\,F_i(u_{i+1}) - \alpha_{i+1}u_{i+1}f_i(u_{i+1})dx \\ &+ \int_B NF_{2^k}(u_1) - \alpha_1 u_1 f_{2^k}(u_1)dxdx \\ &+ \sum_{i=1}^{2^k-1}(N-4-\sum_{l=1}^{l=2^k}\alpha_l)\int_B(\Delta u_i, \Delta u_{i+1})dx.\end{aligned} \tag{18}$$

Proof. Looking at [14, Proposition 4], [15, Theorem 2.1] and by some adaptations, we write the following general identity

$$\begin{aligned}&\frac{\partial}{\partial x_i}\left[x_i L - \left(x_k\frac{\partial u_l}{\partial x_k} + a_l u_l\right)\left(L_{p_i} - \frac{\partial}{\partial x_j}L_{r_{ij}}\right) - \frac{\partial}{\partial x_j}\left(x_k\frac{\partial u_l}{\partial x_k} + a_l u_l\right)L_{r_{ij}}\right] \\ &= NL + x_i L_{x_i} - a_l u_l L_{u_l} - (a_l+1)\frac{\partial u_l}{\partial x_i}L_{p_i} - (a_l+2)\frac{\partial^2 u_l}{\partial x_i \partial x_j}L_{r_{ij}},\end{aligned} \tag{19}$$

where $L = L(x, U, p, r)$ is a lagrangian with $U = (u_1, u_2, \ldots, u_{2^k})$, $p = (p_i^k)$, $p_i^k = \frac{\partial u_k}{\partial x_i}$, $r = (r_{ij})$, $i, j = 1, \ldots N$ and a_l for $l = 1, 2, 3, \ldots, 2^k$, are constants. Applying the identity (19) to the Lagrangian of the problem (1)-(2);

$$L = L(x, U, \nabla U, \Delta U) = \sum_{m=1}^{m=2^k-1}[(\Delta u_m, \Delta u_{m+1}) + F_m(u_{m+1})] + (\Delta u_{2^k}, \Delta u_1) + F_{2^k}(u_1),$$

and $a_l = \alpha_l$ for $l = 1, 2, 3, \ldots, 2^k$.

Integrating (19) over B and using the condition $u_l = 0$, $\frac{\partial u_l}{\partial \nu} = 0$ on ∂B for $l = 1, 2, 3, \ldots, 2^k$, we get (18). □

Remark 2.4. *If we take $\Sigma_{l=1}^{l=2^k}\alpha_l = N-4$ in (18), we remark that the critical conditions on f_i, $i = 1, 2, 3, \ldots, 2^k$ are $NF_{2^k}(u_1) - \alpha_1 u_1 f_{2^k}(u_1) = 0$ and $NF_i(u_{i+1}) - \alpha_{i+1}u_{i+1}f_i(u_{i+1}) = 0$ for $i = 1, 2, 3, \ldots, 2^k-1$ therefore*

$$\frac{f_{2^k}(u_1)}{F_{2^k}(u_1)} = \frac{N/\alpha_1}{u_1} \quad \textit{and} \quad \frac{f_{i+1}(u_i)}{F_{i+1}(u_i)} = \frac{N/(\alpha_{i+1})}{u_{i+1}}, \quad \textit{for } 1, 2, 3, \ldots, 2^k-1.$$

Hence, for some positive constants c_i,

$$f_{2^k}(u_1) = c_{2^k}\,u^{\frac{N}{\alpha_1}-1} \quad \textit{and} \quad f_i(u_{i+1}) = c_i\,u_{i+1}^{\frac{N}{\alpha_{i+1}}-1}, \quad \textit{for } 1, 2, 3, \ldots, 2^k-1.$$

3. Main Results and Proofs

We define the following critical exponents associated to the system (1)-(2) by

$$q_i^\star = \frac{N-\alpha_{i+1}}{\alpha_{i+1}} \text{ and } q_{2^k}^\star = \frac{N-\alpha_1}{\alpha_1}, \quad \text{where } \alpha_i,\ \alpha_{i+1} \in ((N-4)/2,\ N/2), \text{for } i = 1, 2, 3, \ldots, 2^k-1. \tag{20}$$

A simple computation shows that $\displaystyle\sum_{i=1}^{i=2^k}\frac{1}{q_i^*+1} = \frac{N-4}{N}$.

We state our first main result.

Theorem 3.1. *Suppose that f_i for $i = 1,2,3,\dots,2^k$, verify the following conditions*
(I) $\liminf_{s\to\infty} f_i(s)s^{-1} > \lambda_i, \quad \limsup_{s\to 0} f_i(s)s^{-1} < \lambda_i,$

(II) $\begin{array}{ll} NF_i(s) - \alpha_{i+1} s f_i(s) & \geq \theta_{i+1}\, s\, f_i(s),\ s>0,\ \textit{for some } \theta_{i+1}\geq 0,\ i=1,2,3,\dots,2^k-1, \\ NF_{2^k}(s) - \alpha_1 s f_{2^k}(s) & \geq \theta_1\, s\, f_{2^k}(s),\ s>0,\ \textit{for some } \theta_1 \geq 0, \end{array}$ *where* α_j, $j = 1,2,3,\dots,2^k$

are positive reals such that $\sum_{j=1}^{j=2^k} \alpha_j = N-4.$

In addition, we suppose that:

(H) *There exists a constant $C > 0$ such that for every solution $(u_1, u_2, u_3, \dots, u_{2^k})$ of the system (1)-(2) verifies $\|u_i\|_\infty \leq C$, for $i = 1,2,3,\dots,2^k$.* *Then there exists a non-trivial solution of the system (1)-(2).*

Remark 3.2. *Let f_i, $i = 1,2,3,\dots,2^k$ verifying the conditions (I) and (II) of Theorem 3.1.*
we have

$$\lim_{t\to\infty} \frac{f_i(t)}{t^{q^*_{i+1}}} = 0, \textit{for } i = 1,2,3,\dots,2^k-1 \quad \textit{and} \quad \lim_{t\to\infty} \frac{f_{2^k}(t)}{t^{q^*_1}} = 0.$$

Indeed, from condition (I), there exists $t_0 > 0$ such that $f_i(t) > 0$ for $t > t_0$. Then, looking at condition (II) we write

$$NF_i(t) \geq -\theta_i + \eta_i t f_i(t) \quad \text{for } t > t_0, \tag{21}$$

where $\eta_i = \alpha_i + \theta_{i,2}$.
Hence

$$F_i'(t) - \frac{N}{\eta_i t} F_i(t) \geq \frac{\theta_i}{\eta_i t}.$$

Multiplying the last inequalities, respectively, by $t^{-\frac{N}{\eta_i}}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{-\frac{N}{\eta_i}} F_i(t)\right) \leq \frac{\theta_i}{\eta_i} t^{-1-\frac{N}{\eta_i}}.$$

Then, for some positive constants C_i, we have

$$F_i(t) \leq C_i t^{\frac{N}{\eta_i}}. \tag{22}$$

Replacing (22) into (21), we get for t large enough that,

$$f_i(t) \leq C t^{\frac{N}{\eta_i}-1}.$$

for some positive constant C. Since $\sum_{i=1}^{2^k} \alpha_i = N-4$ and $\eta_i = \alpha_i + \theta_{i,2}$, then we have $\sum_{i=1}^{2^k} \eta_i > N-4$.

The proof of Theorem 3.1 relies on a variant of fixed point theorem, see [9] and [12].

Theorem 3.3. *Let C be a cone in a Banach space X and $\Phi : C \to C$ a compact map such that $\Phi(0) = 0$. Assume that there exist numbers $0 < r < R$ such that*

(a) $x \neq \lambda\Phi(x)$ *for* $0 \leq \lambda \leq 1$ *and* $\|x\| = r$,

(b) there exists a compact map $F : \overline{B_R} \times [0, \infty) \to C$ *such that*

$$\begin{cases} F(x,0) = \Phi(x) & \textit{if} \quad \|x\| = R, \\ F(x,\mu) \neq x & \textit{if} \quad \|x\| = R \textit{ and } 0 \leq \mu < \infty, \\ F(x,\mu) \neq x & \textit{if} \quad x \in \overline{B_R} \textit{ and } \mu \geq \mu_0. \end{cases}$$

Then if $U = \{x \in C : r < \|x\| < R\}$ *and* $B_\rho = \{x \in C : \|x\| < \rho\}$, *we have*

$$i_C(\Phi, B_R) = 0, \quad i_C(\Phi, B_r) = 1 \quad i_C(\Phi, U) = -1,$$

where $i_C(\Phi, \Omega)$ *denotes the index of* Φ *with respect to* Ω. *In particular,* Φ *has a fixed point in* U.

Proof of Theorem 3.1 Applying Theorem 3.3, Let $C^*([0,1])$ denote the space of continuous bounded functions defined on $[0,1]$. Consider the Banach space $X = (C^*((0,1)))^{2^k}$ endowed with the norm $\|u\| = \sup_{t\in[0,1]} \{|u(t)|\}$. The cone C is defined by

$$C = \Big\{w \in X : \quad w(t) \geq 0 \quad \text{for all } t \in [0,1]\Big\},$$

where $w = (y_1, \dots, y_{2^k}) \geq 0$ means that $y_i \geq 0$ for $i = 1, \dots, 2^k$.

We define the compact map $\Phi : X \to X$ by

$$\Phi(w)(r) = \int_0^1 G(r,s) h(w(s))\, \mathrm{d}s, \qquad h(w) = (f_1(u_2), \dots, f_{2^k-1}(u_{2^k}), f_{2^k}(u_1)).$$

It's clear that a fixed point of Φ is a solution of (10). So, it will be a solution of (3)-(4) as well.

Verification of condition (a): From hypothesis (I) of Theorem 3.1 we have that $f_i(u_{i+1}(x)) \leq q_i \lambda_i u_{i+1}(x)$, $i = 1, \dots, 2^k - 1$ and $f_{2^k}(u_1(x)) \leq q_{2^k} \lambda_{2^k} u_1(x)$ where $q_i < 1$ for $i = 1, \dots, 2^k$. Then

$$\lambda_1 \int \phi_1 u_2 dx = \int u_2 \Delta^2 \phi_{2^k} dx = \int \Delta^2 u_2\, \phi_{2^k} dx = \int f_2(u_3)\, \phi_{2^k} dx < \lambda_2 q_2 \int \phi_{2^k} u_3 dx, \tag{23}$$

$$\begin{aligned} \lambda_{2^k} \int \phi_{2^k-i+3} u_i dx = \int u_i \Delta^2 \phi_{2^k-i+2} dx \quad &= \quad \int \Delta^2 u_i\, \phi_{2^k-i+2} dx = \int f_i(u_{i+1})\, \phi_{2^k-i+2} dx \\ &< \quad \lambda_i q_i \int \phi_{2^k-i+2} u_{i+1} dx, \text{ for } i = 3, \dots, 2^k - 1, \end{aligned} \tag{24}$$

$$\lambda_3 \int \phi_3 u_{2^k} dx = \int u_{2^k} \Delta^2 \phi_2 dx = \int \Delta^2 u_{2^k}\, \phi_3 dx = \int f_{2^k}(u_1)\, \phi_2 dx < \lambda_{2^k} q_{2^k} \int \phi_2 u_1 dx, \tag{25}$$

$$\lambda_2 \int \phi_2 u_1 dx = \int u_1 \Delta^2 \phi_1 dx = \int \Delta^2 u_1\, \phi_1 dx = \int f_1(u_2)\, \phi_1 dx < \lambda_1 q_1 \int \phi_1 u_2 dx. \tag{26}$$

Multiplying (23), (24) for $i = 3, \dots, 2^k - 1$, (25) and (26) each other. Since the integrals are nonzero, we get, after some simplifications,

$$\prod_{i=1}^{2^k} \lambda_i < \prod_{i=1}^{2^k} q_i \prod_{i=1}^{2^k} \lambda_i,$$

which leads to a contradiction, since $\prod_{i=1}^{2^k} q_i < 1$. Also, if u_i for $i = 1, \dots, 2^k$ are replaced by $\lambda\, u_i$ in the previous inequalities, for $\lambda \in [0,1]$, then similarly a contradiction follows and hence

$$w(t) \neq \lambda \Phi(w(t)) \quad \text{with} \quad \lambda \in [0,1], \quad \|w\| = r, \quad w \in C.$$

Verification of (b): Set the compact mapping $F : C \times [0, \infty) \to C$ such that

$$F(w, \mu)(r) = \Phi(w + \mu)(r) \tag{27}$$

Clearly we have $F(w, 0) = \Phi(w)$. From condition (i) of Theorem 3.1, there exist constants $k_i > \lambda_i$ for $i = 1, \dots, 2^k$, and $\mu_0 > 0$ such that $f_i(y_i + \mu) \geq k_i y_i$ if $\mu \geq \mu_0$ for all $y_i \geq 0$. We have

$$\lambda_1 \int \phi_1 u_2 dx = \int u_2 \Delta^2 \phi_{2^k} dx = \int \Delta^2 u_2\, \phi_{2^k} dx = \int f_2(u_3)\, \phi_{2^k} dx \geq k_2 \int \phi_{2^k} u_3 dx, \tag{28}$$

$$\begin{aligned}\lambda_{2^k}\int \phi_{2^k-i+3}u_i dx = \int u_i\Delta^2\phi_{2^k-i+2}dx &= \int \Delta^2 u_i\,\phi_{2^k-i+2}dx = \int f_i(u_{i+1})\,\phi_{2^k-i+2}dx \\ &\geq k_i q_i\int \phi_{2^k-i+2}u_{i+1}dx, \text{ for } i=3,\dots,2^k-1,\end{aligned} \tag{29}$$

$$\lambda_3\int \phi_3 u_{2^k}dx = \int u_{2^k}\Delta^2\phi_2 dx = \int \Delta^2 u_{2^k}\,\phi_3 dx = \int f_{2^k}(u_1)\,\phi_2 dx \geq k_{2^k}\int \phi_2 u_1 dx, \tag{30}$$

$$\lambda_2\int \phi_2 u_1 dx = \int u_1\Delta^2\phi_1 dx = \int \Delta^2 u_1\,\phi_1 dx = \int f_1(u_2)\,\phi_1 dx \geq k_1\int \phi_1 u_2 dx. \tag{31}$$

Multiplying all the previous inequality each other, since the integrals $\int u_i\phi_i$, for $i,\, j\in\{1,2,\dots,2^k\}$, are nonzero, we obtain

$$\prod_{i=1}^{2^k}\lambda_i \geq \prod_{i=1}^{2^k}k_i.$$

The last inequality leads to a contradiction since $k_i>\lambda_i$ for every $i=1,2,\dots,2^k$.
Then, there exists a constant $\mu_0>0$ such that

$$w(t)\neq F(w,\mu)(t)\quad \text{for all } w\in C \text{ and } \mu\geq\mu_0. \tag{32}$$

Therefore the last condition of (b) is verified. Now, in order to prove the second condition of (b), we take the family of nonlinearities $\big(f_1(y_1+\mu),\dots,f_{2^k}(y_2^k+\mu)\big)$ for $\mu\in[0,\mu_0]$. Using the a priori estimates (H) which does not depend on μ and choosing $R>r$ large enough, we have

$$w(r)\neq F(w,\mu)(r)\quad \text{for all } \mu\in[0,\mu_0],\quad w\in C,\quad \|w\|=R. \tag{33}$$

The relations (32) and (33) prove the second condition of (b).
Finally, all conditions of Theorem 3.3 are fulfilled, then we obtain the existence of a nontrivial positive solution of problem (10). Therefore we deduce the existence of positive solution of problem (1)-(2) as well. □

Theorem 3.4. *Suppose that f_i for $i=1,2,3,\dots,2^k$, satisfy the conditions (I) and (II). Then every solution of the system (1)-(2) is bounded in L^∞, namely the hypothesis (H) is verified.*

Proof. We will proof it in four steps.
Step 1. We claim that there exist positive constants $C_{i,1}$, $C_{i,2}$, for $i=1,\dots,2^k$ such that

$$\int_B f_i(u_{i+1})\phi_i \mathrm{d}x \leq C_{i,1},\ \text{for } i=1,\dots,2^k-1, \tag{34}$$

$$\int_B f_{2^k}(u_1)\phi_{2^k}\mathrm{d}x \leq C_{2^k,1}, \tag{35}$$

$$\int_B u_i\,\phi_i\mathrm{d}x \leq C_{i,2},\ \text{for } i=1,\dots 2^k. \tag{36}$$

Indeed, from the equations (1) and (11) one can write

$$\begin{aligned}\int_B f_i(u_{i+1})\phi_i\mathrm{d}x &= \int_B \Delta^2 u_i\phi_i\mathrm{d}x = \int_B u_i\Delta^2\phi_i\mathrm{d}x \\ &= \lambda_{i+1}\int_B u_i\phi_{i+1}\mathrm{d}x \text{ for } i=1,\dots,2^k-1, \\ \int_B f_{2^k}(u_1)\phi_{2^k}\mathrm{d}x &= \int_B \Delta^2 u_{2^k}\phi_{2^k}\mathrm{d}x = \int_B u_{2^k}\Delta^2\phi_{2^k}\mathrm{d}x \\ &= \lambda_1\int_B u_{2^k}\phi_1\mathrm{d}x.\end{aligned}$$

Next, from condition (I) of Theorem 3.1, there exist $k_i > \lambda_i$ and $A_i > 0$, for every $i \in \{1, \ldots, 2^k\}$, such that $f_i(u_{i+1}) \geq k_i u_{i+1} - A_i$ for $i = 1, \ldots, 2^k - 1$ and $f_{2^k}(u_1) \geq k_{2^k} u_1$. Thus, for generic constant C, we have

$$\int_B f_1(u_2)\phi_1 \mathrm{d}x = \lambda_2 \int_B u_1\phi_2 \mathrm{d}x \leq C + \frac{\lambda_2}{K_{2^k}} \int_B f_{2^k}(u_1)\phi_2 \mathrm{d}x, \tag{37}$$

$$\int_B f_{2^k}(u_1)\phi_2 \mathrm{d}x = \lambda_3 \int_B u_{2^k}\phi_3 \mathrm{d}x \leq C + \frac{\lambda_3}{K_{2^k-1}} \int_B f_{2^k-1}(u_{2^k})\phi_3 \mathrm{d}x, \tag{38}$$

$$\begin{aligned} \int_B f_{2^k-i}(u_{2^k+1-i})\phi_{2+i} \mathrm{d}x &= \lambda_{3+i} \int_B u_{2^k}\phi_3 \mathrm{d}x \\ &\leq C + \frac{\lambda_{3+i}}{K_{2^k-1-i}} \int_B f_{2^k-1-i}(u_{2^k-i})\phi_{3+i} \mathrm{d}x, \text{ for } i = 1, \ldots 2^k - 3, \end{aligned} \tag{39}$$

$$\int_B |f_2(u_3)|\phi_{2^k} \mathrm{d}x = \lambda_1 \int_B u_{2^k}\phi_1 \mathrm{d}x \leq C + \frac{\lambda_1}{K_1} \int_B f_1(u_2)\phi_1 \mathrm{d}x \tag{40}$$

Combining (37)-(40) we get, for a generic constant C,

$$\begin{aligned} \int_B f_{2^k-j}(u_{2^k+1-j})\phi_{2+j} \mathrm{d}x &\leq C + \frac{\lambda_{3+j}\lambda_{3+j-1}}{k_{2^k-1-j}k_{2^k-2-j}} \int_B f_{2^k-2-j}(u_{2^k-1-j})\phi_{3+j-1} \mathrm{d}x \\ &\vdots \\ &\leq C + \frac{\prod_{i=1}^{2^k}\lambda_i}{\prod_{i=1}^{2^k}k_i} \int_B f_{2^k-j}(u_{2^k+1-j})\phi_{2+j} \mathrm{d}x \end{aligned} \tag{41}$$

also

$$\begin{aligned} \int_B f_1(u_2)\phi_1 \mathrm{d}x &\leq C + \frac{\lambda_2\lambda_3}{k_{2^k}k_{2^k-1}} \int_B f_{2^k-1}(u_{2^k})\phi_3 \mathrm{d}x \\ &\vdots \\ &\leq C + \frac{\prod_{i=1}^{2^k}\lambda_i}{\prod_{i=1}^{2^k}k_i} \int_B f_1(u_2)\phi_1 \mathrm{d}x \end{aligned} \tag{42}$$

and

$$\begin{aligned} \int_B f_2(u_3)\phi_{2^k} \mathrm{d}x &\leq C + \frac{\lambda_1\lambda_2}{k_1k_{2^k}} \int_B f_{2^k}(u_1)\phi_2 \mathrm{d}x \\ &\vdots \\ &\leq C + \frac{\prod_{i=1}^{2^k}\lambda_i}{\prod_{i=1}^{2^k}k_i} \int_B f_2(u_3)\phi_{2^k} \mathrm{d}x \end{aligned} \tag{43}$$

Since $\frac{\prod_{i=1}^{2^k}\lambda_i}{\prod_{i=1}^{2^k}k_i} < 1$, this implies (34) and (35).

From condition (I) of Theorem 3.1, (34) and (35) we deduce (36).

Step 2. We claim that, for $i \in \{1, 2, \ldots, 2^k\}$, there exist positive constants $C_{i,1}, \ldots, C_{i,2}$ such that

$$u_i(r) \le C_{i,1} \quad \text{for } \frac{2}{3} \le r \le 1 \tag{44}$$

and

$$u_i''(1) \le C_{i,3}. \tag{45}$$

Indeed, we have

$$u_i(r) = \int_0^1 G(r,s) f_i(u_{i+1}(s))\mathrm{d}s, \quad \text{for } i \in \{1, 2, \ldots, 2^k - 1\},$$

and

$$u_{2^k}(r) = \int_0^1 G(r,s) f_{2^k}(u_1(s))\mathrm{d}s.$$

The fact that $r \to G(r,s)$ is decreasing, (see (8) and (7)), gives that $u_i(r)$, for $i \in 1, 2, \ldots, 2^k$, are decreasing and for arbitrary $\frac{2}{3} \le r \le 1$,

$$u_i(r) \le u_i(\frac{2}{3}) = 3\int_{\frac{1}{3}}^{\frac{2}{3}} u_i(s)\mathrm{d}s \le C\int_0^1 s^{N-1}(1-s)^2 u_i(s)\mathrm{d}s \le C + \int_0^1 s^{N-1}(1-s)^2 u_i(s)\mathrm{d}s.$$

From (5) and Lemma 2.2, we have

$$u_i(r) \le C\left(1 + \int_0^1 s^{N-1}(1-s)^2 u_i(s)\mathrm{d}s\right) \le C\left(1 + \int_B \phi_i u_i dx\right).$$

Using (36) we conclude that $u_i(r) \le C_{i,1}$ for $\frac{2}{3} \le r \le 1$.
To prove (45) we will use the following

$$\begin{aligned} u_i(r) &= \int_0^1 G(r,s) f_i(u_{i+1}(s))\mathrm{d}s, \quad \text{for } i \in 1, 2, \ldots, 2^k - 1, \\ u_{2^k}(r) &= \int_0^1 G(r,s) f_{2^k}(u_1(s))\mathrm{d}s. \end{aligned} \tag{46}$$

We differentiate (46) two times, we get

$$u_i''(r) = \int_0^1 \frac{\partial^2 G(r,s)}{\partial r^2} f_i(u_{i+1}(s))\mathrm{d}s \quad \text{and} \quad u_{2^k}''(r) = \int_0^1 \frac{\partial^2 G(r,s)}{\partial r^2} f_{2^k}(u_1(s))\mathrm{d}s.$$

Taking the limit when r goes to 1, since the integrals converge, we write

$$u_i''(1) = \int_0^1 \frac{\partial^2 G(r,s)}{\partial r^2}\Big|_{r=1} f_i(u_{i+1}(s))\mathrm{d}s \quad \text{and} \quad u_{2^k}''(1) = \int_0^1 \frac{\partial^2 G(r,s)}{\partial r^2}\Big|_{r=1} f_{2^k}(u_1(s))\mathrm{d}s.$$

From (9), we get

$$u_i''(1) = \frac{1}{2}\int_0^1 s^{N-1}(1-s^2)\ f_i(u_{i+1}(s))\mathrm{d}s, \quad \text{and} \quad u_{2^k}''(1) = \frac{1}{2}\int_0^1 s^{N-1}(1-s^2)\ f_{2^k}(u_1(s))\mathrm{d}s.$$

Using (5) and Lemma 2.2, we write, for some positive constant C, that

$$u_i''(1) \le C\int_B \phi_i\, f_i(u_{i+1})\mathrm{d}s \text{ for } i = 1, 2, \ldots, 2^k - 1, \text{ and } \quad u_{2^k}''(1) \le C\int_B \phi_{2^k}\, f_{2^k}(u_1)\mathrm{d}s.$$

Then we obtain (45) using (34) and (35).
Step 3. We claim that, for a small number $0 < l < 1$, there exist positive constants $C_1, \ldots, C_4$ such that

$$\int_0^l s^{N-1} f_i(u_{i+1}(s))\mathrm{d}s \le C_1 \quad \text{for } i = 1, 2, \ldots, 2^k - 1, \quad \int_0^l s^{N-1} f_{2^k}(u_1(s))\mathrm{d}s \le C_2. \tag{47}$$

$$\int_B u_{i+1}\, f_i(u_{i+1})\mathrm{d}x \le C_3 \quad \text{for } i = 1, 2, \ldots, 2^k - 1, \quad \int_B u_1\, f_{2^k}(u_1)\mathrm{d}x \le C_4. \tag{48}$$

Indeed, following Step 1, Lemma 2.1 and Lemma 2.2 we have, for $i = 1, 2, \ldots, 2^k - 1$ and for small $0 < l < 1$,

$$\begin{aligned}
\int_0^l s^{N-1} f_i(u_{i+1}(s))\mathrm{d}s &\le \int_0^l s^{N-1} \frac{(1-s)^2}{(1-l)^2} f_i(u_{i+1}(s))\mathrm{d}s \\
&\le \frac{1}{(1-l)^2} \int_0^l s^{N-1}(1-s)^2 f_i(u_{i+1}(s))\mathrm{d}s \\
&\le C \int_0^1 s^{N-1}\phi_i(s) f_i(u_{i+1}(s))\mathrm{d}s = C\int_B \phi_i f_i(u_{i+1})\mathrm{d}x \le M_i
\end{aligned}$$

and

$$\begin{aligned}
\int_0^l s^{N-1} f_{2^k}(u_1(s))\mathrm{d}s &\le \int_0^l s^{N-1} \frac{(1-s)^2}{(1-l)^2} f_{2^k}(u_1(s))\mathrm{d}s \\
&\le \frac{1}{(1-l)^2} \int_0^l s^{N-1}(1-s)^2 f_{2^k}(u_1(s))\mathrm{d}s \\
&\le \bar{C} \int_0^1 s^{N-1}\phi_{2^k}(s) f_{2^k}(u_1(s))\mathrm{d}s = \bar{C}\int_B \phi_{2^k} f_{2^k}(u_1)\mathrm{d}x \le M_{2^k},
\end{aligned}$$

where M_i, $1 \le i \le 2^k$, are some positive constants. This shows (47).
For the proof of (48), using the identity (18) of Lemma 2.3, considering the fact that $\sum_{i=1}^{2^k} \alpha_i = N - 4$, as

$$\begin{aligned}
\sum_{i=1}^{2^k-1} \int_B NF_i(u_{i+1}) - \alpha_{i+1}u_{i+1}f_i(u_{i+1})\mathrm{d}x \quad &+ \quad \int_B NF_{2^k}(u_1) - \alpha_1 u_1 f_{2^k}(u_1)\mathrm{d}x \\
&= \sum_{i=1}^{2^k-1} \int_{\partial B} (\Delta u_i, \Delta u_{i+1})(x.\nu)\mathrm{d}\sigma_x.
\end{aligned}$$

Using condition (II) of Theorem 3.1 for the left hand side of the last equality and after some computations on the right hand side we obtain, for a positive constant C,

$$\sum_{i=1}^{2^k-1} \theta_{i+1} \int_B u_{i+1} f_i(u_{i+1})dx + \theta_1 \int_B u_1 f_{2^k}(u_1)dx \le C \sum_{i=1}^{2^k} u_i''(1)u_{i+1}''(1),$$

Therefore

$$\sum_{i=1}^{2^k-1} \theta_{i+1} \int_B u_{i+1} f_i(u_{i+1})dx + \theta_1 \int_B u_1 f_{2^k}(u_1)dx \le C,$$

we obtain (48) since all terms in the left hand side are positive.
Step 4. We claim that there exist positive constants C_i for $i = 1, \ldots, 2^k$, such that, for any solution $(u_1, \ldots, u_{2^k})$ of problem (1)-(2),

$$\|u_i\|_\infty \le C_i \quad \text{for } i = 1, \ldots, 2^k. \tag{49}$$

Indeed, for u_{i+1}, $i = 1, \dots, 2^k - 1$, we have

$$\begin{aligned} \|u_{i+1}\|_\infty &\leq u_{i+1}(0) \leq \int_0^1 G(0,s) f_i(u_{i+1}(s))\mathrm{d}s \\ &\leq C\int_0^1 s^3(1-s)^2 f_i(u_{i+1}(s))\mathrm{d}s \\ &\leq C\int_0^1 s^3 f_i(u_{i+1}(s))\mathrm{d}s \\ &\leq C\int_0^t s^3 f_i(u_{i+1}(s))\mathrm{d}s + C\int_t^1 s^3 f_i(u_{i+1}(s))\mathrm{d}s, \end{aligned}$$

where $t \in (0,1)$ is arbitrary and C is a generic positive constant for the rest of this step.
Let $\tilde{f_i}(m) = \max_{s\in[0,m]} f_i(s)$ for $m \in (0,\infty)$, by Hölder's inequality, we obtain

$$\begin{aligned} \|u_i\|_\infty &\leq Ct^4 \|\tilde{f_i}(\|u_{i+1}\|_\infty) + C\left(\int_t^1 s^{\gamma_{i+1}(q_i^*+1)}\mathrm{d}s\right)^{\frac{1}{q_i^*+1}} \left(\int_t^1 s^{N-1} f_i(u_{i+1}(s))^{\frac{q_i^*+1}{q_i^*}}\,\mathrm{d}s\right)^{\frac{q_i^*}{q_i^*+1}} \\ &\leq Ct^4 \tilde{f_i}(\|u_{i+1}\|_\infty) \\ &+ C\left(\int_t^1 s^{\gamma_{i+1}(q_i^*+1)}\mathrm{d}s\right)^{\frac{1}{q_i^*+1}} \left(\int_t^1 s^{N-1}(f_i(u_{i+1}(s)))(f_i(u_{i+1}(s)))^{\frac{1}{q_{i+1}^*}}\,\mathrm{d}s\right)^{\frac{q_{i+1}^*}{q_{i+1}^*+1}}, \end{aligned}$$

where $\gamma_{i+1} = 3 - (N-1)\frac{q_{i+1}^*}{q_{i+1}^*+1}$. From Remark 3.2, we have the existence of a positive constant M such that

$$f_i(s) < M(1+s)^{q_{i+1}^*}, \quad \text{for all } s \geq 0 \tag{50}$$

Then

$$\begin{aligned} \|u_i\|_\infty &\leq Ct^4 \tilde{f_i}(\|u_{i+1}\|_\infty) + C\left(\int_t^1 s^{\gamma_{i+1}(q_{i+1}^*+1)}\mathrm{d}s\right)^{\frac{1}{q_{i+1}^*+1}} \left(\int_t^1 s^{N-1} f_i(u_{i+1}(s))(1+v(s))\mathrm{d}s\right)^{\frac{q_i^*}{q_i^*+1}}, \\ &\leq Ct^4 \tilde{f_i}(\|u_{i+1}\|_\infty) \\ &+ C\left(\int_t^1 s^{\gamma_{i+1}(q_{i+1}^*+1)}\mathrm{d}s\right)^{\frac{1}{q_{i+1}^*+1}} \left(\int_B (f_i(u_{i+1}(s)))dx + \int_B (f_i(u_{i+1}(s)))u_{i+1}(x)dx\right)^{\frac{q_{i+1}^*}{q_{i+1}^*+1}}. \end{aligned}$$

Using (47) and (48), we get

$$\|u_i\|_\infty \leq C\,t^4 \tilde{f_i}(\|u_{i+1}\|_\infty) + C\left(\int_t^1 s^{\gamma_{i+1}(q_{i+1}^*+1)}\mathrm{d}s\right)^{\frac{1}{q_{i+1}^*+1}}, \quad \text{for } i = 1, \dots, 2^k - 1.$$

Similarly, we have for u_{2^k},

$$\|u_{2^k}\|_\infty \leq C\,t^4 \tilde{f}_{2^k}(\|u_1\|_\infty) + C\left(\int_t^1 s^{\gamma_1(q_1^*+1)}\mathrm{d}s\right)^{\frac{1}{q_1^*+1}}, \text{ where } \gamma_1 = 3 - (N-1)\frac{q_1^*}{q_1^*+1}.$$

After some manipulations, we get

$$\|u_i\|_\infty \leq C\,t^4 \tilde{f_i}(\|u_{i+1}\|_\infty) + C\,t^{\frac{4+(4-N)q_{i+1}^*}{q_{i+1}^*+1}}, \quad \text{for } i = 1, \dots, 2^k - 1 \tag{51}$$

and

$$\|u_{2^k}\|_\infty \leq C\,t^4 \tilde{f}_{2^k}(\|u_1\|_\infty) + C\,t^{\frac{4+(4-N)q_1^*}{q_1^*+1}}. \tag{52}$$

Note that if $\tilde{f}_i$, for $i = 1, \ldots, 2^k$, are bounded then (49) comes directly. Nevertheless, if $\tilde{f}_i$ is not bounded then there exist a positive M_i, see (50) such that $\tilde{f}_i(m) \leq M_i\, m^{q_{i+1}^*}$ for $i = 1, \ldots, 2^k - 1$ and $\tilde{f}_{2^k}(m) \leq M_{2^k}\, m^{q_1^*}$ where $m \geq 1$.
Therefore (51) becomes

$$\|u_i\|_\infty \leq C\,t^4 (\|u_{i+1}\|_\infty)^{q_{i+1}^*} + C\,t^{\frac{4+(4-N)q_{i+1}^*}{q_{i+1}^*+1}}, \quad \text{for } i = 1, \ldots, 2^k - 1. \tag{53}$$

We rewrite (52) and all the equations appearing in (53) as

$$\begin{aligned}
\|u_1\|_\infty &\leq C\,t^4 (\|u_2\|_\infty)^{q_2^*} + C\,t^{\frac{4+(4-N)q_2^*}{q_2^*+1}}, && (54)\\
\|u_2\|_\infty &\leq C\,t^4 (\|u_3\|_\infty)^{q_3^*} + C\,t^{\frac{4+(4-N)q_3^*}{q_3^*+1}}, \\
&\vdots \\
\|u_{2^k-1}\|_\infty &\leq C\,t^4 (\|u_{2^k}\|_\infty)^{q_{2^k}^*} + C\,t^{\frac{4+(4-N)q_{2^k}^*}{q_{2^k}^*+1}}, && (55)\\
\|u_{2^k}\|_\infty &\leq C\,t^4 (\|u_1\|_\infty)^{q_1^*} + C\,t^{\frac{4+(4-N)q_1^*}{q_1^*+1}}.
\end{aligned}$$

Combining the previous inequalities and using the inequality $(a+b)^n \leq C_n(a^n + b^n)$ for $a, b, n \geq 0$ where C_n is a positive constant depending only on n, we obtain

$$\|u_1\|_\infty \leq C\,t^{4+4\left(\sum_{j=2}^{2^k-1}\prod_{l=2}^{j} q_l^*\right)} (\|u_{2^k}\|_\infty)^{\prod_{l=2}^{2^k} q_l^*} + C\sum_{j=2}^{2^k-1} t^{m_j^*\left(\prod_{l=2}^{j} q_l^*\right) + 4\left(\sum_{i=2}^{j-1}\prod_{l=2}^{i} q_l^*\right)+4} + C\,t^{m_1^*}, \tag{56}$$

where $m_j^\star = \frac{4+(4-N)q_{j+1}^*}{q_{j+1}^*+1}$ for $j = 1, \ldots, 2^k - 1$.
Now, putting (52) into (56) and using again the inequality $(a+b)^n \leq C_n(a^n + b^n)$, we get

$$\|u_1\|_\infty \leq C\,t^{4+4\left(\sum_{j=2}^{2^k}\prod_{l=2}^{j} q_l^*\right)} \left\{\tilde{f}_{2^k}(\|u_1\|_\infty)\right\}^{\prod_{l=2}^{2^k} q_l^*} + C\sum_{j=3}^{2^k} t^{m_j^*\left(\prod_{l=2}^{j} q_l^*\right) + 4\left(\sum_{i=2}^{j-1}\prod_{l=2}^{i} q_l^*\right)+4} + C t^{m_2^* q_2^* + 4} + C\,t^{m_1^*}. \tag{57}$$

We note $M_j = m_j^*\left(\prod_{l=2}^{j} q_l^*\right) + 4\left(\sum_{i=2}^{j-1}\prod_{l=2}^{i} q_l^*\right) + 4$, for $j = 3, \ldots, 2^k - 1$, $M_2 = m_2^* q_2^* + 4$ and $M_1 = m_1^*$.

We remark that

$$M_2 - M_1 = -N \sum_{\substack{j=2,\\ j\neq 2,3}}^{2^k} \frac{q_2^*}{q_j^*+1},$$

$$\vdots$$

$$M_i - M_{i-1} = -N \sum_{\substack{j=2,\\ j\neq i,i+1}}^{2^k} \frac{\prod_{l=2}^{i} q_l^*}{q_j^*+1},$$

$$\vdots$$

$$M_{2^k-1} - M_{2^k-2} = -N \sum_{\substack{j=2,\\ j\neq 2^k-1,\, 2^k}}^{2^k} \frac{\prod_{l=2}^{2^k-1} q_l^*}{q_j^*+1}.$$

Noting $\gamma_i = N \sum_{\substack{j=2,\\ j\neq i,i+1}}^{2^k} \frac{\prod_{l=2}^{i} q_l^*}{q_j^*+1}$, for $i = 1, \dots, 2^k - 1$.

We deduce that $M_i = M_{2^k-1} + \sum_{j=i+1}^{2^k-1} \gamma_j$ for $i = 1, \dots, 2^k - 1$.
Using this relation with (57) and the fact that $t^\gamma \leq 1$ for $\gamma > 0$, we write

$$\|u_1\|_\infty \leq C\, t^{4+4\left(\sum_{j=2}^{2^k} \prod_{l=2}^{j} q_l^*\right)} \left\{\tilde{f}_{2^k}(\|u_1\|_\infty)\right\}^{\prod_{l=2}^{2^k} q_l^*} + C t^{M_{2^k-1}}. \tag{58}$$

For convenient calculations, we define $r = \frac{M_{2^k-1}(1-\prod_{l=1}^{2^k} q_l^*)}{q_1^*}$. Since $t \in (0,1)$, we write

$$\|u_1\|_\infty \leq C\, t^r \left\{\tilde{f}_{2^k}(\|u_1\|_\infty)\right\}^{\prod_{l=2}^{2^k} q_l^*} + C t^{M_{2^k-1}}. \tag{59}$$

In order to have the best estimate for $\|u_1\|_\infty$ we take the infimum with respect to t in the right expression of (59). Then we define the function

$$h(t) = t^r \left\{\tilde{f}_{2^k}(\|u_1\|_\infty)\right\}^{\prod_{l=2}^{2^k} q_l^*} + t^{M_{2^k-1}}. \tag{60}$$

The function h attains its infimum at $t_0 = C\,(\tilde{f}_{2^k}(\|u_1\|_\infty))^{\frac{\prod_{l=2}^{2^k} q_l^*}{M_{2^k-1}-r}}$ and has the following value

$$h(t_0) \leq C\,(\tilde{f}_{2^k}(\|u_1\|_\infty)^{\frac{r \prod_{l=2}^{2^k} q_l^*}{M_{2^k-1}-r} + \prod_{l=2}^{2^k} q_l^*} + C\,(\tilde{f}_{2^k}(\|u_1\|_\infty)^{\frac{M_{2^k-1} \prod_{l=2}^{2^k} q_l^*}{M_{2^k-1}-r}}.$$

The choice of r gives that $\dfrac{r\prod_{l=2}^{2^k}q_l^*}{M_{2^k-1}-r}+\prod_{l=2}^{2^k}q_l^*=\dfrac{M_{2^k-1}\prod_{l=2}^{2^k}q_l^*}{M_{2^k-1}-r}=\dfrac{1}{q_1^*}$. Then

$$h(t_0)\le C+C\,(\tilde{f}_{2^k}(\|u_1\|_\infty))^{\frac{1}{q_1^*}}.$$

From Remark 3.2 we have $\tilde{f}_{2^k}(x)=o(x^{q_1^*})$ for $x\to+\infty$ then we obtain

$$\|u_1\|_\infty\le C\,(1+o(\|u_1\|_\infty)),$$

this shows that $\|u_1\|_\infty$ is bounded. Replacing the bound of $\|u_1\|_\infty$ into (56) we deduce that $\|u_i\|_\infty$ is bounded for $i=1,\dots,2^k$.
This finish Step 4 and complete the prove of Theorem 3.4. $\square$

We end this section by giving a non-existence theorem.

Theorem 3.5. *Assume that f_i, for $i=1,2,3,\dots,2^k$, verify for $t>0$*

$$NF_i(t)-\alpha_{i+1}t\,f_i(t)\le 0,\ \ i=1,2,3,\dots,2^k-1,\quad \textit{and}\quad NF_{2^k}(t)-\alpha_1 t\,f_{2^k}(t)\le 0. \tag{61}$$

Then there is no nontrivial solution of the system (1)-(2) in $\left(C^2(B)\cap C^1(\bar{B})\right)^2$.

Proof. Taking $\sum_{i=1}^{2^k}\alpha_i=N-4$ in the identity (18). Since $u_i=0=\frac{\partial u_i}{\partial\nu}$ for $i=1,\dots,2^k$, we have $(\Delta u_i,\Delta u_{i+1})=\frac{\partial^2 u_i}{\partial\nu^2}\,\frac{\partial^2 u_{i+1}}{\partial\nu^2}$ for $i=1,\dots,2^k-1$.
If $(u_1,\dots,u_{2^k})$ is a nontrivial solution of (1)-(2), since B is star-shaped domain about 0, then $x.\nu\ge 0$ on ∂B. Then the identity (18) gives a contradiction in the case of the condition (61). This finishes the proof. $\square$

4. Examples of Some Numerical Solutions

In this section, we give some examples to illustrate the study of the general system (1)-(2). We fix the dimension of the space $N=5$ and $k=2$ that means we consider the following system.

$$\begin{cases} u_1^{(4)}+\frac{2(5-1)}{r}u_1^{(3)}+\frac{(5-1)(5-3)}{r^2}u_1''-\frac{(5-1)(5-3)}{r^3}u_1'=f_1(u_2),\\ u_2^{(4)}+\frac{2(5-1)}{r}u_2^{(3)}+\frac{(5-1)(5-3)}{r^2}u_2''-\frac{(5-1)(5-3)}{r^3}u_2'=f_2(u_3),\\ u_3^{(4)}+\frac{2(5-1)}{r}u_3^{(3)}+\frac{(5-1)(5-3)}{r^2}u_3''-\frac{(5-1)(5-3)}{r^3}u_3'=f_3(u_4),\\ u_4^{(4)}+\frac{2(5-1)}{r}u_4^{(3)}+\frac{(5-1)(5-3)}{r^2}u_4''-\frac{(5-1)(5-3)}{r^3}u_4'=f_4(u_1), \end{cases} \tag{62}$$

with the boundary conditions

$$\begin{cases} u_1'(0)=0, u_1^{(3)}(0)=0,\ \ u_1(1)=c_1,\ \ u_1'(1)=0,\\ u_2'(0)=0, u_2^{(3)}(0)=0,\ \ u_2(1)=c_2,\ \ u_2'(1)=0,\\ u_3'(0)=0, u_3^{(3)}(0)=0,\ \ u_3(1)=c_3,\ \ u_3'(1)=0,\\ u_4'(0)=0, u_4^{(3)}(0)=0,\ \ u_4(1)=c_4,\ \ u_4'(1)=0, \end{cases} \tag{63}$$

where c_1, c_2, c_3, and c_4 are constants.

In order to obtain a numerical solution, we write the system (62)-(63) as a system of first order ODEs

$$\begin{cases} u_1' = u_{1,1} \\ u_{1,1}' = u_{1,2} \\ u_{1,2}' = u_{1,3} \\ u_{1,3}' = f_1(u_2) - \left(u_{1,3} + \frac{(N-1)(N-3)}{r^2} u_{1,2} - \frac{(N-1)(N-3)}{r^3} u_{1,1}\right), \\ u_2' = u_{2,1} \\ u_{2,1}' = u_{2,2} \\ u_{2,2}' = u_{2,3} \\ u_{2,3}' = f_2(u_3) - \left(u_{2,3} + \frac{(N-1)(N-3)}{r^2} u_{2,2} - \frac{(N-1)(N-3)}{r^3} u_{2,1}\right), \\ u_3' = u_{3,1} \\ u_{3,1}' = u_{3,2} \\ u_{3,2}' = u_{3,3} \\ u_{3,3}' = f_3(u_4) - \left(u_{3,3} + \frac{(N-1)(N-3)}{r^2} u_{3,2} - \frac{(N-1)(N-3)}{r^3} u_{3,1}\right), \\ u_4' = u_{4,1} \\ u_{4,1}' = u_{4,2} \\ u_{4,2}' = u_{4,3} \\ u_{4,3}' = f_4(u_1) - \left(u_{4,3} + \frac{(N-1)(N-3)}{r^2} u_{4,2} - \frac{(N-1)(N-3)}{r^3} u_{4,1}\right), \end{cases} \tag{64}$$

subject to boundary conditions

$$\begin{cases} u_{1,1}(0) = 0 & u_{1,3}(0) = 0 & u_1(1) = c_1, & u_{1,1}(1) = 0, \\ u_{2,1}(0) = 0 & u_{2,3}(0) = 0 & u_2(1) = c_2, & u_{2,1}(1) = 0, \\ u_{3,1}(0) = 0 & u_{3,3}(0) = 0 & u_3(1) = c_3, & u_{3,1}(1) = 0, \\ u_{4,1}(0) = 0 & u_{4,3}(0) = 0 & u_4(1) = c_4, & u_{4,1}(1) = 0. \end{cases} \tag{65}$$

The numerical solutions obtained using the Matlab program bvp5c [16] which requires initial guess for the solution on a given mesh.

Example 4.1.
Let $f_1(u) = u^2$, $f_2(u) = u^3$, $f_3(u) = u^4$, $f_4(u) = u^5$ and $c_1 = 1$, $c_2 = \frac{1}{2}$, $c_3 = \frac{1}{4}$, $c_4 = \frac{1}{8}$. Easily we see that the f_i, $1 \leq i \leq 4$ verify the condition (I) and (II) of Theorem 3.1. The numerical solution computed by choosing the initial guess

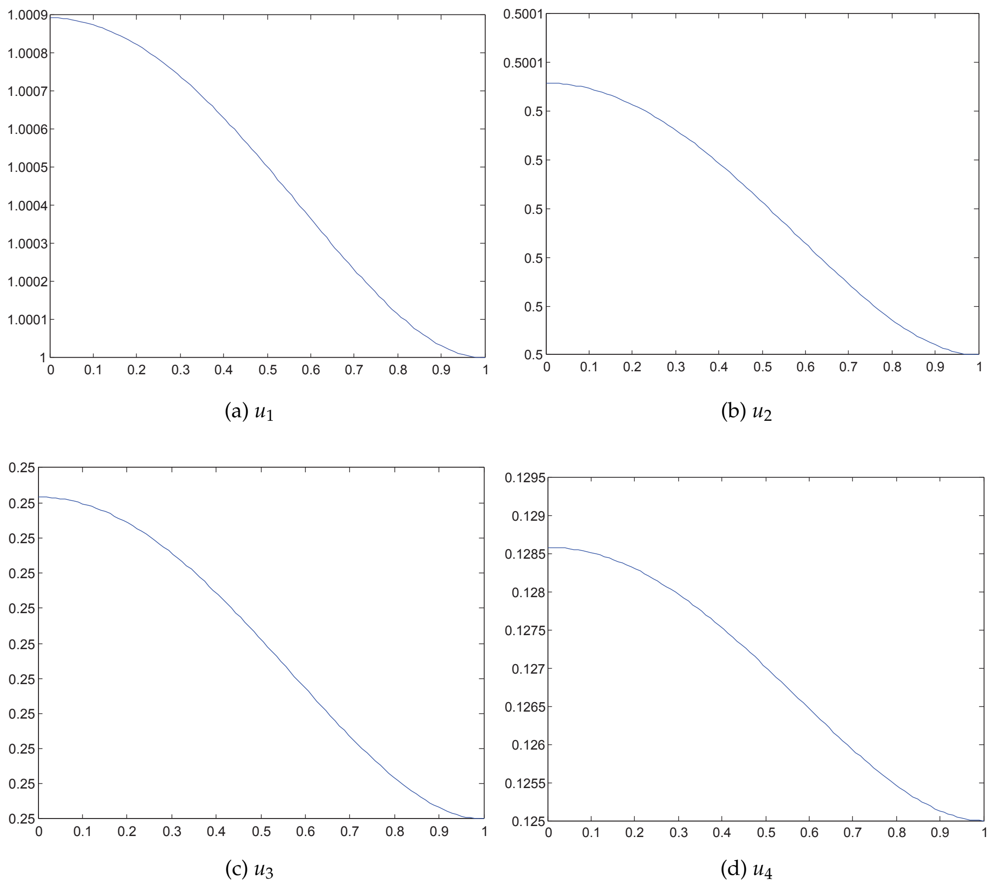
$$u_1 = x,\ u_2 = \frac{x^2}{2}, u_3 = \frac{x^3}{4},\ \text{and}\ u_4 = \frac{x^4}{8}$$

and is presented in Figure 1 on a mesh of 100 *points and relative error tolerance RelTol* = 10^{-9}.

Example 4.2. *Let $f_1(u) = u^2 + u$, $f_2(u) = u^3 + u^2 + u$, $f_3(u) = u^4 + u^3 + u^2 + u$, $f_4(u) = u^5 + u^4 + u^3 + u^2 + u$. We note that the functions f_i, $1 \leq i \leq 4$ verify the conditions (I) and (II) of Theorem 3.1. Therefore, the numerical solution computed by choosing the initial guess*

$$u_1 = \frac{1}{x+1},\ u_2 = \frac{1}{x^2+1}, u_3 = \frac{1}{x^3+1},\ \text{and}\ u_4 = \frac{1}{x^4+1}$$

and is presented in Figure 2 on a mesh of 1000 *points and maximum error* 10^{-3}.



(a) u_1 (b) u_2

(c) u_3 (d) u_4

Figure 1: Numerical Solution for Example 1 obtained on a mesh of 100 points and $RelTol = 10^{-9}$

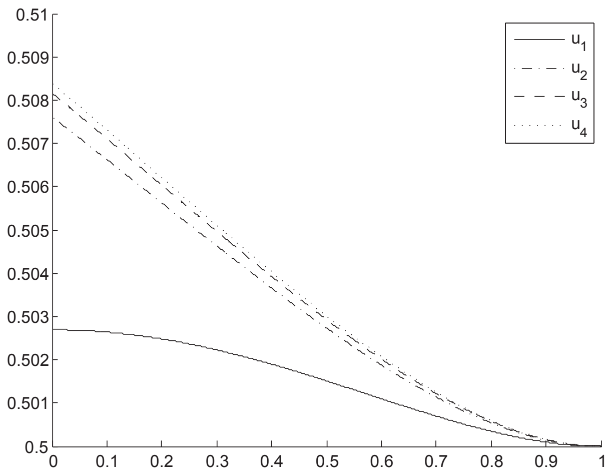


Figure 2: Numerical Solution for Example 2 obtained on a mesh of 1000 points

References

[1] H. Afshari, H.R. Marasi and H. Aydi, Existence and uniqueness of positive solutions for boundary value problems of fractional diferential equations, Filomat 31 :9 (2017), 2675-2782.

[2] H. Brezis and R.E.L. Turner, On a class of superlinear elliptic problems, Comm. in P.D.E. 2 (1977) 601–614.

[3] L. Boccardo and D.G. de Figueiredo, Some remarks on a system of quasilinear elliptic equations, Nonl. Diff. Eq. Appl. 9 (2002) 309–323.

[4] Ph. Clement, D.G. de Figueiredo and E. Mitidieri, Positive solutions of semilinear elliptic systems, Comm. Part. Diff. Eq. 17 (1992) 923–940.

[5] C. Cosner, Positive solutions for superlinear elliptic systems without variational structure, Nonlinear. Anal. T.M.A. 8 (1984) 1427–1436.

[6] L. Dung, Global L^∞ estimates for a class of reaction-diffusion systems, Journal of Math. Ana. and Apl. 217 (1998) 72–94.

[7] R. Demarque, O. Miyagaki, Radial solutions of inhomogeneous fourth order elliptic equations and weighted Sobolev embeddings, Adv. Nonlinear Anal. **4** (2015), no. 2, 135–151.

[8] D.E. Edmunds, D. Fortunato and E. Janelli, Critical exponents, critical dimensions and the biharmonic operator, Arch. Rat. Mech. Anal. **112** (1990) 269–289.

[9] D.G. de Figueiredo, P.L. Lions and R.D. Nussbaum, A Priori estimates and existence of positive solutions of semilinear elliptic equations, J. Math. P. Appl. **61** (1982) 41–63.

[10] M. Jleli, V. Cojbasic Rajic, B. Samet, C. Vetro, Fixed point theorems on ordered metric spaces and applications to nonlinear elastic beam equations, Journal of Fixed Point Theory and Applications 12 (2012) 175-192.

[11] F. Gazzola, H.C. Grunau and G. Sweers, Polyharmonic boundary value problems, Lecture notes in Mathematics 1991, Springer, Berlin (2010).

[12] D.D. Joseph and T.S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Rat. Mech. Anal. 49 (1972) 241–269.

[13] P. Pucci, V. Radulescu, Remarks on a polyharmonic eigenvalue problem, C. R. Math. Acad. Sci. Paris 348 (2010), no. 3-4, 161–164.

[14] P. Pucci and J. Serrin, A general variational identity, Indiana Univ. Math., J. 35 (1986) 681–703.

[15] L.A. Peletier and R.C.A.M. Van Der Vost, Existence and non-existence of positive solutions of non-linear elliptic systems and The Biharmonic equation, Diff. Int. Eq. **5**(4) (1992) 447–467.

[16] L. F. Shampine, J. Kierzenka and M. W. Reichelt, Solving Boundary Value Problems for Ordinary Differential Equations in Matlab with bvp4c, October 26, 2000.

[17] F. Soltani and H. Yazidi, Existence of solutions for a non-variational system of elliptic PDE's via topological methods, Electronic Journal of Diferential Equations 2016 (2016), No. 309, 1–13.

[18] P. Oswald, On a priori estimates for positive solutions of a semilinear biharmonic equation in a Ball, Comm. Math. Univ. Carol. 26(3) (1985) 565–577.

[19] R.C.A.M. Van Der Vost, Variational identities and applications to differential systems, Arch. Rat. Mech. Anal. 116 (1990) 375–398.