Filomat 32:12 (2018), 4419–4433
https://doi.org/10.2298/FIL1812419E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New Weighted Inequalities for Higher Order Derivatives and Applications

Samet Erden[a], Mehmet Zeki Sarikaya[b], Huseyin Budak[b]

[a]*Department of Mathematics, Faculty of Science, Bartın University, Bartın-Turkey*
[b]*Department of Mathematics, Faculty of Science and Arts, Düzce University, Konuralp Campus, Düzce-TURKEY*

Abstract. We establish a new Ostrowski type inequality for $(n+1)$-times differentiable mappings which are bounded. Then, some new inequalities of Hermite-Hadamard type are obtained for functions whose $(n+1)\,th$ derivatives in absolute value are convex. Spacial cases of these inequalities reduce some well known inequalities. With the help of obtained inequalities, we give applications for the $kth-$moment of random variables.

1. Introduction

In 1938, Ostrowski established the integral inequality which is one of the fundemental inequalities of mathematics as follows (see, [20]):

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\left\|f'\right\|_\infty=\sup\limits_{t\in(a,b)}\left|f'(t)\right|<\infty$. Then, the inequality holds:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty \tag{1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [7]):

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{2}$$

where $f:I\subset\mathbb{R}\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$.

2010 *Mathematics Subject Classification*. Primary 26D07; Secondary 26D15, 60E15.
Keywords. Hermite-Hadamard inequality; Ostrowski inequality; Midpoint inequality; Convex function; Random variables.
Received: 21 November 2016; Revised: 21 October 2018; Accepted: 15 November 2018
Communicated by Dragan S. Djordjević
Email addresses: erdensmt@gmail.com (Samet Erden), sarikayamz@gmail.com (Mehmet Zeki Sarikaya), hsyn.budak@gmail.com (Huseyin Budak)

Inequalities (1) and (2) have wide applications in numerical analysis and in the theory of some special means; estimating error bounds for some special means, some mid-point, trapezoidal and Simpson rules and other quadrature rules, etc. Hence, inequality (1) and (2) have attracted considerable attention and interest from mathematicans and researchers. Now, we give some inequalities related to (1) and (2) which were proved in recent years (see, [7], [8], [11], [21], [24], [26]).

In [8], Cerone et.al. proved the following inequalities of Ostrowski type and Hadamard type respectively.

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping on (a,b) and $f'':(a,b)\to\mathbb{R}$ is bounded, i.e., $\|f''\|_\infty=\sup\limits_{t\in(a,b)}|f''(t)|<\infty$. Then we have the inequality:*

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\left(x-\frac{a+b}{2}\right)f'(x)\right| &\le \left[\frac{1}{24}(b-a)^2+\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right]\|f''\|_\infty\\ &\le \frac{(b-a)^2}{6}\|f''\|_\infty\end{aligned}$$

for all $x\in[a,b]$.

Corollary 1.2. *Under the assumptions of Theorem 1.1, we have the mid-point inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)^2}{24}\|f''\|_\infty \tag{3}$$

In [11], Kırmacı proved the following results connected with the left part of (2).

Theorem 1.3. *Let $f:I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right|\le\frac{b-a}{8}\left(|f'(a)|+|f'(b)|\right). \tag{4}$$

Theorem 1.4. *Let $f:I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$, and let $p>1$. If the mapping $|f'|^{\frac{p}{p-1}}$ is convex on $[a,b]$, then we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right| \qquad (5)\\ &\le\frac{b-a}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(3|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}+\left(|f'(a)|^{\frac{p}{p-1}}+3|f'(b)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right].\end{aligned}$$

Sarikaya et. al. pointed out some inequalities in [24], as follows:

Theorem 1.5. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, with $f''\in L_1[a,b]$. If $|f''|$ is convex on $[a,b]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)^2}{24}\left(\frac{|f''(a)|+|f''(b)|}{2}\right). \tag{6}$$

Theorem 1.6. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable function on I° such that $f''\in L_1[a,b]$ where $a,b\in I$, $a<b$, If $|f''|^q$ is convex on $[a,b]$, $q>1$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\left[\frac{|f''(a)|^q+|f''(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{7}$$

In a recent paper, in [4], Barnett and Dragomir obtained a variety of bounds for the variance and expected value of a continuous random variable whose p.d.f. is defined over a finite interval base on the identity:

$$\int_a^b (t-m_1)^2 f(t)\,dt + (m_1-b)(m_1-a) = \int_a^b (t-a)(t-b) f(t)\,dt$$

where $m_1 = \int_a^b u f(u)\,du$.

In recent years, researchers have studied some integral inequalities by using n-times differentiable functions. For example, Authors gave some Ostrowski type inequalities for mappings whose nth derivatives are bounded in [6] and [29]. Sofo established integral inequalities on L_p norm in [27]. In [22] and [23], the authors deduced midpoint and trapezoidal formulas for n-times differentiable mappings, respectively. In [1], [2], [9], [18] and [28], researchers obtained some integral inequalities for functions whose obsolute value of nth derivatives are convex, $s-$convex, $m-$convex and $(\alpha, m)-$convex. Kechriniotis and Theodorou proved some integral inequalities via n-times differentiable functions and gave some applications for probability density function in [10]. In [16], [17] and [19], Latif and Dragomir established Hermite-Hadamard type inequalities for n-times differentiable.

In this study, first of all, we derive an identity for $(n+1)$-times differentiable functions. Then, some weighted integral inequalities are obtained by using this identity. Some results presented in earlier works related to these inequalities are given. Finally, some applications for random variable whose probability density functions are bounded and their derivatives in absolute are convex on the interval of real numbers.

2. Some inequalities for the moments

In order to prove weighted integral inequalities, we need the following lemma:

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be $(n+1)-$times differentiable function on I°, $a, b \in I^\circ$ with $a < b$, $f^{(n+1)}$ is absolutely continuous on $[a, b]$ and let $w : [a, b] \to \mathbb{R}$ be nonnegative and continuous on $[a, b]$. Then the following equality holds:*

$$\sum_{k=0}^{n} \frac{M_k(x)}{k!} f^{(k)}(x) - \int_a^b w(t) f(t)\,dt = \int_a^b P_w(x,t) f^{(n+1)}(t)\,dt \tag{8}$$

where $n \in \mathbb{N}$, $M_k(x)$ is defined by

$$M_k(x) = \int_a^b (u-x)^k w(u)\,du, \quad k = 0, 1, 2, \ldots$$

and

$$P_w(x,t) := \begin{cases} \frac{1}{n!} \int_a^t (u-t)^n w(u)\,du, & a \le t < x \\ \\ \frac{1}{n!} \int_b^t (u-t)^n w(u)\,du, & x \le t \le b. \end{cases} \tag{9}$$

Proof. By integration by parts, we have

$$\begin{aligned}
&\int_a^b P_w(x,t) f^{(n+1)}(t)\,dt \\
&= \frac{1}{n!}\int_a^x \left(\int_a^t (u-t)^n w(u)\,du\right) f^{(n+1)}(t)\,dt + \frac{1}{n!}\int_x^b \left(\int_b^t (u-t)^n w(u)\,du\right) f^{(n+1)}(t)\,dt \\
&= \frac{1}{n!}\left(\int_a^b (u-x)^n w(u)\,du\right) f^{(n)}(x) + \frac{1}{(n-1)!}\int_a^x \left(\int_a^t (u-t)^{n-1} w(u)\,du\right) f^{(n)}(t)\,dt \\
&\quad + \frac{1}{(n-1)!}\int_x^b \left(\int_b^t (u-t)^{n-1} w(u)\,du\right) f^{(n)}(t)\,dt.
\end{aligned}$$

By integration by parts $n-$times, we get

$$\begin{aligned}
\int_a^b P_w(x,t) f^{(n+1)}(t)\,dt &= \frac{M_n(x)}{n!} f^{(n)}(x) + \frac{M_{n-1}(x)}{(n-1)!} f^{(n-1)}(x) + \dots + \frac{M_2(x)}{2!} f''(x) \\
&\quad + M_1(x) f'(x) + M_0(x) f(x) - \int_a^b w(t) f(t)\,dt
\end{aligned}$$

which is the required identity in (8). Hence, the proof is completed. □

We establish a new inequality for functions whose $(n+1)-$th derivatives are bounded

Theorem 2.2. *Suppose that all the assumptions of Lemma 2.1 hold. Additionally, we assume that* $f^{(n+1)} : (a,b) \to \mathbb{R}$ *is bounded, i.e.,* $\left\|f^{(n+1)}\right\|_\infty = \sup_{t\in(a,b)} \left|f^{(n+1)}(t)\right| < \infty$, *then we have the inequality*

$$\begin{aligned}
&\left|\sum_{k=0}^{n} \frac{M_k(x)}{k!} f^{(k)}(x) - \int_a^b w(t) f(t)\,dt\right| \qquad (10) \\
&\leq \frac{\left\|f^{(n+1)}\right\|_{[a,b],\infty}}{(n+1)!} \\
&\quad \times \begin{cases} M_{n+1}(x) & \text{,if } n \text{ is an odd number} \\ \left[\int_x^b (u-x)^{n+1} w(u)du - \int_a^x (u-x)^{n+1} w(u)du\right] & \text{, if } n \text{ is an even number} \end{cases}
\end{aligned}$$

for all $x \in [a,b]$.

Proof. If we take absolute value of both sides of the equality (8), because $f^{(n+1)}$ is a bounded function, we can write

$$\begin{aligned}
&\left|\sum_{k=0}^{n}\frac{M_k(x)}{k!}f^{(k)}(x)-\int_a^b w(t)f(t)\,dt\right| \\
&\leq \frac{1}{n!}\int_a^x\left(\int_a^t (t-u)^n w(u)du\right)\left|f^{(n+1)}(t)\right|dt+\frac{1}{n!}\int_x^b\left(\int_t^b (u-t)^n w(u)du\right)\left|f^{(n+1)}(t)\right|dt \\
&\leq \frac{\left\|f^{(n+1)}\right\|_{[a,x],\infty}}{n!}\int_a^x\left(\int_a^t (t-u)^n w(u)du\right)dt+\frac{\left\|f^{(n+1)}\right\|_{[x,b],\infty}}{n!}\int_x^b\left(\int_t^b (u-t)^n w(u)du\right)dt.
\end{aligned}$$

By using the change of order of integration and the fact that n is an odd number, we get

$$\begin{aligned}
&\left|\sum_{k=0}^{n}\frac{M_k(x)}{k!}f^{(k)}(x)-\int_a^b w(t)f(t)\,dt\right| \\
&\leq \frac{\left\|f^{(n+1)}\right\|_{[a,x],\infty}}{(n+1)!}\int_a^x (x-u)^{n+1} w(u)du+\frac{\left\|f^{(n+1)}\right\|_{[x,b],\infty}}{(n+1)!}\int_x^b (u-x)^{n+1} w(u)du \\
&\leq \frac{\left\|f^{(n+1)}\right\|_{[a,b],\infty}}{(n+1)!}M_{n+1}(x).
\end{aligned}$$

Hence, the proof is completed. □

Remark 2.3. *If we choose $n=1$ in Theorem 2.2, then we obtain*

$$\left|M_1(x)f'(x)+M_0(x)f(x)-\int_a^b w(t)f(t)\,dt\right|\leq\frac{\left\|f''\right\|_{[a,b],\infty}}{2}M_2(x).$$

which was given by Sarikaya and Yaldiz in [25].

Remark 2.4. *If we choose $w(u)=1$ in Theorem 2.2, then we have the inequality*

$$\begin{aligned}
&\left|\sum_{k=0}^{n}\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)-\int_a^b f(t)\,dt\right| \\
&\leq \frac{\left\|f^{(n+1)}\right\|_{[a,b],\infty}}{(n+2)!}\left[(b-x)^{n+2}+(x-a)^{n+2}\right].
\end{aligned}$$

for all $n\geq 0$. This inequality was proved by Cerone et al. in [6].

Remark 2.5. *If we take $w(u)=1$ and $n=0$ in Theorem 2.2, then we get the clasical Ostrowski inequality.*

Remark 2.6. *If we take $w(u)=1$ and $n=1$ in Theorem 2.2, then the Theorem 2.2 reduces to the Theorem 1.1 which is proved by Cerone et.al. in [8].*

Remark 2.7. *If we choose $w(u)=1$ and $x=\frac{a+b}{2}$ in Theorem 2.2, then we have the inequality*

$$\begin{aligned}
&\left|\sum_{k=0}^{n}\frac{(b-a)^{k+1}\left[1+(-1)^k\right]}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt\right| \\
&\leq \frac{\left\|f^{(n+1)}\right\|_{\infty}(b-a)^{n+2}}{2^{n+1}(n+2)!}.
\end{aligned}$$

for all $n \geq 0$. This inequality was proved by Cerone et al. in [6].

Remark 2.8. *If we take $w(u) = 1$, $x = \frac{a+b}{2}$ and $n = 1$ in Theorem 2.2, then the inequality (10) becomes the inequality (3) which was given by Cerone et.al. in [8].*

Now, we give an inequality for mappings whose absolute value of $(n+1)-$th derivatives are convex.

Theorem 2.9. *Suppose that all the assumptions of Lemma 2.1 hold. If $\left|f^{(n+1)}\right|$ is convex on $[a,b]$, then, for all $x \in [a,b]$, the following inequality holds:*

$$\left|\sum_{k=0}^{n} \frac{M_k(x)}{k!} f^{(k)}(x) - \int_a^b w(t) f(t)\, dt\right| \tag{11}$$

$$\leq \frac{\|w\|_{[a,b],\infty}}{(n+1)!\,(b-a)} \left[\left((b-a)\frac{(x-a)^{n+2}}{n+2} + \frac{(b-x)^{n+3}-(x-a)^{n+3}}{n+3}\right)\left|f^{(n+1)}(a)\right| + \left(\frac{(x-a)^{n+3}-(b-x)^{n+3}}{n+3} + (b-a)\frac{(b-x)^{n+2}}{n+2}\right)\left|f^{(n+1)}(b)\right|\right]$$

where $\|w\|_\infty = \sup_{t\in[a,b]} |w(t)|$.

Proof. By taking absolute value of (8) and using the boundedness of mapping w, we find that

$$\left|\sum_{k=0}^{n} \frac{M_k(x)}{k!} f^{(k)}(x) - \int_a^b w(t) f(t)\, dt\right|$$

$$\leq \frac{\|w\|_{[a,x],\infty}}{(n+1)!} \int_a^x (t-a)^{n+1} \left|f^{(n+1)}(t)\right| dt + \frac{\|w\|_{[x,b],\infty}}{(n+1)!} \int_x^b (b-t)^{n+1} \left|f^{(n+1)}(t)\right| dt.$$

Since $\left|f^{(n+1)}(t)\right|$ is convex on $[a,b] = [a,x] \cup [x,b]$

$$\left|f^{(n+1)}\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b\right)\right| \leq \frac{b-t}{b-a}\left|f^{(n+1)}(a)\right| + \frac{t-a}{b-a}\left|f^{(n+1)}(b)\right|. \tag{12}$$

Utilizing the inequality (12), we write

$$\left|\sum_{k=0}^{n} \frac{M_k(x)}{k!} f^{(k)}(x) - \int_a^b w(t) f(t)\, dt\right| \tag{13}$$

$$\leq \frac{\|w\|_{[a,x],\infty}}{(n+1)!\,(b-a)} \left(\left|f^{(n+1)}(a)\right| \int_a^x (t-a)^{n+1}(b-t)\, dt + \left|f^{(n+1)}(b)\right| \int_a^x (t-a)^{n+2}\, dt\right)$$

$$+ \frac{\|w\|_{[x,b],\infty}}{(n+1)!\,(b-a)} \left(\left|f^{(n+1)}(a)\right| \int_x^b (b-t)^{n+2}\, dt + \left|f^{(n+1)}(b)\right| \int_x^b (b-t)^{n+1}(t-a)\, dt\right).$$

If we calculate the above four inetgrals and also substitute the results in (13), because of $\|w\|_{[a,x],\infty}, \|w\|_{[x,b],\infty} \leq \|w\|_{[a,b],\infty}$, we obtain desired inequality (11) which completes the proof. $\square$

Remark 2.10. *Under the same assumptions of Theorem 2.9 with $n = 0$, then the following inequality holds:*

$$\begin{aligned}\left| f(x)\int_a^b w(t)\,dt - \int_a^b w(t)\,f(t)\,dt \right| &\le \frac{\|w\|_{[a,b],\infty}}{(b-a)}\left[\left((b-a)\frac{(x-a)^2}{2}+\frac{(b-x)^3-(x-a)^3}{3}\right)\left|f'(a)\right|\right.\\ &\left.+\left(\frac{(x-a)^3-(b-x)^3}{3}+(b-a)\frac{(b-x)^2}{2}\right)\left|f'(b)\right|\right]\end{aligned}$$

*which is "**weighted Ostrowski**" inequality provided that $|f'|$ is convex on $[a,b]$. This inequality was given by Sarikaya and Erden in [26].*

Remark 2.11. *Under the same assumptions of Theorem 2.9 with $n = 0$ and $x = \frac{a+b}{2}$, then the following inequality hols:*

$$\left| f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt - \int_a^b w(t)\,f(t)\,dt \right| \le \frac{\|w\|_{[a,b],\infty}(b-a)^2}{4}\left[\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right]$$

*which is "**weighted mid-point**" inequality provided that $|f'|$ is convex on $[a,b]$. This inequality was given by Sarikaya and Erden in [26].*

Remark 2.12. *If we choose $n = 1$ in Theorem 2.9, then we obtain*

$$\begin{aligned}\left| M_1(x)\,f'(x) + M_0(x)f(x) - \int_a^b w(t)\,f(t)\,dt \right| &\le \frac{\|w\|_{[a,b],\infty}}{2(b-a)}\left[\left((b-a)\frac{(x-a)^3}{3}+\frac{(b-x)^4-(x-a)^4}{4}\right)\left|f''(a)\right|\right.\\ &\left.+\left(\frac{(x-a)^4-(b-x)^4}{4}+(b-a)\frac{(b-x)^3}{3}\right)\left|f''(b)\right|\right]\end{aligned}$$

which was given by Sarikaya and Yaldiz in [25].

Corollary 2.13. *Under the same assumptions of Theorem 2.9 with $w(u) = 1$, then we have the inequality*

$$\begin{aligned}&\left|\sum_{k=0}^{n}\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)-\int_a^b f(t)\,dt\right|\\ &\le\left[\left(\frac{(x-a)^{n+2}}{(n+2)!}+\frac{(b-x)^{n+3}-(x-a)^{n+3}}{(b-a)(n+1)!(n+3)}\right)\left|f^{(n+1)}(a)\right|+\left(\frac{(x-a)^{n+3}-(b-x)^{n+3}}{(b-a)(n+1)!(n+3)}+\frac{(b-x)^{n+2}}{(n+2)!}\right)\left|f^{(n+1)}(b)\right|\right].\end{aligned}$$

Corollary 2.14. *If we take $w(u) = 1$ and $n = 0$ in Theorem 2.9, then we have*

$$\left|(b-a)f(x)-\int_a^b f(t)\,dt\right| \le \left[\left(\frac{(x-a)^2}{2}+\frac{(b-x)^3-(x-a)^3}{3(b-a)}\right)\left|f'(a)\right|+\left(\frac{(x-a)^3-(b-x)^3}{3(b-a)}+\frac{(b-x)^2}{2}\right)\left|f'(b)\right|\right].$$

Remark 2.15. *If we take $w(u) = 1$ and $n = 1$ in Theorem 2.9, then we get*

$$\begin{aligned}&\left|(b-a)f(x)+(b-a)\left(\frac{a+b}{2}-x\right)f'(x)-\int_a^b f(t)\,dt\right|\\ &\le\left[\left(\frac{(x-a)^3}{6}+\frac{(b-x)^4-(x-a)^4}{8(b-a)}\right)\left|f''(a)\right|+\left(\frac{(x-a)^4-(b-x)^4}{8(b-a)}+\frac{(b-x)^3}{6}\right)\left|f''(b)\right|\right]\end{aligned}$$

which was given by Sarikaya and Yaldiz in [25].

Remark 2.16. *If we choose $w(u) = 1$ and $x = \frac{a+b}{2}$ in Theorem 2.9, then we have the inequality*

$$\left|\sum_{k=0}^{n}\frac{(b-a)^{k+1}\left[1+(-1)^{k}\right]}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\int_{a}^{b}f(t)\,dt\right| \leq \frac{(b-a)^{n+2}}{2^{n+1}(n+2)!}\left[\frac{\left|f^{(n+1)}(a)\right|+\left|f^{(n+1)}(b)\right|}{2}\right]$$

which was derived by Ozdemir and Yildiz in [22].

Remark 2.17. *If we take $w(u) = 1$, $x = \frac{a+b}{2}$ and $n = 0$ in Theorem 2.9, then the inequality (11) reduce to the inequality (4).*

Remark 2.18. *If we take $w(u) = 1$, $x = \frac{a+b}{2}$ and $n = 1$ in Theorem 2.9, then the inequality (11) becomes the inequality (6).*

We prove some inequalities by using convexity of $\left|f^{(n+1)}\right|^{q}$.

Theorem 2.19. *Suppose that all the assumptions of Lemma 2.1 hold. If $\left|f^{(n+1)}\right|^{q}$ is convex on $[a,b]$, $q>1$, then, for all $x \in [a,b]$, we have the inequality*

$$\left|\sum_{k=0}^{n}\frac{M_k(x)}{k!}f^{(k)}(x)-\int_{a}^{b}w(t)f(t)\,dt\right| \tag{14}$$

$$\leq \frac{\|w\|_{[a,b],\infty}}{(n+1)!}(b-a)^{\frac{1}{q}}\left[\frac{(b-x)^{(n+1)p+1}+(x-a)^{(n+1)p+1}}{(n+1)p+1}\right]^{\frac{1}{p}}\left[\frac{\left|f^{(n+1)}(a)\right|^{q}+\left|f^{(n+1)}(b)\right|^{q}}{2}\right]^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$, and $\|w\|_{\infty}=\sup_{t\in[a,b]}|w(t)|$.

Proof. By similar methods in the proof of Theorem 2.9 and from Hölder's inequality, we find that

$$\left|\sum_{k=0}^{n}\frac{M_k(x)}{k!}f^{(k)}(x)-\int_{a}^{b}w(t)f(t)\,dt\right| \leq \left(\int_{a}^{b}|P_w(x,t)|^{p}\,dt\right)^{\frac{1}{p}}\left(\int_{a}^{b}\left|f^{(n+1)}(t)\right|^{q}dt\right)^{\frac{1}{q}}. \tag{15}$$

By simple calculations, we obtain

$$\begin{aligned}\left(\int_{a}^{b}|P_w(x,t)|^{p}\,dt\right)^{\frac{1}{p}} &= \frac{1}{n!}\left[\int_{a}^{x}\left(\int_{a}^{t}(t-u)^{n}w(u)du\right)^{p}dt+\int_{x}^{b}\left(\int_{t}^{b}(u-t)^{n}w(u)du\right)^{p}dt\right]^{\frac{1}{p}} \\ &\leq \frac{\|w\|_{[a,b],\infty}}{(n+1)!}\left[\int_{a}^{x}(t-a)^{(n+1)p}\,dt+\int_{x}^{b}(b-t)^{(n+1)p}\,dt\right]^{\frac{1}{p}} \\ &= \frac{\|w\|_{[a,b],\infty}}{(n+1)!}\left[\frac{(b-x)^{(n+1)p+1}+(x-a)^{(n+1)p+1}}{(n+1)p+1}\right]^{\frac{1}{p}}.\end{aligned} \tag{16}$$

Since $\left|f^{(n+1)}(t)\right|^{q}$ is convex on $[a,b]=[a,x]\cup[x,b]$, we have

$$\left|f^{(n+1)}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|^{q} \leq \frac{b-t}{b-a}\left|f^{(n+1)}(a)\right|^{q}+\frac{t-a}{b-a}\left|f^{(n+1)}(b)\right|^{q}. \tag{17}$$

Using the inequality (17), it follows that

$$\left(\int_a^b \left|f^{(n+1)}(t)\right|^q dt\right)^{\frac{1}{q}} \leq (b-a)^{\frac{1}{q}} \left[\frac{\left|f^{(n+1)}(a)\right|^q + \left|f^{(n+1)}(b)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{18}$$

Hence, the proof of theorem is completed. □

Corollary 2.20. *Under the same assumptions of Theorem 2.19 with $n = 0$, then the following inequality hols:*

$$\left|f(x)\int_a^b w(t)\,dt - \int_a^b w(t) f(t)\,dt\right| \leq \|w\|_{[a,b],\infty} (b-a)^{\frac{1}{q}} \left[\frac{(b-x)^{p+1}+(x-a)^{p+1}}{p+1}\right]^{\frac{1}{p}} \left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}$$

*which is "**weighted Ostrowski**" inequality provided that $\left|f'\right|^q$ is convex on $[a,b]$.*

Corollary 2.21. *Under the same assumptions of Theorem 2.19 with $n = 0$ and $x = \frac{a+b}{2}$, then the following inequality hols:*

$$\left|f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt - \int_a^b w(t) f(t)\,dt\right| \leq \frac{\|w\|_{[a,b],\infty} (b-a)^2}{2\cdot(p+1)^{\frac{1}{p}}} \left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}$$

*which is "**weighted mid-point**" inequality provided that $\left|f'\right|$ is convex on $[a,b]$.*

Remark 2.22. *If we choose $n = 1$ in Theorem 2.19, then we obtain*

$$\begin{aligned} &\left|M_1(x) f'(x) + M_0(x) f(x) - \int_a^b w(t) f(t)\,dt\right| \\ &\leq \frac{\|w\|_{[a,b],\infty}}{2} (b-a)^{\frac{1}{q}} \left[\frac{(b-x)^{2p+1}+(x-a)^{2p+1}}{2p+1}\right]^{\frac{1}{p}} \left[\frac{\left|f''(a)\right|^q + \left|f''(b)\right|^q}{2}\right]^{\frac{1}{q}} \end{aligned}$$

which was given by Sarikaya and Yaldiz in [25].

Corollary 2.23. *Under the same assumptions of Theorem 2.19 with $w(u) = 1$, then we have the inequality*

$$\begin{aligned} &\left|\sum_{k=0}^{n} \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} f^{(k)}(x) - \int_a^b f(t)\,dt\right| \\ &\leq \frac{(b-a)^{\frac{1}{q}}}{(n+1)!} \left[\frac{(b-x)^{(n+1)p+1}+(x-a)^{(n+1)p+1}}{(n+1)p+1}\right]^{\frac{1}{p}} \left[\frac{\left|f^{(n+1)}(a)\right|^q + \left|f^{(n+1)}(b)\right|^q}{2}\right]^{\frac{1}{q}}. \end{aligned}$$

Remark 2.24. *If we take $w(u) = 1$ and $n = 0$ in Theorem 2.19, then we get*

$$\left|(b-a) f(x) - \int_a^b f(t)\,dt\right| \leq (b-a)^{\frac{1}{q}} \left[\frac{(b-x)^{p+1}+(x-a)^{p+1}}{p+1}\right]^{\frac{1}{p}} \left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}.$$

Remark 2.25. *If we take $w(u) = 1$ and $n = 1$ in Theorem 2.19, then we have*

$$\left|(b-a)f(x)+(b-a)\left(\frac{a+b}{2}-x\right)f'(x)-\int_a^b f(t)\,dt\right|$$
$$\leq \frac{(b-a)^{\frac{1}{q}}}{2}\left[\frac{(b-x)^{2p+1}+(x-a)^{2p+1}}{2p+1}\right]^{\frac{1}{p}}\left[\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{2}\right]^{\frac{1}{q}}$$

which was given by Sarikaya and Yaldiz in [25].

Corollary 2.26. *If we choose $w(u) = 1$ and $x = \frac{a+b}{2}$ in Theorem 2.19, then we have the inequality*

$$\left|\sum_{k=0}^{n}\frac{(b-a)^{k+1}\left[1+(-1)^k\right]}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt\right|$$
$$\leq \frac{(b-a)^{n+2}}{2^{n+1}(n+1)!\left[(n+1)p+1\right]^{\frac{1}{p}}}\left[\frac{\left|f^{(n+1)}(a)\right|^q+\left|f^{(n+1)}(b)\right|^q}{2}\right]^{\frac{1}{q}}.$$

Remark 2.27. *If we take $w(u) = 1$, $x = \frac{a+b}{2}$ and $n = 0$ in Theorem 2.19, then we have*

$$\left|(b-a)f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^2}{2(p+1)^{\frac{1}{p}}}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}.$$

Remark 2.28. *If we take $w(u) = 1$, $x = \frac{a+b}{2}$ and $n = 1$ in Theorem 2.19, then the inequality (14) becomes the inequality (7).*

Theorem 2.29. *Suppose that all the assumptions of Lemma 2.1 hold. If $\left|f^{(n+1)}\right|^q$ is convex on $[a,b]$, $q > 1$, then for all $x \in [a,b]$, we have the inequality*

$$\left|\sum_{k=0}^{n}\frac{M_k(x)}{k!}f^{(k)}(x)-\int_a^b w(t)f(t)\,dt\right| \tag{19}$$
$$\leq \frac{\|w\|_{[a,b],\infty}}{(b-a)^{\frac{1}{q}}(n+1)!\left[(n+1)p+1\right]^{\frac{1}{p}}}$$
$$\times\left\{(x-a)^{n+1+\frac{1}{p}}\left[\frac{(b-a)^2-(b-x)^2}{2}\left|f^{(n+1)}(a)\right|^q+\frac{(x-a)^2}{2}\left|f^{(n+1)}(b)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+(b-x)^{n+1+\frac{1}{p}}\left[\frac{(b-x)^2}{2}\left|f^{(n+1)}(a)\right|^q+\frac{(b-a)^2-(x-a)^2}{2}\left|f^{(n+1)}(b)\right|^q\right]^{\frac{1}{q}}\right\}$$

where $\frac{1}{p}+\frac{1}{q} = 1$, and $\|w\|_\infty = \sup_{t\in[a,b]}|w(t)|$.

Proof. Using similar methods in the proof of Theorem 2.19 and from Hölder's inequality, we obtain the inequality (19). Hence, the proof is completed. □

Remark 2.30. *Under the same assumptions of Theorem 2.29 with $n = 0$, then the following inequality holds:*

$$\begin{aligned}&\left| f(x)\int_a^b w(t)\,dt - \int_a^b w(t) f(t)\,dt \right| \\ &\leq \frac{\|w\|_{[a,b],\infty}}{(b-a)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left\{ (x-a)^{1+\frac{1}{p}}\left[\frac{(b-a)^2-(b-x)^2}{2}\left|f'(a)\right|^q + \frac{(x-a)^2}{2}\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right. \\ &\left. + (b-x)^{1+\frac{1}{p}}\left[\frac{(b-x)^2}{2}\left|f'(a)\right|^q + \frac{(b-a)^2-(x-a)^2}{2}\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}\end{aligned}$$

*which is "**weighted Ostrowski**" inequality provided that $\left|f'\right|^q$ is convex on $[a,b]$. This inequality was given by Sarikaya and Erden in [26].*

Remark 2.31. *Under the same assumptions of Theorem 2.29 with $n = 0$ and $x = \frac{a+b}{2}$, then the following inequality hols:*

$$\begin{aligned}&\left| f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt - \int_a^b w(t) f(t)\,dt \right| \\ &\leq \frac{(b-a)^2}{4[p+1]^{\frac{1}{p}}}\left\{\left[\frac{3\left|f'(a)\right|^q + \left|f'(b)\right|^q}{4}\right]^{\frac{1}{q}} + \left[\frac{\left|f'(a)\right|^q + 3\left|f'(b)\right|^q}{4}\right]^{\frac{1}{q}}\right\}\|w\|_{[a,b],\infty}\end{aligned}$$

*which is "**weighted mid-point**" inequality provided that $\left|f'\right|$ is convex on $[a,b]$. This inequality was given by Sarikaya and Erden in [26].*

Remark 2.32. *If we choose $n = 1$ in Theorem 2.29, then we obtain*

$$\begin{aligned}&\left| M_1(x) f'(x) + M_0(x) f(x) - \int_a^b w(t) f(t)\,dt \right| \\ &\leq \frac{\|w\|_{[a,b],\infty}}{2(b-a)^{\frac{1}{q}}[2p+1]^{\frac{1}{p}}}\left\{ (x-a)^{2+\frac{1}{p}}\left[\frac{(b-a)^2-(b-x)^2}{2}\left|f''(a)\right|^q + \frac{(x-a)^2}{2}\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right. \\ &\left. + (b-x)^{2+\frac{1}{p}}\left[\frac{(b-x)^2}{2}\left|f''(a)\right|^q + \frac{(b-a)^2-(x-a)^2}{2}\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\}\end{aligned}$$

which was established by Sarikaya and Yaldiz in [25].

Corollary 2.33. *Under the same assumptions of Theorem 2.29 with $w(u) = 1$, then we have the inequality*

$$\begin{aligned}&\left| \sum_{k=0}^{n} \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} f^{(k)}(x) - \int_a^b f(t)\,dt \right| \\ &\leq \frac{1}{(b-a)^{\frac{1}{q}}(n+1)!\,[(n+1)p+1]^{\frac{1}{p}}} \\ &\times\left\{ (x-a)^{n+1+\frac{1}{p}}\left[\frac{(b-a)^2-(b-x)^2}{2}\left|f^{(n+1)}(a)\right|^q + \frac{(x-a)^2}{2}\left|f^{(n+1)}(b)\right|^q\right]^{\frac{1}{q}}\right. \\ &\left. + (b-x)^{n+1+\frac{1}{p}}\left[\frac{(b-x)^2}{2}\left|f^{(n+1)}(a)\right|^q + \frac{(b-a)^2-(x-a)^2}{2}\left|f^{(n+1)}(b)\right|^q\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Remark 2.34. *If we take $w(u) = 1$ and $n = 1$ in Theorem 2.29, then we get*

$$\left|(b-a)\,f(x)+(b-a)\left(\frac{a+b}{2}-x\right)f'(x)-\int_a^b f(t)\,dt\right|$$
$$\leq \frac{1}{2\,(b-a)^{\frac{1}{q}}\,[2p+1]^{\frac{1}{p}}}\left\{(x-a)^{2+\frac{1}{p}}\left[\frac{(b-a)^2-(b-x)^2}{2}\left|f''(a)\right|^q+\frac{(x-a)^2}{2}\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+(b-x)^{2+\frac{1}{p}}\left[\frac{(b-x)^2}{2}\left|f''(a)\right|^q+\frac{(b-a)^2-(x-a)^2}{2}\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\}$$

which was given by Sarikaya and Yaldiz in [25].

Remark 2.35. *If we choose $w(u) = 1$ and $x = \frac{a+b}{2}$ in Theorem 2.29, then we have the inequality*

$$\left|\sum_{k=0}^{n}\frac{(b-a)^{k+1}\left[1+(-1)^k\right]}{2^{k+1}\,(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt\right|$$
$$\leq \frac{(b-a)^{n+2}}{2^{n+2}\,(n+1)!\,[(n+1)\,p+1]^{\frac{1}{p}}}\left\{\left[\frac{3\left|f^{(n+1)}(a)\right|^q+\left|f^{(n+1)}(b)\right|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{\left|f^{(n+1)}(a)\right|^q+3\left|f^{(n+1)}(b)\right|^q}{4}\right]^{\frac{1}{q}}\right\}$$

which was proved by Ozdemir and Yildiz in [22].

Remark 2.36. *If we take $w(u) = 1$, $x = \frac{a+b}{2}$ and $n = 0$ in Theorem 2.29, then the inequality (19) reduce to the inequality (5).*

Remark 2.37. *If we take $w(u) = 1$, $x = \frac{a+b}{2}$ and $n = 1$ in Theorem 2.29, then we have the inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^2}{16\,[2p+1]^{\frac{1}{p}}}\left\{\left[\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{4}\right]^{\frac{1}{q}}\right\}$$

which was given by Sarikaya and Yaldiz in [25].

Theorem 2.38. *Suppose that all the assumptions of Lemma 2.1 hold. If $\left|f^{(n+1)}\right|^q$ is convex on $[a,b]$, $q \geq 1$, then, for all $x \in [a,b]$, the following inequality holds:*

$$\left|\sum_{k=0}^{n}\frac{M_k(x)}{k!}f^{(k)}(x)-\int_a^b w(t)\,f(t)\,dt\right| \tag{20}$$
$$\leq \frac{\|w\|_{[a,b],\infty}}{(n+1)!\,(b-a)^{\frac{1}{q}}}\left(\frac{(b-x)^{n+2}+(x-a)^{n+2}}{(n+2)}\right)^{\frac{1}{p}}$$
$$\times\left[\left((b-a)\,\frac{(x-a)^{n+2}}{n+2}+\frac{(b-x)^{n+3}-(x-a)^{n+3}}{n+3}\right)\left|f^{(n+1)}(a)\right|^q\right.$$
$$\left.+\left(\frac{(x-a)^{n+3}-(b-x)^{n+3}}{n+3}+(b-a)\,\frac{(b-x)^{n+2}}{n+2}\right)\left|f^{(n+1)}(b)\right|^q\right]^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\|w\|_\infty = \sup_{t\in[a,b]}|w(t)|$.

Proof. From (10), using the properties of modulus and from Hölder's inequality, we get

$$\left|\sum_{k=0}^{n}\frac{M_k(x)}{k!}f^{(k)}(x)-\int_a^b w(t)f(t)\,dt\right| \tag{21}$$

$$\leq\left(\int_a^b|P_w(x,t)|\,dt\right)^{\frac{1}{p}}\left(\int_a^b|P_w(x,t)|\left|f^{(n+1)}(t)\right|^q dt\right)^{\frac{1}{q}}.$$

By simple calculations, we obtain

$$\int_a^b|P_w(x,t)|\,dt\leq\|w\|_{[a,b],\infty}\frac{(b-x)^{n+2}+(x-a)^{n+2}}{(n+2)!}. \tag{22}$$

Because of convexity of $\left|f^{(n+1)}\right|^q$ and bounded of w, appliying similar methods in the proof of Theorem 2.9 and using the inequality (17), we find that

$$\int_a^b|P_w(x,t)|\left|f^{(n+1)}(t)\right|^q dt \tag{23}$$

$$\leq\frac{\|w\|_{[a,b],\infty}}{(+1)!\,(b-a)}\left[\left((b-a)\frac{(x-a)^{n+2}}{n+2}+\frac{(b-x)^{n+3}-(x-a)^{n+3}}{n+3}\right)\left|f^{(n+1)}(a)\right|^q\right.$$
$$\left.\left(\frac{(x-a)^{n+3}-(b-x)^{n+3}}{n+3}+(b-a)\frac{(b-x)^{n+2}}{n+2}\right)\left|f^{(n+1)}(b)\right|^q\right].$$

Substituting the inequalities (22) and (23) in (21), we easily deduce required inequality (20) which completes the proof. □

Remark 2.39. *In case $(p,q)=(\infty,1)$, if we take limit as $p\to\infty$ in Theorem 2.38, then the inequality (20) becomes the inequality (11). Thus, we obtain all of the results which are similar to Theorem 2.9.*

Remark 2.40. *If we take $w(u)=1$, $x=\frac{a+b}{2}$ and $n=0$ in Theorem 2.38, then we obtain*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{b-a}{4}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}$$

which was established by Pearce and Pečarić in [21].

3. Some applications for the moments

Distribution functions and density functions provide complete descriptions of the distribution of probality for a given random variable. However, they do not allow us to easily make comparisons between two different distributions. The set of moments that uniquely characterizes the distribution under reasonable conditions are useful in making comparisons. Knowing the probability function, we can determine moments if they exist. Applying the mathematical inequalities, some estimations for the moments of random variables were recently studied (see, [3]-[5], [12]-[15], [25]).

Set X to denote a random variable whose probability function is $w : [a,b] \to \mathbb{R}$ is a integrable and nonnegative function on the interval of real numbers I and let $a, b \in I$, $(a < b)$. Denote by $M_r(x)$ the *rth* moment about any arbitrary point x of the random variable $X, r \geq 0$, defined as

$$M_r(x) = \int_a^b (u-x)^r w(u)\, du, \quad r = 0, 1, 2, ...$$

Now, we reconsider the identity (8) by changing conditions given in Lemma 2.1. Herewith, we deduce an identity involving r^{th} moment.

Lemma 3.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be $(n+1)-$times differentiable function on I°, $a, b \in I^\circ$ with $a < b$, $f^{(n+1)}$ is absolutely continuous on $[a,b]$ and let X be random variable whose probability function is $w : [a,b] \to [0,\infty)$ is integrable on $[a,b]$. Then the following equality holds:*

$$\sum_{k=0}^{n} \frac{M_k(x)}{k!} f^{(k)}(x) - \int_a^b w(t) f(t)\, dt = \int_a^b P_w(x,t) f^{(n+1)}(t)\, dt$$

where $n \in \mathbb{N}$, $M_k(x)$ is the kth moment and $P_w(x,t)$ is defined as (9).

Similarly, using boundedness of $f^{(n+1)}$, convexity of $\left|f^{(n+1)}\right|$ or convexity of $\left|f^{(n+1)}\right|^q$ in addition to conditions of Lemma 3.1, we obtain same of the inequalities given in previous section for random variable.

References

[1] M.A. Ardic, Inequalities via $n-$times differentiable convex functions, arXiv:1310.0947v1, 2013.

[2] S.-P. Bai, S.-H. Wang and F. Qi, Some Hermite-Hadamard type inequalities for n-time differentiable (α, m)-convex functions, Journal of Inequalities and Applications 2012, 2012:267.

[3] N.S. Barnett, P. Cerone, S.S. Dragomir and J. Roumeliotis, Some inequalities for the dispersion of a random variable whose pdf is defined on a finite interval, J. Ineq. Pure Appl. Math, 2 (1) (2001).

[4] N.S. Barnett and S.S. Dragomir, Some elemantary inequalities for the expectation and aariance of a random variable whose pdf is defined on a finite interval, RGMIA Res. Rep. Coll., 2(7), Article 12.

[5] P. Cerone and S.S. Dragomir, On some inequalities for the expectation and variance, Korean J. Comp. & Appl. Math., 8(2) (2000), 357–380.

[6] P. Cerone, S.S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, Demonstratio Math., 32 (1999), No. 4, 697-712.

[7] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.

[8] P. Cerone, S.S. Dragomir and J. Roumeliotis, An inequality of Ostrowski type for mappings whose second derivatives are bounded and applications, RGMIA Res. Rep. Coll., 1(1) (1998), Article 4.

[9] W.-D. Jiang, D.-W. Niu, Y. Hua, and F. Qi, Generalizations of Hermite-Hadamard inequality to n-time differentiable functions which are s-convex in the second sense, Analysis (Munich), vol. 32, pp. 1001–1012, 2012.

[10] A. I. Kechriniotis and Y. A. Theodorou, New integral inequalities for n-time differentiable functions with applications for pdfs, Applied Mathematical Sciences, Vol. 2, 2008, no. 8, 353-362.

[11] U.S. Kirmaci, "Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula," Appl. Math. Comp., vol. 147, pp. 137–146, 2004.

[12] P. Kumar, Moments inequalities of a random variable defined over a finite interval, J. Inequal. Pure and Appl. Math. vol.3, ıss.3, article 41, 2002.

[13] P. Kumar, Inequalities involving moments of a continuous random variable defined over a finite interval, Computers and Mathematics with Applications 48 (2004) 257-273.

[14] P. Kumar, Hermite-Hadamard inequalities and their applications in estimating moments, In Inequality Theory and Applications, Volume 2, (Edited by Y.J. Cho, J.K. Kim and S.S. Dragomir), Nova Science, (2003).

[15] P. Kumar and S.S. Dragomir, Some inequalities for the gamma functions and moment ratios of gamma variables, Indian Jour. Math., (2001).

[16] M.A. Latif and S.S. Dragomir, On Hermite-Hadamard type integral inequalities for n-times differentiable Log-Preinvex functions, Filomat, 29(7) (2015), 1651–1661.

[17] M.A. Latif and S.S. Dragomir, Generalization of Hermite-Hadamard type inequalities for n-times differentiable functions which are s-preinvex in the second sense with applications, Hacettepe J. of Math. and Stat., 44(4) (2015), 389-853.

[18] M.A. Latif and S.S. Dragomir, New inequalities of Hermite-Hadamard type for n-times differentiable convex and concav functions with applications, Submited, 2014.

[19] M. A. Latif and S. S. Dragomir, On Hermite-Hadamard type integral inequalities for n-times differentiable (α, m)-logarithmically convex functions, RGMIA Research Report Collection, vol. 17, Article 14, 16 pages, 2014.
[20] A. M. Ostrowski, Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert, Comment. Math. Helv. 10(1938), 226-227.
[21] C. E. M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., vol. 13, no. 2, pp. 51–55, 2000.
[22] M. E. Özdemir and Ç. Yıldız, A new generalization of the midpoint formula for n-time differentiable mappings which are convex, arXiv:1404.5128v1, 2014.
[23] B.G. Pachpatte, New inequalities of Ostrowski and Trapezoid type for n-time differentiable functions, Bull. Korean Math. Soc. 41 (2004), No. 4, pp. 633-639.
[24] M. Z. Sarikaya, A. Saglam and H. Yildirim, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex, Int. Jour. of Open Problems in Computer Sci. and Math. (IJOPCM). 5(3), 2012, pp.
[25] M. Z. Sarikaya and H. Yaldiz, Some inequalities associated with the probability density function, Submited, 2015.
[26] M. Z. Sarikaya and S. Erden, On the weighted integral inequalities for convex function, Acta Universitatis Sapientiae, Mathematica, 6(2), (2014) 194-208
[27] A. Sofo, Integral inequalities for n- times differentiable mappings, with multiple branches,on the L_p norm, Soochow Journal of Mathematics, Volume 28, No. 2, pp. 179-221, 2002.
[28] S.H. Wang, B. Y. Xi, and F. Qi, Some new inequalities of Hermite-Hadamard type for n-time differentiable functions which are m-convex, Analysis (Munich), vol. 32, 247–262, 2012.
[29] M. Wang and X. Zhao, Ostrowski type inequalities for higher-order derivatives, J. of Inequalities and App., Vol. 2009, Article ID 162689, 8 pages.