Solution of Volterra Integral Equation in Metric Spaces via New Fixed Point Theorem

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Abstract. The aim of this article is to study the existence of coincidences and fixed points of generalized hybrid contractions involving single-valued mappings and left total relations in the context of complete metric spaces. Some special cases are also discussed to derive some well known results of the literature. Finally, some examples and applications are also presented to verify the effectiveness and applicability of our main results.

1. Introduction and Preliminaries

One of the simplest and most useful results in fixed point theory is the Banach contraction principle [9], a powerful tool in analysis for establishing existence and uniqueness of solution of problems in different fields. Over the years, this principle has been generalized in numerous directions in different spaces. These generalizations have been obtained either by extending the domain of the mapping or by considering a more general contractive condition on the mappings.

Very recently, Jleli and Samet [24] introduced a new type of contraction and established some new fixed point theorems for such contraction in the context of generalized metric spaces.

Definition 1.1. Let $\psi : (0, \infty) \to (1, \infty)$ be a function satisfying:

(\psi_1) $\psi$ is nondecreasing;

(\psi_2) for each sequence $\{a_n\} \subseteq R^+, \lim_{n \to \infty} \psi(a_n) = 1$ if and only if $\lim_{n \to \infty}(a_n) = 0$;

(\psi_3) there exists $0 < k < 1$ and $l \in (0, \infty]$ such that $\lim_{a \to 0^+} \frac{\psi(a) - 1}{a} = l$.

A mapping $F : X \to X$ is said to be JS-contraction if there exist the function $\psi$ satisfying (\psi_1)-(\psi_3) and a constant $\alpha \in (0, 1)$ such that for all $x, y \in X$,

$$d(Fx, Fy) \not= 0 \implies \psi(d(Fx, Fy)) \leq [\psi(d(x, y))]^\alpha.$$ (1.1)
**Theorem 1.2.** [24] Let $(X,d)$ be a complete metric space and $F : X \rightarrow X$ be a JS-contraction, then $F$ has a unique fixed point.

To be consistent with Samet et al. [24], we denote by $\Psi$ the set of all functions $\psi : (0, \infty) \rightarrow (1, \infty)$ satisfying the above conditions.

Hussain et al. [18] modified and extended the above result and proved the following fixed point theorem for $\psi$-contractive condition in the setting of complete metric spaces.

**Theorem 1.3.** [18] Let $(X,d)$ be a complete metric space and $F : X \rightarrow X$ be a self-mapping. If there exist a function $\psi \in \Psi$ and positive real numbers $\alpha, \beta, \gamma, \delta$ with $0 \leq \alpha + \beta + \gamma + 2\delta < 1$ such that

$$
\psi(d(Fx, Fy)) \leq [\psi(d(x, y))]^2 \cdot [\psi(d(y, Fy))]^2 \cdot [\psi(d(x, Fx)) + d(y, Fx)]^\delta
$$

(1.2)

for all $x, y \in X$, then $F$ has a unique fixed point.

**Main Results**

In the following we always suppose $X$ is a nonempty set and $Y$ be such that $R$ is left-total, Ran $Fy = \{y \in Y : y \in R \{y\} \text{ for some } x \in \text{dom } (R)\}$.

For convenience, we denote $R \{x\}$ by $R x$. The class of relations from $A$ to $B$ is denoted by $\mathcal{R}(A, B)$. Thus the collection $\mathcal{M}(A, B)$ of all mappings from $A$ to $B$ is a proper sub collection of $\mathcal{R}(A, B)$. An element $w \in A$ is called coincidence point of $F : A \rightarrow B$ and $R : A \rightsquigarrow B$ if $Fw \in R \{w\}$. In the following we always suppose that $X$ is nonempty set and $(Y, d)$ is a metric space. For $R : X \rightsquigarrow Y$ and $u, v \in \text{dom } (R)$, we define

$$
D(R u, R v) = \inf_{uRx \supseteq y} d(x, y).
$$

The aim of this paper is to prove coincidence fixed point results of a pair of self mappings and left total relation satisfying a generalized $\psi$-contractive condition in the framework of complete metric spaces.

**2. Main Results**

Now we state and prove our main results of this section.

**Theorem 2.1.** Let $X$ be a nonempty set and $(Y, d)$ be a metric space. Let $F : X \rightarrow Y$ be a single-valued mapping, $R : X \rightsquigarrow Y$ be such that $R$ is left-total, Range($F$) $\subseteq$ Range($R$) and Range($F$) or Range($R$) is complete. If there exist a mapping $\psi \in \Psi$ and a constant $k \in (0, 1)$ such that

$$
\psi(d(Fx, Fy)) \leq [\psi(D(Rx, Ry))]^k
$$

(2.1)
for all \( x, y \in X \). Then there exists \( w \in X \) such that \( Fw \in R[w] \).

**Proof.** Let \( x_0 \in X \) be an arbitrary, but fixed element. We define the sequences \( \{x_n\} \subset X \) and \( \{y_n\} \subset \text{Range}(R) \). Let \( y_1 = Fx_0 \), since \( \text{Range}(F) \subseteq \text{Range}(R) \). We may choose \( x_1 \in X \) such that \( x_1y_1 \), since \( R \) is left-total. Let \( y_2 = Fx_1 \), since \( \text{Range}(F) \subseteq \text{Range}(R) \). If \( Fx_0 = Fx_1 \), then we have \( x_1y_2 \). This implies that \( x_1 \) is the required point that is \( Fx_1 \in R[x_1] \). So we assume that \( Fx_0 \neq Fx_1 \), then from (2.1) we get

\[
1 < \psi(d(y_1, y_2)) = \psi(d(Fx_0, Fx_1)) \leq [\psi(D(R[x_0], R[x_1]))]^k. \tag{2.2}
\]

We may choose \( x_2 \in X \) such that \( x_2y_2 \), since \( R \) is left-total. Let \( y_3 = Fx_2 \), since \( \text{Range}(F) \subseteq \text{Range}(R) \). If \( Fx_1 = Fx_2 \), then we have \( x_2y_3 \). This implies that \( Fx_2 \in R[x_2] \) and \( x_2 \) is the coincidence point. So \( Fx_1 \neq Fx_2 \), then from (2.1), we get

\[
1 < \psi(d(y_2, y_3)) = \psi(d(Fx_1, Fx_2)) \leq [\psi(D(R[x_1], R[x_2]))]^k. \tag{2.3}
\]

By induction, we can construct sequences \( \{x_n\} \subset X \) and \( \{y_n\} \subset \text{Range}(R) \) such that

\[
y_n = Fx_{n-1} \quad \text{and} \quad x_ny_n \tag{2.4}
\]

for all \( n \in \mathbb{N} \). If there exists \( n_0 \in \mathbb{N} \) for which \( Fx_{n_0-1} = Fx_{n_0} \), then \( x_{n_0}y_{n_0+1} \). Thus \( Fx_{n_0} \in R[x_{n_0}] \) and the proof is finished. So we suppose now that \( Fx_{n-1} \neq Fx_n \) for every \( n \in \mathbb{N} \). Then from (2.2),(2.3) and (2.4), we deduce that

\[
1 < \psi(d(y_n, y_{n+1})) = \psi(d(Fx_{n-1}, Fx_n)) \leq [\psi(D(R[x_{n-1}], R[x_n]))]^k \tag{2.5}
\]

for all \( n \in \mathbb{N} \). Since \( x_ny_n \) and \( x_{n+1}y_{n+1} \), therefore by the definition of \( D \), we get \( D(R[x_{n-1}], R[x_n]) \leq d(y_{n-1}, y_n) \). Thus from (2.5), we have

\[
1 < \psi(d(y_n, y_{n+1})) \leq [\psi(d(y_{n-1}, y_n))]^k \tag{2.6}
\]

which further implies that

\[
1 < \psi(d(y_n, y_{n+1})) \leq [\psi(d(y_{n-1}, y_{n+1}))]^k \leq [\psi(d(y_{n-2}, y_{n-1}))]^k \leq ... \leq [\psi(d(y_0, y_1))]^{k^n}. \tag{2.7}
\]

From (2.7), we obtain

\[
\lim_{n \to \infty} \psi(d(y_n, y_{n+1})) = 1. \tag{2.8}
\]

Then from (\( \psi_2 \)), we get

\[
\lim_{n \to \infty} d(y_n, y_{n+1}) = 0. \tag{2.9}
\]

From the condition (\( \psi_3 \)), there exist \( 0 < k < 1 \) and \( l \in (0, \infty) \) such that

\[
\lim_{n \to \infty} \frac{\psi(d(y_n, y_{n+1})) - 1}{d(y_n, y_{n+1})^k} = l.
\]

Suppose that \( l < \infty \). In this case, let \( B = \frac{1}{k} > 0 \). From the definition of the limit, there exists \( n_1 \in \mathbb{N} \) such that

\[
\frac{|\psi(d(y_n, y_{n+1})) - 1|}{d(y_n, y_{n+1})^k} - l \leq B
\]

for all \( n > n_1 \). This implies that

\[
\frac{\psi(d(y_n, y_{n+1})) - 1}{d(y_n, y_{n+1})^k} \geq 1 - B = \frac{l}{k} = 2B
\]
for all $n > n_1$. Then
\[ n(d(y_n, y_{n+1}))^k \leq An[\psi(d(y_n, y_{n+1})) - 1] \]
for all $n > n_1$, where $A = \frac{1}{B}$. Now we suppose that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that
\[ B \leq \frac{\psi(d(y_n, y_{n+1})) - 1}{(d(y_n, y_{n+1}))^k} \]
for all $n > n_1$. This implies that
\[ n(d(y_n, y_{n+1}))^k \leq An[\psi(d(y_n, y_{n+1})) - 1] \]
for all $n > n_1$, where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_1 \in \mathbb{N}$ such that
\[ n(d(y_n, y_{n+1}))^k \leq An[\psi(d(y_n, y_{n+1})) - 1] \]
for all $n > n_1$. Thus by (2.7), we get
\[ n(d(y_n, y_{n+1}))^k \leq An[\psi(d(y_n, y_{n+1}))^k - 1]. \]
Letting $n \to \infty$ in the above inequality, we obtain
\[ \lim_{n \to \infty} n(d(y_n, y_{n+1}))^k = 0. \]
Thus, there exists $n_2 \in \mathbb{N}$ such that
\[ d(y_n, y_{n+1}) \leq \frac{1}{n^{1/k}} \] (2.10)
for all $n > n_2$. Now we prove that $\{y_n\}$ is a Cauchy sequence. For $m > n > n_2$ we have,
\[ d(y_n, y_m) \leq \sum_{i=n}^{m-1} d(y_i, y_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}}. \] (2.11)
Since, $0 < k < 1$, then $\sum_{i=n}^{m-1} \frac{1}{i^{1/k}}$ converges. Therefore, $d(y_n, y_m) \to 0$ as $m, n \to \infty$. Thus we proved that $\{y_n\}$ is a Cauchy sequence in $\text{Range}(R)$. Completeness of $\text{Range}(R)$ ensures that there exist $z \in \text{Range}(R)$ such that, $y_n \to z$ as $n \to \infty$. Now since $R$ is left-total, so $wRz$ for some $w \in X$. Now
\[ 1 < \psi(d(y_n, Fw)) = \psi(d(F_{x_{n-1}}, Fw)) \leq \psi(D(R(x_{n-1}), R[w]))^k \leq [\psi(d(y_{n-1}, z))]^k. \]
Since $\lim_{n \to \infty} d(y_{n-1}, z) = 0$, so by (2.5), we have $\lim_{n \to \infty} \psi(d(y_{n-1}, z)) = 1$. This implies that $\lim_{n \to \infty} \psi(d(y_n, Fw)) = 1$, which further implies that $\lim_{n \to \infty} d(y_n, Fw) = 0$. Hence $d(z, Fw) = 0$. It follows that $z = Fw$. Hence $Fw \in R[w]$. In the case when $\text{Range}(F)$ is complete. Since $\text{Range}(F) \subseteq \text{Range}(R)$, so there exists an element $z' \in \text{Range}(R)$ such that $y_n \to z'$. The remaining part of the proof is same as in previous case. \(\square\)

**Example 2.2.** Let $X = Y = \mathbb{R}$, $d(x, y) = |x - y|$. Define $F : \mathbb{R} \to \mathbb{R}$, $R : \mathbb{R} \rightsquigarrow \mathbb{R}$ as follows:
\[ Fx = \begin{cases} 0 & \text{if } x \in \mathbb{Q}' \\
1 & \text{if } x \in \mathbb{Q} \end{cases} \]
\[ R = (\mathbb{Q} \times [0, 3]) \cup (\mathbb{Q}' \times [7, 9]) \]
Then $\text{Range}(F) = [0, 1] \subseteq \text{Range}(R) = [0, 3] \cup [7, 9]$. Let $\psi(t) = e^{\sqrt{t}}$. 

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For \( x \in Q, y \in Q' \) or either \( y \in Q, x \in Q' \), we have \( d(Fx, Fy) \neq 0 \) implies
\[
\psi(d(Fx, Fy)) \leq \lceil \psi(D(R[x], R[y])) \rceil^k
\]
with \( k = \frac{1}{2} \). Thus all conditions of the above theorem are satisfied and 1 is the coincidence point of \( F \) and \( R \).

From Theorem 2.1, we deduce the following result immediately.

**Theorem 2.3.** Let \( X \) be a nonempty set and \((Y, d)\) be a metric space. Let \( F, R : X \to Y \) be two mappings such that \( \text{Range}(F) \subseteq \text{Range}(R) \) and \( \text{Range}(F) \) or \( \text{Range}(R) \) is complete. If there exist a mapping \( \psi \in \Psi \) and a constant \( k \in (0, 1) \) such that
\[
\psi(d(Fx, Fy)) \leq \lceil \psi(d(Rx, Ry)) \rceil^k
\]
for all \( x, y \in X \). Then \( F \) and \( R \) have a coincidence point in \( X \). Moreover, if either \( F \) or \( R \) is injective, then \( R \) and \( F \) have a unique coincidence point in \( X \).

**Proof.** By Theorem 2.1, we obtain that there exists \( w \in X \) such that \( Fw = Rw \), where,
\[
Rw = \lim_{n \to \infty} Rx_n = \lim_{n \to \infty} Fx_{n-1}, x_0 \in X.
\]
For uniqueness, assume that \( w_1, w_2 \in X, w_1 \neq w_2 \), \( Fw_1 = Rw_1 \) and \( Fw_2 = Rw_2 \). Then
\[
1 < \psi(d(Fw_1, Fw_2)) \leq [\psi(d(Rw_1, Rw_2))]^k
\]
for any \( k \in (0, 1) \). If \( R \) or \( F \) is injective, then
\[
d(Rw_1, Rw_2) > 0
\]
and
\[
1 < \psi(d(Rw_1, Rw_2)) = \psi(d(Fw_1, Fw_2)) \leq [\psi(d(Rw_1, Rw_2))]^k < \psi(d(Rw_1, Rw_2)),
\]
a contradiction. Thus proved. \( \Box \)

**Corollary 2.4.** [24] Let \((X, d)\) be a complete metric space and \( F : X \to X \) be a self mapping. If there exist a function \( \psi \) and a constant \( k \in (0, 1) \) such that for all \( x, y \in X \),
\[
d(Fx, Fy) \neq 0 \implies \psi(d(Fx, Fy)) \leq [\psi(d(x, y))]^k.
\]
then \( F \) has a unique fixed point.

**Proof.** Choosing \( X = Y \) and \( R = I \) (the identity mapping on \( X \)). \( \Box \)

**Corollary 2.5.** Let \( F : X \to Y, R : X \to Y \) be such that \( R \) is left-total, \( \text{Range}(F) \subseteq \text{Range}(R) \) and \( \text{Range}(F) \) or \( \text{Range}(R) \) is complete. If there exists some \( k \in [0, 1) \) such that for all \( x, y \in X \)
\[
d(Fx, Fy) \leq kD(R[x], R[y]).
\]
Then there exists \( w \in X \) such that \( Fw \in R[w] \).

**Proof.** Consider the mapping \( \psi(t) = e^{\sqrt{t}} \), for \( t > 0 \). Then obviously \( \psi \) satisfies \( (\psi_1)-(\psi_3) \). From Theorem 2.1, we obtain the desired conclusion. \( \Box \)

**Corollary 2.6.** Let \( X \) be nonempty set and \((Y, d)\) be a metric space. \( F, R : X \to Y \) be two mappings such that \( \text{Range}(F) \subseteq \text{Range}(R) \) and \( \text{Range}(F) \) or \( \text{Range}(R) \) is complete. If there exists some \( k \in [0, 1) \) such that for all \( x, y \in X \)
\[
d(Fx, Fy) \leq kd(Rx, Ry).
\]
Then \( R \) and \( F \) have a coincidence point in \( X \). Moreover, if either \( F \) or \( R \) is injective, then \( R \) and \( F \) have a unique coincidence point in \( X \).
Let $X$ be a nonempty set and 

**Theorem 2.9.**

Clearly, the Banach contraction is not satisfied. In fact, we can check easily that $F$ for all $x, y \in X$.

**Example 2.11.** Consider the sequence 

$$\begin{align*}
S_1 &= 1 \times 2 \\
S_2 &= 1 \times 2 + 3 \times 4 \\
S_3 &= 1 \times 2 + 3 \times 4 + 5 \times 6 \\
S_n &= 1 \times 2 + 3 \times 4 + \ldots + (2n-1)(2n) = \frac{n(n+1)(4n-1)}{3}.
\end{align*}$$

Let $X = \{S_n : n \in \mathbb{N}\}$ and $d(x, y) = |x - y|$. Then $(X, d)$ is a complete metric space. Define the mapping $F : X \to X$ by 

$$F(S_1) = S_{1}, \quad F(S_n) = S_{n-1}, \quad \text{for all } n \geq 2.$$ 

Clearly, the Banach contraction is not satisfied. In fact, we can check easily that 

$$\lim_{n \to \infty} \frac{d(F(S_n), F(S_1))}{d(S_n, S_1)} = 1.$$
Let us consider the mapping \( \psi : (0, \infty) \to (1, \infty) \) defined by
\[
\psi(t) = e^{\sqrt{t}}.
\]
We can easily show that \( \psi \in \Psi \). Now we shall prove that \( F \) satisfies the conditions of Corollary 2.4, that is \( d(F(S_n), F(S_m)) \neq 0 \) implies that
\[
d(F(S_n), F(S_m)) \neq 0 \implies e^{\sqrt{d(F(S_n), F(S_m))}} \leq e^{\sqrt{d(S_n, S_m)}}
\]
for some \( k \in (0, 1) \). The above condition is equivalent to
\[
d(F(S_n), F(S_m))e^{d(F(S_n), F(S_m))} \leq k^2 d(S_n, S_m) e^{d(S_n, S_m)}.
\]
So, we have to check that
\[
\frac{d(F(S_n), F(S_m))e^{d(F(S_n), F(S_m))}}{d(S_n, S_m) e^{d(S_n, S_m)}} \leq k^2
\]
for some \( k \in (0, 1) \). We consider two cases,

**Case 01.** For \( 1 = n \) and \( m > 2 \), we have
\[
d(F(S_1), F(S_m))e^{d(F(S_1), F(S_m)) - d(S_1, S_m)}
\]
\[
= \frac{4m^3 - 9m^2 + 5m - 6}{4m^3 + 3m^2 - m - 6} e^{2(2m-1)m}
\]
\[
\leq e^{-1}
\]

**Case 02.** For \( m > n > 1 \), we have
\[
d(F(S_m), F(S_n))e^{d(F(S_m), F(S_n)) - d(S_m, S_n)}
\]
\[
= \frac{(2m - 3)(2m - 2) + (2n - 1)(2n)}{(2m - 1)(2m) + (2n + 1)(2n + 2)} \frac{2(2n-1)m - 2(2m-1)m}{e^{2(2m-1)m - 2(2m-1)m}}
\]
\[
\leq e^{-1}
\]

with \( k = e^{-1} \). Thus all the conditions of Corollary 2.4 are satisfied and \( F \) has a unique fixed point. In this example \( S_1 \) is a unique fixed point of \( F \).

Now we discuss the existence and uniqueness of solution of a general class of the following Volterra type integral equation under various assumptions on the functions involved. Let \( C(0, \Theta) \) denote the space of all continuous functions on \([0, \Theta]\), where \( \Theta > 0 \) and for an arbitrary \( \|x\|_1 = \sup_{t \in [0, \Theta]} |x(t)| e^{-\lambda t} \), where \( \lambda > 0 \) is taken arbitrary. Note that \( \| \cdot \|_1 \) is a norm equivalent to supremum norm and \( (C([0, \Theta], \mathbb{R}), \| \cdot \|_1) \) endowed with the metric \( d_1 \) defined by
\[
d_1(x, y) = \sup_{t \in [0, \Theta]} |x(t) - y(t)| e^{-\lambda t}
\]
for all \( x, y \in C([0, \Theta], \mathbb{R}) \) is a Banach space.

Consider the integral equation:
\[
(f y)(t) = \int_0^t K(t, s, h x(s)) ds + g(t) \quad (2.12)
\]
where \( x : [0, \Theta] \to \mathbb{R} \) is unknown, \( g : [0, \Theta] \to \mathbb{R} \) and \( h, f : \mathbb{R} \to \mathbb{R} \) are given functions. The kernel \( K \) of the integral equation is defined on \([0, \Theta] \times [0, \Theta] \times \mathbb{R}\).
Theorem 2.12. Assume that the following conditions are satisfied:

(i) $K : [0, \Theta] \times [0, \Theta] \times \mathbb{R} \to \mathbb{R}$, $g : [0, \Theta] \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous
(ii) $\int_0^t K(t, s, :) : \mathbb{R} \to \mathbb{R}$ is increasing, for all $t, s \in [0, \Theta]$,
(iii) there exists $\lambda \in (0, +\infty)$ such that

$$|K(t, s, hx(s)) - K(t, s, hy(s))| \leq \lambda |hx(s) - hy(s)|$$

for all $t, s \in [0, \Theta]$ and $hx, hy \in \mathbb{R}$.

(iv) If $f$ is injective, there exists $k \in (0, 1)$ such that for all $x, y \in \mathbb{R}$;

$$|hx - hy| \leq k^2 |fx - fy|$$

and $\{fx : x \in \mathbb{R}\}$ is complete. Then there exist $w \in C([0, \Theta], \mathbb{R})$ such that for $x_0 \in \mathbb{R}$,

$$fw(t) = \lim_{n \to \infty} fx_n(t) = \lim_{n \to \infty} \left[ g(t) + \int_0^t K(t, s, hx_{n-1}(s))ds \right]$$

and $w$ is the unique solution of (2.12).

Proof. Let $X = Y = C([0, \Theta], \mathbb{R})$ and

$$d_\lambda(x, y) = \sup_{t \in [0, \Theta]} \left| |x(t) - y(t)| e^{-\lambda t} \right|$$

for all $x, y \in X$. Let $F, R : X \to X$ be defined as follows:

$$(Fx)(t) = g(t) + \int_0^t K(t, s, hx(s))ds \quad \text{and} \quad Rx = fx.$$ 

Then by assumptions $RX = \{Rx : x \in X\}$ is complete. Let $x^* \in FX$, then $x^* = Fx$ for $x \in X$ and $x^*(t) = Fx(t)$. By assumptions there exists $y \in X$ such that $Fx(t) = fy(t)$, hence $RX \subseteq FX$. Since

$$|Fx(t) - (Fy)(t)| = \left| \int_0^t [K(t, s, hx(s))]ds - \int_0^t [K(t, s, hy(s))]ds \right|$$

$$\leq \int_0^t |K(t, s, hx(s)) - K(t, s, hy(s))|ds$$

$$\leq \int_0^t \lambda |hx(s) - hy(s)|ds$$

$$\leq \int_0^t \lambda k^2 |fx(s) - fy(s)|ds$$

$$= \int_0^t \lambda k^2 |Rx(s) - (Ry)(s)| e^{-\lambda s}e^{\lambda s}ds$$

$$\leq \lambda k^2 \|Rx - Ry\|_1 \int_0^t e^{\lambda s}ds$$

$$\leq \lambda k^2 \|Rx - Ry\|_1 \frac{e^{\lambda t}}{\lambda}$$

$$= k^2 \|Rx - Ry\|_1 e^{\lambda t}.$$
This implies that
\[ |(Fx(t) - (Fy(t))|e^{\lambda t} \leq k^2|Rx(t) - Ry(t)|, \]
or equivalently,
\[ d_\lambda(Fx, Fy) \leq k^2 d_\lambda(Rx, Ry). \]
Taking exponential, we have
\[ e^{d_\lambda(Fx, Fy)} \leq e^{k^2 d_\lambda(Rx, Ry)}. \]
Now, we observe that mapping \( \psi : (0, \infty) \to (1, \infty) \) defined by
\[ \psi(t) = e^{kt}. \]
for each \( t \in [0, \Theta] \) and \( k \in (0, 1). \) Thus all conditions of Theorem 2.1 are satisfied. Hence, there exists a unique \( w \in X \) such that
\[ f(w) = \lim_{n \to \infty} Rx_n(t) = \lim_{n \to \infty} Fx_{n-1}(t) = F(w)(t), \quad x_0 \in X \]
for all \( t, \) which is the unique solution of (2.12). \( \square \)

References


