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On λ Statistical Upward Compactness and Continuity

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Abstract. A sequence (α_k) of real numbers is called λ-statistically upward quasi-Cauchy if for every $\varepsilon > 0$ $\lim_{n\to\infty}\frac{1}{\lambda_n}|\{k \in I_n : \alpha_k - \alpha_{k+1} \geq \varepsilon\}| = 0$, where (λ_n) is a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, and $I_n = [n - \lambda_n + 1, n]$ for any positive integer n. A real valued function f defined on a subset of $\mathbb{R}$, the set of real numbers is λ-statistically upward continuous if it preserves λ-statistical upward quasi-Cauchy sequences. λ-statistically upward compactness of a subset in real numbers is also introduced and some properties of functions preserving such quasi Cauchy sequences are investigated. It turns out that a function is uniformly continuous if it is λ-statistical upward continuous on a λ-statistical upward compact subset of $\mathbb{R}$.

Introduction

The concept of continuity and any concept involving continuity play a very important role not only in pure mathematics but also in other branches of sciences involving mathematics especially in computer science, information theory, economics, and biological science.

Let $\boldsymbol{\lambda} = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. The generalized de la Vallée-Pousin mean of a sequence $\boldsymbol{\alpha} = (\alpha_k)$ is defined by

$$t_n(\boldsymbol{\alpha}) := \frac{1}{\lambda_n}\sum_{k\in I_n}\alpha_k$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $\boldsymbol{\alpha} = (\alpha_k)$ is said to be (V, λ)-summable to a number L if

$$t_n(\boldsymbol{\alpha}) \longrightarrow L \quad as\ n \longrightarrow \infty,$$

which is denoted by $V_\lambda - lim\alpha_k = L$. A sequence $\boldsymbol{\alpha} = (\alpha_k)$ is said to be $[V, \lambda]$-summable to a number L or strongly (V, λ)-summable to a number L (see [27]) if

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\in I_n}|\alpha_k - L| = 0,$$

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which is denoted by $[V_\lambda] - lim\alpha_k = L$, and a sequence $\boldsymbol{\alpha} = (\alpha_k)$ of points in $\mathbb{R}$ is called to be λ-statistically convergent to an element L of $\mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |\alpha_k - L| \geq \varepsilon\}| = 0$$

for every positive real number ε ([28]). This is denoted by $S_\lambda - lim\alpha_k = \alpha_0$. If $\lambda_n = n$ for all $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers, then we get statistical convergence. Throughout this paper, S , and S_λ will denote the set of statistically convergent sequences, and the set of λ-statistically convergent sequences in $\mathbb{R}$, respectively. If $S_\lambda - lim(\alpha_k - \alpha_{k+1}) = 0$, then (α_k) is called λ-statistically quasi-Cauchy.

Modifying the definitions of a forward Cauchy sequence introduced in [34] and [4], Palladino ([30]) gave the concept of upward half Cauchyness in the following way: a real sequence (α_k) is called upward half Cauchy if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ so that $\alpha_n - \alpha_m < \varepsilon$ for $m \geq n \geq n_0$ (see also [34]). A sequence (α_k) of points in $\mathbb{R}$, the set of real numbers, is called statistically upward quasi-Cauchy if $\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \alpha_k - \alpha_{k+1} \geq \varepsilon\}| = 0$ for every $\varepsilon > 0$ ([13]).

Recently, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: slowly oscillating continuity ([7], [35]), quasi-slowly oscillating continuity ([25]), ward continuity ([16], [2]), δ-ward continuity ([8]), p-ward continuity ([14]), statistical ward continuity ([10]), λ-statistical ward continuity ([22], Abel continuity ([19]), which enabled some authors to obtain conditions on the domain of a function for some characterizations of uniform continuity in terms of sequences in the sense that a function preserves a certain kind of sequences (see [35, Theorem 6], [2, Theorem 1 and Theorem 2], [25, Theorem 2.3], and [12, Theorem 3.2 and Theorem 3.5]).

The aim of this paper is to introduce, and investigate the concepts of λ-statistical upward continuity and λ-statistical upward compactness.

1. λ-statistical upward compactness

The definition of a Cauchy sequence is often misunderstood by the students who first encounter it in an introductory real analysis course. In particular, some fail to understand that it involves far more than that the distance between successive terms is tending to zero. Nevertheless, sequences which satisfy this weaker property are interesting in their own right. In [2] the authors call them "quasi-Cauchy", while they were called "forward convergent to 0" sequences in [16], where a sequence (α_k) is called quasi-Cauchy if for given any $\varepsilon > 0$, there exists an integer $K > 0$ such that $k \geq K$ implies that $|\alpha_k - \alpha_{k+1}| < \varepsilon$. A subset E of $\mathbb{R}$ is upward compact if any sequence of points in E has an upward quasi-Cauchy subsequence, and a subset E of $\mathbb{R}$ is statistically upward compact if any sequence of points in E has a statistically upward quasi-Cauchy subsequence. A subset E of $\mathbb{R}$ is compact if and only if any sequence in E has a convergent subsequence whose limit is in E. Boundedness of a subset E of $\mathbb{R}$ coincides with that any sequence of points in E has either a Cauchy subsequence, or a quasi-Cauchy subsequence. What is the case for below boundedness? λ-statistical upward quasi Cauchy sequences provide with the answer.

Weakening the condition on the definition of a λ-statistical quasi-Cauchy sequence, omitting the absolute value symbol, i.e. replacing $|\alpha_k - \alpha_{k+1}| < \varepsilon$ with $\alpha_k - \alpha_{k+1} < \varepsilon$ in the definition of a λ-statistical quasi-Cauchy sequence given in [22], we introduce the following definition.

Definition 1.1. *A sequence (α_k) of points in $\mathbb{R}$ is called λ-statistically upward quasi-Cauchy if*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : \alpha_k - \alpha_{k+1} \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$.

$\Delta\lambda^+_\lambda$ will denote the set of all λ-statistically upward quasi-Cauchy sequences of points in $\mathbb{R}$. As an interesting example, the sequence (λ_n) is a λ-statistically upward quasi-Cauchy sequence. Any λ-statistically convergent sequence is λ-statistically upward quasi-Cauchy. Any λ-statistically quasi-Cauchy sequence is λ-statistically upward quasi-Cauchy, so any slowly oscillating sequence is λ-statistically upward quasi-Cauchy, so any Cauchy sequence is, so any convergent sequence is. Any upward Cauchy sequence is λ-statistically upward quasi-Cauchy.

Now we give some interesting examples that show importance of the interest.

Example 1.2. *Let n be a positive integer. In a group of n people, each person selects at random and simultaneously another person of the group. All of the selected people are then removed from the group, leaving a random number $n_1 < n$ of people which form a new group. The new group then repeats independently the selection and removal thus described, leaving $n_2 < n_1$ people, and so forth until either one person remains, or no person remains. Denote by α_n the probability that, at the end of this iteration initiated with a group of n people, one person remains. Then the sequence $(\alpha_1, \alpha_2, , \alpha_n, ...)$ is a λ-statistically upward quasi-Cauchy sequence for the special $\boldsymbol{\lambda}$ defined as $\lambda_n = n + \frac{1}{2}$ for each positive integer n (see also [36]).*

Example 1.3. *In a group of k people, k is a positive integer, each person selects independently and at random one of three subgroups to which to belong, resulting in three groups with random numbers k_1, k_2, k_3 of members; $k_1 + k_2 + k_3 = k$. Each of the subgroups is then partitioned independently in the same manner to form three sub subgroups, and so forth. Subgroups having no members or having only one member are removed from the process. Denote by α_k the expected value of the number of iterations up to complete removal, starting initially with a group of k people. Then the sequence $(\alpha_1, \frac{\alpha_2}{2}, \frac{\alpha_3}{3}, ..., \frac{\alpha_n}{n}, ...)$ is a bounded non-convergent λ-statistically upward quasi-Cauchy sequence for the special $\boldsymbol{\lambda}$ defined as $\lambda_n = n + \frac{1}{n}$ for each positive integer n ([26]).*

It is well known that a subset of $\mathbb{R}$ is compact if and only if any sequence of points in E has a convergence subsequence, whose limit is in E. By using this idea in the definition of sequential compactness, now we introduce a definition of λ-statistically upward compactness of a subset of $\mathbb{R}$.

Definition 1.4. *A subset E of $\mathbb{R}$ is called λ-statistically upward compact if any sequence of points in E has a λ-statistically upward quasi-Cauchy subsequence.*

First, we note that any finite subset of $\mathbb{R}$ is λ-statistically upward compact, the union of finite number of λ-statistically upward compact subsets of $\mathbb{R}$ is λ-statistically upward compact, and the intersection of any family of λ-statistically upward compact subsets of $\mathbb{R}$ is λ-statistically upward compact. Furthermore any subset of a λ-statistically upward compact set is λ-statistically upward compact, any compact subset of $\mathbb{R}$ is λ-statistically upward compact, any bounded subset of $\mathbb{R}$ is λ-statistically upward compact, and any slowly oscillating compact subset of $\mathbb{R}$ is λ-statistically upward compact (see [7] for the definition of slowly oscillating compactness). The sum of finite number of λ-statistically upward compact subsets of $\mathbb{R}$ is λ-statistically upward compact. Any bounded belove subset of $\mathbb{R}$ is λ-statistically upward compact. These observations suggest to us giving the following result.

Theorem 1.5. *A subset of $\mathbb{R}$ is λ-statistically upward compact if and only if it is bounded below.*

Proof. Let E be a bounded below subset of $\mathbb{R}$. If E is also bounded above, then it follows from [10, Lemma 2] and [11, Theorem 3] that any sequence of points in E has a quasi Cauchy subsequence which is also λ-statistically upward half quasi-Cauchy. If E is unbounded above, and (α_n) is an unbounded above sequence of points in E, then for $k = 1$ we can find an α_{n_1} greater than 0. For k=2 we can pick an α_{n_2} such that $\alpha_{n_2} > \lambda_2 + \alpha_{n_1}$. We can successively find for each $k \in \mathbb{N}$ an α_{n_k} such that $\alpha_{n_{k+1}} > \lambda_{k+1} + \alpha_{n_k}$. Then $\alpha_{n_k} - \alpha_{n_{k+1}} < -\lambda_{k+1}$ for each $k \in \mathbb{N}$. Therefore we see that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : \alpha_{n_k} - \alpha_{n_{k+1}} \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$. Conversely, suppose that E is not bounded below. Pick an element α_1 of E. Then we can choose an element α_2 of E such that $\alpha_2 < -\lambda_2 + \alpha_1$. Similarly we can choose an element α_3 of E such that $\alpha_3 < -\lambda_3 + \alpha_2$. We can inductively choose α_k satisfying $\alpha_{k+1} < -\lambda_k + \alpha_k$ for each positive integer k. Then the sequence (α_k) does not have any λ-statistically upward half quasi-Cauchy subsequence. Thus E is not λ-statistically upward compact. This contradiction completes the proof. □

It follows from the above theorem that if a closed subset E of $\mathbb{R}$ is λ-statistically upward compact, and $-A$ is λ-statistically upward compact, then any sequence of points in E has a (P_n, s)-absolutely almost convergent subsequence (see [24], [37], and [38]).

Corollary 1.6. *A subset of $\mathbb{R}$ is λ-statistically upward compact if and only if it is statistically upward compact.*

Proof. The proof of this corollary follows from Theorem 1.5 and [13, Theorem 3.3], so it is omitted. □

Corollary 1.7. *A subset of $\mathbb{R}$ is λ-statistically upward compact if and only if it is lacunary statistically upward compact.*

Proof. The proof of this corollary follows from Theorem 1.5 and [20, Theorem 1.9], so is omitted. □

Corollary 1.8. *A subset of $\mathbb{R}$ is λ-statistically upward compact if and only if it is upward compact.*

Proof. The proof of this corollary follows from Theorem 1.5 and [15, Theorem 2.8], so is omitted. □

Corollary 1.9. *A subset A of $\mathbb{R}$ is bounded if and only if the sets A and $-A$ are λ-statistically upward compact.*

Proof. The proof of this corollary follows from the fact that a subset A of $\mathbb{R}$ is bounded if and only if the sets A and $-A$ are bounded below, so is omitted. □

2. λ-statistical upward continuity

A real valued function f defined on a subset of $\mathbb{R}$ is λ-statistically continuous, or S_λ-continuous if for each point ℓ in the domain, $S_\lambda - \lim_{n\to\infty} f(\alpha_k) = f(\ell)$ whenever $S_\lambda - \lim_{n\to\infty}\alpha_k = \ell$ ([22]). This is equivalent to the statement that $(f(\alpha_k))$ is a convergent sequence whenever (α_k) is. This is also equivalent to the statement that $(f(\alpha_k))$ is a Cauchy sequence whenever (α_k) is Cauchy provided that the domain of the function is complete. A real valued function f is called λ-statistically ward continuous on a subset E of $\mathbb{R}$ if it preserves λ-statistically quasi-Cauchy sequences. These known results for λ-statistically-continuity and continuity for real functions in terms of sequences might suggest to us introducing a new type of continuity, namely, λ-statistically-upward continuity, weakening the condition on the definition of a λ-statistically ward continuity, omitting the absolute value symbol, i.e. replacing "$|\alpha_k - \alpha_{k+1}|$" with $\alpha_k - \alpha_{k+1}$" in the definition of λ-statistically ward continuity given in [22].

Definition 2.1. *A real valued function f is called λ-statistically upward continuous, or S^+_λ-continuous on a subset E of $\mathbb{R}$ if it preserves λ-statistically upward quasi-Cauchy sequences, i.e. the sequence $(f(\alpha_k))$ is λ-statistically-upward quasi-Cauchy whenever (α_k) is a λ-statistically-upward quasi-Cauchy sequence of points in E.*

It should be noted that λ-statistically-upward continuity cannot be given by any A-continuity in the manner of [5]. We see that the composition of two λ-statistically-upward continuous functions is λ-statistically-upward continuous, and for every positive real number c, cf is λ-statistically-upward continuous, if f is λ-statistically-upward continuous.
We see in the following that the sum of two λ-statistically-upward continuous functions is λ-statistically-upward continuous

Proposition 2.2. *If f and g are λ-statistically-upward continuous functions, then $f + g$ is λ-statistically-upward continuous.*

Proof. Let f, g be λ-statistically-upward continuous functions on a subset E of $\mathbb{R}$. To prove that $f+g$ is λ-statistically-upward continuous on E, take any λ-statistically-upward quasi-Cauchy sequence (α_k) of points in E. Then $(f(\alpha_k))$ and $(g(\alpha_k))$ are λ-statistically-upward quasi-Cauchy sequences. Let $\varepsilon > 0$ be given. Since $(f(\alpha_k))$ and $(g(\alpha_k))$ are λ-statistically-upward quasi-Cauchy, we have

$$\mathsf{lim}_{n\to\infty}\frac{1}{\lambda_n}|\{k \in I_n : f(\alpha_k) - f(\alpha_{k+1}) \geq \frac{\varepsilon}{2}\}| = 0$$

and

$$\mathsf{lim}_{n\to\infty}\frac{1}{\lambda_n}|\{k \in I_n : g(\alpha_k) - g(\alpha_{k+1}) \geq \frac{\varepsilon}{2}\}| = 0.$$

Hence

$$\mathsf{lim}_{n\to\infty}\frac{1}{\lambda_n}|\{k \in I_n : [f(\alpha_k) - f(\alpha_{k+1})] + [g(\alpha_k - g(\alpha_{k+1})]) \geq \varepsilon\}| = 0$$

which follows from the inclusion
$\{k \in I_n : (f+g)(\alpha_k) - (f+g)(\alpha_{k+1}) \geq \varepsilon\}$
$\subseteq \{k \in I_n : f(\alpha_k) - f(\alpha_{k+1}) \geq \frac{\varepsilon}{2}\} \cup \{k \in I_n : g(\alpha_k) - g(\alpha_{k+1}) \geq \frac{\varepsilon}{2}\}$.
This completes the proof. □

Proposition 2.3. *The composition of two λ-statistically-upward continuous functions is λ-statistically-upward continuous, i.e. if f and g are λ-statistically-upward continuous functions on $\mathbb{R}$, then the composition gof of f and g is λ-statistically-upward continuous.*

Proof. Let f and g be λ-statistically-upward continuous functions on $\mathbb{R}$, and (α_n) be a λ-statistically-upward quasi Cauchy sequence of points in $\mathbb{R}$. As f is λ-statistically-upward continuous, the transformed sequence $(f(\alpha_n))$ is a λ-statistically upward quasi Cauchy sequence. Since g is λ-statistically-upward continuous, the transformed sequence $g(f(\alpha_n))$ of the sequence $(f(\alpha_n))$ is a λ-statistically upward quasi Cauchy sequence. This completes the proof of the theorem.
□

Remark 2.4. *In connection with λ-statistically-upward quasi-Cauchy sequences, λ-statistically-quasi-Cauchy sequences, λ-statistically-statistical convergent sequences, and convergent sequences the problem arises to investigate the following types of continuity of functions on $\mathbb{R}$.*

(δS_λ^+) $(\alpha_k) \in \Delta S_\lambda^+ \Rightarrow (f(\alpha_k)) \in \Delta S_\lambda^+$

($\delta S_\lambda^+ c$) $(\alpha_k) \in \Delta S_\lambda^+ \Rightarrow (f(\alpha_k)) \in c$

(c) $(\alpha_k) \in c \Rightarrow (f(\alpha_k)) \in c$

$(c\delta S_\lambda^+)$ $(\alpha_k) \in c \Rightarrow (f(\alpha_k)) \in \Delta S_\lambda^+$

(S_λ) $(\alpha_k) \in S_\lambda \Rightarrow (f(\alpha_k)) \in S_\lambda$

(δS_λ) $(\alpha_k) \in \Delta S_\lambda \Rightarrow (f(\alpha_k)) \in \Delta S_\lambda$.

We see that (δS_λ^+) is λ-statistically upward continuity of f, (S_λ) is the λ-statistical continuity, and (δS_λ) is the λ-statistically-ward continuity. It is easy to see that $(\delta S_\lambda^+ c)$ implies (δS_λ^+); (δS_λ^+) does not imply $(\delta S_\lambda^+ c)$; (δS_λ^+) implies $(c\delta S_\lambda^+)$; $(c\delta S_\lambda^+)$ does not imply (δS_λ^+); $(\delta S_\lambda^+ c)$ implies (c), and (c) does not imply $(\delta S_\lambda^+ c)$; and (c) implies $(c\delta S_\lambda^+)$. We see that (c) can be replaced by not only λ-statistically-continuity ([22]), but also statistical continuity ([11]), I-sequential continuity ([3]), and more generally G-sequential continuity ([6], [9]).
Now we give the implication (δS_λ^+) implies (δS_λ), i.e. any λ-statistically-upward continuous function is λ-statistically-ward continuous.

Theorem 2.5. *If f is λ-statistically-upward continuous on a subset E of $\mathbb{R}$, then it is λ-statistically-ward continuous on E.*

Proof. Let (α_k) be any λ-statistically-quasi-Cauchy sequence of points in E. Then the sequence

$$(\alpha_1, \alpha_2, \alpha_1, \alpha_2, \alpha_3, \alpha_2, \alpha_3, ..., \alpha_{n-1}, \alpha_k, \alpha_{n-1}, \alpha_k, \alpha_{n+1}, \alpha_k, \alpha_{n+1}, ...)$$

is also λ-statistically quasi-Cauchy. Then it is λ-statistically-upward quasi-Cauchy. As f is λ-statistically-upward continuous, the sequence

$$(f(\alpha_1), f(\alpha_2), f(\alpha_1), f(\alpha_2), f(\alpha_3), f(\alpha_2), f(\alpha_3), ..., f(\alpha_{n-1}),$$
$$f(\alpha_k), f(\alpha_{n-1}), f(\alpha_k), f(\alpha_{n+1}), f(\alpha_k), f(\alpha_{n+1}), ...)$$

is λ-statistically-upward quasi-Cauchy. It follows from this that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |f(\alpha_k) - f(\alpha_{k+1})| \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$. This completes the proof of the theorem. □

We note that the converse of the preceding theorem is not always true, i.e. there are λ-statistically-ward continuous functions which are not λ-statistically-upward continuous.

Example 2.6. *Write $\boldsymbol{\lambda} = (\lambda_n)$ as $\lambda_n = n + \frac{1}{n}$ for each $n > 1$, $\lambda_1 = 1$, and consider the sequence $\boldsymbol{\alpha} = (n)$. Then we see that the function f defined by $f(x) = -x$ for every $x \in \mathbb{R}$ is an λ-statistically-ward continuous function, but not λ-statistically-upward continuous, since $\boldsymbol{\alpha}$ is λ-statistically-upward quasi-Cauchy, but $f(\boldsymbol{\alpha})$ is not λ-statistically-upward quasi-Cauchy.*

Now we give the implication (δS^+_λ) implies (S_λ), i.e. any λ-statistically-upward continuous function is λ-statistically-continuous.

Corollary 2.7. *If f is λ-statistically-upward continuous on a subset E of $\mathbb{R}$, then it is λ-statistically-continuous on E.*

Proof. Although the proof follows from [22, Theorem 3], the preceding theorem, and the fact that λ-statistical continuity coincides with ordinary continuity, we give a direct proof in the following for completeness. Let (α_k) be any λ-statistically convergent sequence with $S_\lambda - \lim_{k\to\infty} \alpha_k = \ell$. Then

$$(\alpha_1, \ell, \alpha_1, \ell, \alpha_2, \ell, \alpha_2, \ell, ..., \alpha_k, \ell, \alpha_k, \ell, ...)$$

is also λ-statistically convergent to ℓ. Thus it is λ-statistically-upward quasi-Cauchy. Hence

$$(f(\alpha_1), f(\ell), f(\alpha_1), f(\ell), f(\alpha_2), f(\ell), f(\alpha_2), f(\ell), ..., f(\alpha_k), f(\ell), f(\alpha_k), f(\ell), ...)$$

is λ-statistically-upward quasi-Cauchy. It follows from this that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |f(\alpha_k) - f(\ell)| \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$. This completes the proof. □

Observing that λ-statistically-continuity implies ordinary continuity, we note that it follows from Theorem 2.7 that λ-statistically-upward continuity implies not only ordinary continuity, but also some other kinds of continuities, namely, lacunary statistical continuity, statistical continuity ([22]), N_θ-sequential continuity, I-continuity for any non-trivial admissible ideal I of $\mathbb{N}$ ([3, Theorem 4]), and G-continuity for any regular subsequential method G (see [5], [6], and [9]).

Theorem 2.8. *λ-statistically-upward continuous image of any λ-statistically-upward compact subset of $\mathbb{R}$ is λ-statistically-upward compact.*

Proof. Let E be a subset of $\mathbb{R}$, $f : E \longrightarrow \mathbb{R}$ be an λ-statistically-upward continuous function, and A be an λ-statistically-upward compact subset of E. Take any sequence $\boldsymbol{\beta} = (\beta_n)$ of points in $f(A)$. Write $\beta_n = f(\alpha_k)$, where $\alpha_k \in A$ for each $n \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_k)$. λ-statistically-upward compactness of A implies that there is an λ-statistically-upward quasi-Cauchy subsequence $\boldsymbol{\xi}$ of the sequence of $\boldsymbol{\alpha}$. Write $\boldsymbol{\eta} = (\eta_k) = f(\boldsymbol{\xi}) = (f(\xi_k))$. Then $\boldsymbol{\eta}$ is an λ-statistically-upward quasi-Cauchy subsequence of the sequence $f(\boldsymbol{\beta})$. This completes the proof of the theorem. □

Theorem 2.9. *Any λ-statistically-upward continuous real valued function on a λ-statistically-upward compact subset of $\mathbb{R}$ is uniformly continuous.*

Proof. Let E be an λ-statistically-upward compact subset of $\mathbb{R}$ and let $f : E \longrightarrow \mathbb{R}$. Suppose that f is not uniformly continuous on E so that there exists an $\varepsilon_0 > 0$ such that for any $\delta > 0$, there are $x, y \in E$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$. For each positive integer n, there are α_n and β_n such that $|\alpha_n - \beta_n| < \frac{1}{n}$, and $|f(\alpha_n) - f(\beta_n)| \geq \varepsilon_0$. Since E is λ-statistically-upward compact, there exists an λ-statistically-upward quasi-Cauchy subsequence (α_{n_k}) of the sequence (α_k). It is clear that the corresponding subsequence (β_{n_k}) of the sequence (β_n) is also λ-statistically-upward quasi-Cauchy, since $(\beta_{n_k} - \beta_{n_{k+1}})$ is a sum of three λ-statistically-upward quasi-Cauchy sequences, i.e.

$$\beta_{n_k} - \beta_{n_{k+1}} = (\beta_{n_k} - \alpha_{n_k}) + (\alpha_{n_k} - \alpha_{n_{k+1}}) + (\alpha_{n_{k+1}} - \beta_{n_{k+1}}).$$

Then the sequence

$$(\beta_{n_1}, \alpha_{n_1}, \beta_{n_2}, \alpha_{n_2}, \beta_{n_3}, \alpha_{n_3}, ..., \beta_{n_k}, \alpha_{n_k}, ...)$$

is λ-statistically-upward quasi-Cauchy, since the sequence $(\beta_{n_k} - \alpha_{n_{k+1}})$ is an λ-statistically-upward quasi-Cauchy sequence which follows from the equality

$$\beta_{n_k} - \alpha_{n_{k+1}} = \beta_{n_k} - \beta_{n_{k+1}} + \beta_{n_{k+1}} - \alpha_{n_{k+1}}.$$

But the sequence

$$(f(\beta_{n_1}), f(\alpha_{n_1}), f(\beta_{n_2}), f(\alpha_{n_2}), f(\beta_{n_3}), f(\alpha_{n_3}), ..., f(\beta_{n_k}), f(\alpha_{n_k}), ...)$$

is not λ-statistically-upward quasi-Cauchy. Thus f does not preserve λ-statistically-upward quasi-Cauchy sequences. This contradiction completes the proof of the theorem. □

Theorem 2.10. *If a function f is uniformly continuous on a subset E of $\mathbb{R}$, then $(f(\alpha_k))$ is λ-statistically-upward quasi Cauchy whenever (α_k) is a quasi-Cauchy sequence of points in E.*

Proof. Let E be a subset of $\mathbb{R}$ and let f be a uniformly continuous function on E. Take any quasi-Cauchy sequence (α_k) of points in A, and let ε be any positive real number in $]0,1[$. By uniform continuity of f, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$ and $x, y \in E$. Since (α_k) is a quasi-Cauchy sequence, there exists a positive integer k_0 such that $|\alpha_k - \alpha_{k+1}| < \varepsilon$ for $k \geq k_0$, therefore $f(\alpha_k) - f(\alpha_{k+1}) < \varepsilon$ for $k \geq k_0$. Thus the number of indices $k \in I_n$ that satisfy $f(\alpha_k) - f(\alpha_{k+1}) \geq \varepsilon$ is less than or equal to k_0. Hence

$$\frac{1}{\lambda_n}|\{k \in I_n : f(\alpha_k) - f(\alpha_{k+1}) \geq \varepsilon\}| \leq \frac{k_0}{\lambda_n}.$$

It follows from this that

$$\lim_{n\to\infty} \frac{1}{\lambda_n}|\{k \in I_n : f(\alpha_k) - f(\alpha_{k+1}) \geq \varepsilon\}| = 0.$$

Thus $(f(\alpha_k))$ is a λ-statistically-upward quasi-Cauchy sequence. This completes the proof of the theorem. □

Now we have the following result related to uniform convergence, namely, uniform limit of a sequence of λ-statistically-upward continuous functions is λ-statistically-upward continuous.

Theorem 2.11. *If (f_n) is a sequence of λ-statistically-upward continuous functions defined on a subset E of $\mathbb{R}$ and (f_n) is uniformly convergent to a function f, then f is λ-statistically-upward continuous on E.*

Proof. Let ε be a positive real number and (α_k) be any λ-statistically-upward quasi-Cauchy sequence of points in E. By uniform convergence of (f_n) there exists a positive integer N such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in E$ whenever $n \geq N$. As f_N is λ-statistically-upward continuous on E, we have

$$\mathrm{lim}_{n\to\infty}\frac{1}{\lambda_n}|\{k \in I_n : f_N(\alpha_k) - f_N(\alpha_{k+1}) \geq \frac{\varepsilon}{3}\}| = 0.$$

On the other hand, we have
$\{k \in I_n : f(\alpha_k) - f(\alpha_{k+1}) \geq \varepsilon\} \subseteq \{k \in I_n : f(\alpha_k) - f_N(\alpha_k) \geq \frac{\varepsilon}{3}\} \cup \{k \in I_n : f_N(\alpha_k) - f_N(\alpha_{k+1}) \geq \frac{\varepsilon}{3}\} \cup \{k \in I_n : f_N(\alpha_{k+1}) - f(\alpha_{k+1}) \geq \frac{\varepsilon}{3}\}$
Hence
$\frac{1}{\lambda_n}|\{k \in I_n : f(\alpha_k) - f(\alpha_{k+1}) \geq \varepsilon\}| \leq \frac{1}{\lambda_n}|k \in I_n : f(\alpha_k) - f_N(\alpha_k) \geq \frac{\varepsilon}{3}\}| + \frac{1}{\lambda_n}|\{k \in I_n : f_N(\alpha_k) - f_N(\alpha_{k+1}) \geq \frac{\varepsilon}{3}\}| + \frac{1}{\lambda_n}|\{k \in I_n : f_N(\alpha_{k+1}) - f(\alpha_{k+1}) \geq \frac{\varepsilon}{3}\}|$
Now it follows from this inequality that

$$lim_{n\to\infty}\frac{1}{\lambda_n}|\{k \in I_n : f(\alpha_k) - f(\alpha_{k+1}) \geq \varepsilon\}| = 0.$$

This completes the proof of the theorem. □

3. Conclusion

The results in this paper include not only the related theorems on statistical upward continuity studied in [13] as special cases, i.e. $\lambda_n = n$ for each $n \in \mathbb{N}$, but also include results which are also new for statistical upward continuity. In this paper, mainly, a new type of continuity, namely the concept of λ-statistical upward continuity of a real function, and λ-statistical upward compactness of a subset of $\mathbb{R}$ are introduced and investigated. In this investigation we have obtained theorems related to λ-statistical upward continuity, and uniform continuity. We also introduced and studied some other continuities involving λ-statistical upward quasi-Cauchy sequences, λ-statistical upward quasi-Cauchy sequences, and convergent sequences of points in $\mathbb{R}$. It turns out that the set of continuous functions on a below bounded subset of $\mathbb{R}$ is contained in the set of uniformly continuous functions. We suggest to investigate λ-statistical upward continuity of fuzzy functions or soft functions (see [21] for the definitions and related concepts in fuzzy setting, and see [1] related concepts in soft setting). We also suggest to investigate λ-statistical upward continuity via double sequences (see for example [32], [31], and [33] for the definitions and related concepts in the double sequences case). For another further study, we suggest to investigate λ-statistical upward continuity in an asymmetric cone metric space since in a cone metric space the notion of an λ-statistical upward quasi Cauchy sequence coincides with the notion of an λ-statistically quasi Cauchy sequence, and therefore λ-statistical upward continuity coincides with λ-statistically-ward continuity (see [23], and [29]).

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