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An Extension of Egoroff's and Lusin's Theorems in Operator-valued Case

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Abstract. Here, we extend three basic facts from classical measure theory to operator-valued case. At first we show that operator-valued measurable functions may be approximated by simple ones. In the sequel, two fundamental theorems *Egoroff and Lusin* are extended in operator-valued case.

1. Introduction

Throughout this current discussion, we assume $\mathcal{H}$ is a Hilbert space. As usual $\mathbf{B}(\mathcal{H})$ stands for the set of all bounded linear operators on $\mathcal{H}$. We also assume (Ω, M) is a measurable space. In this paper, three basic results from classical measure theory, concerning measurable functions, will be generalized in operator-valued case. In order to clarify what are supposed to be concerned on and determine its illustration and significance, we give a brief history. Let us come back to the beginning of the story of measure theory. The primary framework is formed by the following three facts:

(F1) The set of measurable functions $f : \Omega \to \mathbb{C}$ forms a complex involutive algebra where the complex conjugation $f \to \overline{f}$ plays the role of the involution here.

(F2) Simple functions are well-handed ones among measurable functions. Moreover, these functions exist much enough to approximate each measurable function by simple functions (up to pointwise convergence topology).

(F3) Egoroff's and Lusin's theorems. Roughly speaking, the first one says that pointwise convergence is nearly uniformly convergent and the second one says that every measurable function is nearly continuous.

In operator-valued case, functions $\varphi : \Omega \to \mathbf{B}(\mathcal{H})$ replace the complex-valued ones. Variety of well-known topologies on $\mathbf{B}(\mathcal{H})$ may force us to face different types of measurability on the set of operator-valued functions $\varphi : \Omega \to \mathbf{B}(\mathcal{H})$. This point makes the discussion interesting and challenging. Indeed checking facts (F1), (F2) and (F3) may be troublesome. Let us give a little documentary. Except the norm topology on $\mathbf{B}(\mathcal{H})$, a number of well known topologies are given along in the diagram below with relationships among them:

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$$\begin{array}{ccccccc} \text{Arens-Mackey} & \supset & \sigma\text{-strong}^* & \supset & \sigma\text{-strong} & \supset & \sigma\text{-weak} \\ & & \bigcup & & \bigcup & & \\ & & \text{strong}^* & \supset & \text{strong} & \supset & \text{weak} \end{array} \tag{1}$$

For details concerning this diagram, we refer to [9].

The first thing that needs to be illustrated is finding an answer to this question: How many different Borel σ-algebras exist coming from the well known locally convex topologies given in the diagram (1)? Solving this issue is necessary, because it will be cleared up how many different types of measurability are implemented by the topologies mentioned in the diagram (1). A brief survey shows that the answer to this question is not only straightforward but also it will be very mysterious and complicated when $\mathcal{H}$ is non-separable. One may find a comprehensive discussion concerning this question in both separable and non-separable cases in [3] and [4]. We mention that the main result in [3] says that in separable case we may just focus on the weak operator topology, since all Borel σ-algebras generated by these seven topologies coincide. It means that in separable case, we deal with the only one kind of measurability, fortunately.

After clarifying the status of measurability of operator-valued functions, in the next step we need to examine (F1), (F2) and (F3) in operator-valued case. The first one (F1) has been thoroughly investigated in [3] and [4]. The purpose of this discussion is to address the next two facts (F2) and (F3). We check, just like in the classical case, that operator-valued measurable functions may be approximated by simple functions. We will also deal with fundamental classical theorems of *Egoroff and Lusin* and propose a generalization of them in operator-valued case. It should be mentioned that, the result on an extension of classical Egoroffs (and Lusins) theorem to operator-valued setting is not the first one in the literature. One may take a look at [6–8] to find some of them.

Some notations are frequently used in this paper. Let ξ and η be in $\mathcal{H}$. We consider the (rank one) operator $\xi \otimes \eta$ in $\mathbf{B}(\mathcal{H})$ given by

$$\xi \otimes \eta(\gamma) = \langle \gamma, \eta \rangle \xi \quad (\gamma \in \mathcal{H}),$$

where $\langle , \rangle$ is the inner product of $\mathcal{H}$. We denote the set of trace class operators on $\mathcal{H}$ by $L^1(\mathcal{H})$, which is the unique pre-dual space of $\mathbf{B}(\mathcal{H})$.

2. Some Basic Theorems In The Operator-Valued Measure Theory

Considering τ is each one of the locally convex vector topologies on $\mathbf{B}(\mathcal{H})$ mentioned in digram (1), we denote by M_τ the Borel σ-algebra on $\mathbf{B}(\mathcal{H})$ generated by τ. Let M be a σ-algebra of subsets of a non-empty set Ω. Recall that for a given operator-valued function $\varphi : \Omega \to \mathbf{B}(\mathcal{H})$, we say φ is τ-measurable if $\varphi^{-1}(E)$ is a measurable subset of Ω for every measurable set E in M_τ. Since the σ-algebras generated by all topologies mentioned in diagram (1) are coinsided when $\mathcal{H}$ is separable [3, theorem (2.1)], we apply the statement of "measurable" instead of "τ-measurabe" in this case. To make a suitable model of (F2) in operator-valued case, a logical interpretation of pointwise convergence and simple function is needed in this setting:

- Let $\{\varphi_n\}_{n\in\mathbb{N}}$ and φ be operator-valued functions on a measurable space (Ω, M). Trivially the pointwise convergence of φ_n depends on the topology of $\mathbf{B}(\mathcal{H})$. For a given topology τ on $\mathbf{B}(\mathcal{H})$, we say φ_n is pointwise-τ convergent provided that the sequence $\{\varphi_n(t)\}_{n\in\mathbb{N}}$ converges to $\varphi(t)$ with respect to topology τ, for every $t \in \Omega$. According to this, when we say φ_n's converge pointwise-weakly to φ, we mean

$$\langle (\varphi_n(t) - \varphi(t))\zeta, \eta \rangle \longrightarrow 0 \quad (t \in \Omega, \quad \zeta, \eta \in \mathcal{H}),$$

as well as the pointwise-strongly convergence of φ_n's to φ means

$$\|(\varphi_n(t) - \varphi(t))\zeta\| \longrightarrow 0 \quad (t \in \Omega, \quad \zeta \in \mathcal{H}).$$

- An operator-valued function $\psi : \Omega \to \mathbf{B}(\mathcal{H})$ is called finite-valued if there exist operators $a_1, ..., a_n \in \mathbf{B}(\mathcal{H})$ and pairwise disjoint subsets $E_1, ..., E_n \subseteq \Omega$ such that

$$\varphi(\cdot) = \sum_{i=1}^{n} \chi_{E_i}(\cdot) a_i.$$

It is also called simple, if E_i's are all measurable subsets of Ω.

We present an analogy of item (F2) in operator-valued case, as follows:

Theorem 2.1. *Let $\mathcal{H}$ be a separable Hilbert spase. For every operator-valued measurable function φ on (Ω, M), there exists a sequence of operator-valued simple functions $\{\psi_n\}_{n\in\mathbb{N}}$ such that converges pointwise-weakly to φ.*

Proof. We split the proof in two steps.

Step1. We first show that a sequence of vector-valued simple functions converges pointwise-weakly to identity function $I : \mathbf{B}(\mathcal{H}) \longrightarrow \mathbf{B}(\mathcal{H})$. Since $\mathcal{H}$ is separable, relative weak operator topology on bounded set is metrizable by the metric

$$d_w(x, y) := \sum_{i,j=1}^{\infty} \frac{|\langle (x-y)e_i, e_j\rangle|}{2^{i+j}} \quad (x, y \in \mathbf{B}(\mathcal{H})),$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for separable Hilbert space $\mathcal{H}$. Let n be an arbitrary positive integer. For every positive integer k, by the compactness of the closed ball $B_n := \{x \in \mathbf{B}(\mathcal{H}) : \|x\| \leq n\}$ in weak operator topology, there are operators $x_{n1}, \cdots, x_{nm_{(n,k)}}$ of B_n such that $B_n = \bigcup_{j=1}^{m_{(n,k)}} B(x_{nj}, k)$, where

$$B(x_{nj}, k) := \{x \in B_n : d_w(x_{nj}, x) < \frac{1}{k}\} \quad (1 \leq j \leq m_{(n,k)}).$$

By assumption $A^n_{1,k} := B(x_{n1}, k)$, take the following measurable sets:

$$A^n_{j,k} := B(x_{nj}, k) \setminus \bigcup_{i=1}^{j-1} B(x_{ni}, k) \quad (1 \leq j \leq m_{(n,k)}).$$

For every $1 \leq j \leq m_{(n,k)}$, choose an arbitrary element $\tilde{x}^n_{j,k}$ of $A^n_{j,k}$ and then fix it (If $A^n_{j,k}$ is empty take zero as $\tilde{x}^n_{j,k}$). Define the simple operator-valued function on closed balls:

$$S^n_k(\cdot) := \sum_{j=1}^{m_{(n,k)}} \tilde{x}^n_{j,k} \chi_{A^n_{j,k}}(\cdot) \quad (k, n \in \mathbb{N}).$$

Therefore by taking $B_0 := \{0\}$, for every positive integer n, the function

$$S_n(\cdot) := \sum_{j=1}^{n} \chi_{B_j \setminus B_{j-1}}(\cdot) S^j_n(\cdot),$$

is a simple operator-valued function from $\mathbf{B}(\mathcal{H})$ to $\mathbf{B}(\mathcal{H})$. Now let x be an operator in $\mathbf{B}(\mathcal{H})$. Suppose m is the least positive integer such that the closed ball B_m contains x. By noting that

$$S_n(x) = \begin{cases} 0 & n < m \\ S^m_n(x) & n \geq m, \end{cases}$$

the sequence of operators $\{S_n(x)\}_{n\in\mathbb{N}}$ is bouneded and is contained in B_m. It is straightforward to see that $d_w(S_n(x), x) \to 0$. Thus the sequence $\{S_n\}_{n\in\mathbb{N}}$ converges pointwise-weakly to identity function I on $\mathbf{B}(\mathcal{H})$.

Step2. In general case, let $\varphi : (\Omega, M) \longrightarrow (\mathbf{B}(\mathcal{H}), M_\tau)$ be measurable. By taking $\psi_n := S_n \circ \varphi$, for all integer n, it is easy to see that the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ of operator-valued simple functions from Ω to $\mathbf{B}(\mathcal{H})$ has the desired property. □

To show τ-measurability of an operator-valued simple function is straightforward. Hence, as an immediate consequence of Theorem 2.1 and [3, Theorem 1.3] we arrive at the following corollary:

Corollary 2.2. *Let $\mathcal{H}$ be a seperable Hilbert space. Then the following statements are equeivalent:*

1) $\varphi : (\Omega, M) \longrightarrow (\mathbf{B}(\mathcal{H}), M_\tau)$ *is τ-measurable.*
2) φ *is a pointwise-weak limit of some sequence of operator-valued simple functions on* Ω.

To propose a generalization of Egoroff's theorem in operator-valued case, we need to have a review both commutative and non-commutative case of this theorem in the literature.

Theorem 2.3. ***(Egoroff's theorem)*** *Let (Ω, M, μ) be a finite measure space. Assume $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions converging pointwisely to f. For a given $\epsilon > 0$ there exists a measurable subset E of Ω with $\mu(E^c) < \epsilon$ such that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent to f on E.*

Theorem 2.4. ***(Non-commutative Egoroff's theorem** [2]) Let $\mathcal{A}$ be a von Neumann algebra in $\mathbf{B}(\mathcal{H})$. Let $A \subseteq \mathcal{A}$ be a subset of $\mathcal{A}$ such that x lies in the strong closure of A ($x \in \overline{A}^s$). Then, for any positive normal functional $\omega \in \mathcal{A}_*$ and any $\epsilon > 0$, there exist a projection p in $\mathcal{A}$ and a sequence $\{a_n\}$ in A such that $\omega(1-p) < \epsilon$ and $\|(a_n - x)p\| \to 0$.*

Combination of these two items suggests the following generalization of Egoroff's theorem in operator-valued case.

Theorem 2.5. ***(Operator-valued Egoroff's theorem)*** *Assume that $\mathcal{H}$ is a separable Hilbert space and (Ω, M, μ) is a finite measure space. Let $\{\varphi_n\}$ be a sequence of operator-valued measurable functions on Ω converging pointwise-strongly to φ. Then for any arbitrary positive operator $\omega \in L^1(\mathcal{H})$ and positive real numbers ϵ and δ, there exist a measurable set $E \subseteq \Omega$ and a projection $p \in \mathbf{B}(\mathcal{H})$ satisfying $\mu(E) < \epsilon$ and $\omega(1-p) < \delta$, such that*

$$\sup_{t\in E^c} \|(\varphi_n(t) - \varphi(t))p\| \longrightarrow 0.$$

Proof. Let ϵ and δ be arbitrary positive real numbers. We prove the assertion in three steps:

Step 1. Let ω be a positive operator in $L^1(\mathcal{H})$. By [9, Theorem II.1.6], there is a sequence of non-negative numbers $\{\alpha_i\}_{i\in\mathbb{N}} \in \ell^1$ ($\sum_{i=1}^\infty \alpha_i < \infty$) and an orthonormal set $\{e_i\}_{i\in\mathbb{N}}$ of $\mathcal{H}$ such that

$$\omega = \sum_{i=1}^{\infty} \alpha_i e_i \otimes e_i.$$

Therefore, for some $N \in \mathbb{N}$ we have $\sum_{i=N+1}^\infty \alpha_i < \delta$. Without loss of generality, we may assume that $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$. By considering the projection $p = \sum_{i=1}^N e_i \otimes e_i$ we conclude that $\omega(1-p) < \delta$.

Step 2. For any $1 \le i \le N$, we consider the following sequence of positive functions:

$$g_i^n(t) = \|(\varphi_n(t) - \varphi(t))e_i\| = \left(\sum_{j=1}^{\infty} |\langle(\varphi_n(t) - \varphi(t))e_i, e_j\rangle|^2\right)^{\frac{1}{2}} \qquad (t \in \Omega).$$

All functions g_i^n's are measurable (see [3, Theorem 1.2, Theorem 1.3]). Moreover, for every $1 \le i \le N$ the sequence $\{g_i^n\}_{n\in\mathbb{N}}$ is pointwise convergent to zero, since φ_n's converge pointwise-strongly to φ. By the classical Egoroff's theorem, there exists a measurable set $E_i \subseteq \Omega$ with $\mu(E_i) < \frac{\epsilon}{2^i}$ such that $\{g_i^n\}_{n\in\mathbb{N}}$ is converging uniformly to 0 on E_i^c, for $1 \le i \le N$. It means that

$$\sup_{t\in E_i^c} \|(\varphi_n(t) - \varphi(t))e_i\| \longrightarrow 0.$$

By taking $E = \bigcup_{i=1}^N E_i$, we have then $\mu(E) < \epsilon$.

Step 3. For any t in E^c and any ξ in the closed unit ball of $\mathcal{H}$, we have

$$\begin{aligned}\|(\varphi_n(t)-\varphi(t))p\xi\| &= \|\sum_{i=1}^{N}\langle\xi,e_i\rangle(\varphi_n(t)-\varphi(t))e_i\| \\ &\leq \sum_{i=1}^{N}\|(\varphi_n(t)-\varphi(t))e_i\|.\end{aligned}$$

The above statement yields that:

$$\sup_{t\in E^c}\|(\varphi_n(t)-\varphi(t))p\| \longrightarrow 0.$$

□

Similar to the Egoroff's theorem, a glance at the classical Lusin's Theorem [5, Theorem 7.10] and the noncommutative one [9, Theorem II.4.15], the following operator-valued case of Lusin's theorem is proposed. We emphasize that the following theorem does not need the condition of 'separability' of the Hilbert space $\mathcal{H}$ and it holds in general case.

Theorem 2.6. ***(Operator-valued Lusin's Theorem)*** *Let Ω be a locally compact and Hausdorff space and μ be a finite Radon measure on Ω. Let $\varphi : \Omega \to \mathbf{B}(\mathcal{H})$ be a measurable function. Then for every positive operator ω in $L^1(\mathcal{H})$ and two arbitrary positive real numbers ϵ and δ, there exist a measurable set $E \subseteq \Omega$ with $\mu(E) < \epsilon$, a projection $p \in \mathbf{B}(\mathcal{H})$ with $\omega(p) < \delta$ and a function $g \in C_c(\Omega, (\mathbf{B}(\mathcal{H}))$ such that*

$$(1-p)(\varphi(t)-g(t))(1-p) = 0 \quad (t \in E^c).$$

Proof. Let ϵ and δ be arbitrary positive real numbers. Let ω be a positive operator in $L^1(\mathcal{H})$. Then by [9, Theorem II.1.6], there are some sequence of non-negative real numbers $\{\alpha_i\}_{i\in\mathbb{N}} \in \ell^1$ and an orthonormal set $\{e_i\}_{i\in\mathbb{N}} \subseteq \mathcal{H}$ such that

$$\omega = \sum_{i=1}^{\infty}\alpha_i e_i \otimes e_i.$$

Therefore, for some $N \in \mathbb{N}$ we have $\sum_{i=N+1}^{\infty}\alpha_i < \delta$. Let us consider the projection $p := 1 - \sum_{i=1}^{N} e_i \otimes e_i$, we have then $\omega(p) < \delta$. The complex-valued function φ_{ij} given by $t \to \langle\varphi(t)e_i, e_j\rangle$ is measurable. Indeed φ_{ij} is combination of the τ_{wot}-continuous functional on $\mathbf{B}(\mathcal{H})$ given by $x \to \langle xe_i, e_j\rangle$ and φ which implies the measurability of φ_{ij}'s. So, by the classical Lusin's theorem, there exists a measurable set E_{ij} in Ω with $\mu(E_{ij}) < \frac{\epsilon}{2^{i+j}}$ and $g_{ij} \in C_c(\Omega)$ such that

$$(\varphi_{ij} - g_{ij})\chi_{E_{ij}^c} \equiv 0.$$

By taking $E := \bigcup_{i,j=1}^{N} E_{ij}$, we conclude that $\mu(E) < \epsilon$. Define the operator-valued function g as follows:

$$g : \Omega \longrightarrow \mathbf{B}(\mathcal{H}) \quad ; \quad t \longrightarrow \sum_{i,j=1}^{N} g_{ij}(t)e_j \otimes e_i.$$

It is easy to see that g is an operator-valued continuous function whose support is compact. Moreover, for any $t \in E^c$

$$(1-p)[g(t)-\varphi(t)](1-p) = \sum_{i,j=1}^{N}(g_{ij}(t)-\varphi_{ij}(t))e_j \otimes e_i = 0,$$

which completes the proof. □

For both fundamental Egoroff's and Lusin's theorems , the measure space is assumed to be finite. They will be not valid, even in the classical case, if this assumption is removed. To see this, let us consider the characteristic function $f_n = \chi_{[-n,n]}$ on the real line. Obviously $\{f_n\}_{n\in\mathbb{N}}$ is pointwise convergent to the constant function 1, however it can not converge uniformly to 1 on any unbounded subset of the real line. It means that Egoroff theorem is not valid on the domain $\mathbb{R}$. To check why Lusin theorem is not valid in the real line, let us consider the constant function $f \equiv 1$ which is clearly a measurable function. On any infinite measure set E of $\mathbb{R}$, it is impossible to find a continuous function with compact support g satisfying the condition $(g - f)\chi_E = 0$. Based on this observation, finding any approach to the following question makes sense.

Problem 2.7. *On the measurable space whose measure is infinite, does there exist any condition(s) such that both these fundamental Theorems 1.5 and 1.6 still hold?*

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