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New Method of Investigation of the Fredholm Property of Three-Dimensional Helmholtz Equation with Nonlocal Boundary Value Conditions

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Abstract. The paper is dedicated to the investigation of the Fredholm property of boundary value problems with nonlocal boundary conditions for a three-dimensional Helmholtz equation. The basic relationships giving complete system of necessary conditions for the solvability of the boundary value problem are obtained on the assumption of well-known fundamental solution of Helmholtz equation. Some of these conditions contain singularities the regularization of which cannot be done by conventional methods and is conducted by an original scheme.

1. Introduction

As is known, for an ordinary differential equation the number of additional conditions (Cauchy conditions or boundary conditions) always coincides with the order of the equation in question .

In the course of equations of mathematical physics and partial differential equations, the canonical form of an equation of elliptic type is the Laplace equation (second-order equation) for which one local boundary condition (Dirichlet, Neumann or Poincare) is specified.

The non-local boundary conditions free us from the above misunderstanding between ordinary differential equations and partial differential equations. For nonlocal boundary value problems the authors have found the possibility of proving Fredholm property with the help of so-called necessary conditions.

It should be noted that for an ordinary differential equation these necessary conditions similar to nonlocal boundary conditions are mentioned by A.A.Dezin [1]-[3] (who came to these conditions by an artificial way and so couldn't build them for partial differential equations).

The idea of necessary conditions for partial differential equations was first used by A.V. Bitsadze for the Laplace equation [4, p.185] both in two-dimensional and three-dimensional cases. But the regularization of the singularities in the necessary conditions was artificial particularly in three-dimensional and contained some uncertainties.

Finally, Begehr derived these necessary conditions for Cauchy-Riemann equation [5]-[6].

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Some of the necessary conditions obtained for the posed three-dimensional problem contain singular multiple integrals. But the regularization of these singularities doesn't subject to the conventional scheme [7-11].

As is known the regularization of singular integral equations in general case is conducted by the method of successive substitutions: after the first substitution a double singular integral is obtained and when changing the order of integration in the double integral Poincare-Bertrand formula is applied to get a regular kernel and a jump which doesn't "eat" the external function. Thus, a Fredholm integral equation of second kind with a regular kernel is obtained.

In the considered problem the obtained necessary conditions, or integral equations, are in spectrum so when they are regularized by the mentioned scheme we come to Fredholm integral equations of first kind what is a "deadlock".

By the suggested new scheme the singular necessary conditions are regularized with aid of the given boundary conditions what is principally new. As a result the posed problem is reduced to a system of Fredholm integral equations of second kind.

2. Problem statement

Let us consider the three-dimensional Helmholtz equation in a convex in the direction x_3 domain $D \subset R^3$ whose projection onto plane $Ox_1x_2 = Ox'$ is domain $S \subset Ox_1x_2$, Γ is the boundary (surface) of the domain D:

$$\Delta u + a^2 u(x) = -f(x),\ x = (x_1, x_2, x_3) \in D \subset R^3 \tag{1}$$

with nonlocal boundary conditions

$$l_i u = \sum_{j=1}^{3} \left[\alpha_{ij}^{(1)}(x') \frac{\partial u(x)}{\partial x_j} \Big|_{x_3=\gamma_1(x')} + \alpha_{ij}^{(2)}(x') \frac{\partial u(x)}{\partial x_j} \Big|_{x_3=\gamma_2(x')} \right] +$$

$$+ \sum_{k=1}^{2} \alpha_i^{(k)}(x') u(x', \gamma_k(x')) = 0,\ i = 1, 2;\ \ x' \in S, \tag{2}$$

and additional Dirichlet's condition on the equator L of the surface Γ

$$u(x) = f_0(x),\ \ x \in L = \bar{\Gamma}_1 \bigcap \bar{\Gamma}_2. \tag{3}$$

where S is the projection of the domain D onto the plane $Ox_1x_2 = Ox'$, $\Gamma = \partial D$ is Lyapunov's surface; L is the equator connecting the upper and lower semi-surfaces Γ_1 and Γ_2: $\Gamma_k = \{\xi = (\xi_1, \xi_2, \xi_3) : \xi_3 = \gamma_k(\xi'),\ \xi' = (\xi_1, \xi_2) \in S\}, k = 1, 2,$ where $\xi_3 = \gamma_k(\xi_1, \xi_2), k = 1, 2,$ are the equations of the semi-surfaces Γ_1 and Γ_2 (the convexity of the domain D in the direction of Ox_3 provides the existence of such equations), functions $\gamma_k(\xi'), k = 1, 2,$ are twice differentiable with respect to the both of variables ξ_1, ξ_2; the coefficients $\alpha_{ij}^{(k)}(x')$ satisfy Hölder condition in S; $\alpha_i^{(k)}(x'),\ i, k = 1, 2, f(x)$ and $f_0(x)$ are continuous functions.

The fundamental solution for the three-dimensional Helmholtz operator $(\Delta + a^2 I)U(x) = \delta(x))$ has the form of [12] $U(x) = -\frac{e^{ia|x|}}{4\pi|x|}$ or $U(x) = -\frac{e^{-ia|x|}}{4\pi|x|}$, from which we choose the following

$$U(x - \xi) = -\frac{e^{ia|x-\xi|}}{4\pi\,|x - \xi|}. \tag{4}$$

3. Necessary conditions

To get necessary conditions of the solvability of boundary value problem (1)-(3) for three-dimensional Helmholtz equation we'll multiply both sides of equation (1) scalarly by fundamental equation (4) and integrate over domain D:

$$\int_D (\Delta u + a^2 u(x))U(x-\xi)dx = -\int_D f(x)U(x-\xi)dx\,,$$

or

$$-\int_D (\Delta u + a^2 u(x))\frac{e^{ia|x-\xi|}}{4\pi\,|x-\xi|}dx = \int_D f(x)\frac{e^{ia|x-\xi|}}{4\pi\,|x-\xi|}dx\,. \tag{5}$$

Integrating (5) by parts we'll obtain the following:

$$\int_D (\Delta u + a^2 u(x))U(x-\xi)dx = \sum_{j=1}^{3}\int_D \frac{\partial^2 u(x)}{\partial x_j^2}U(x-\xi)dx + \int_D a^2 u(x))U(x-\xi)dx =$$

$$= \sum_{j=1}^{3}\left[\int_\Gamma \frac{\partial u(x)}{\partial x_j}U(x-\xi)\cos(\nu_x, x_j)dx - \int_D \frac{\partial u(x)}{\partial x_j}\frac{\partial U(x-\xi)}{\partial x_j}dx\right] + \int_D a^2 u(x))U(x-\xi)dx =$$

$$= \sum_{j=1}^{3}\int_\Gamma \left(\frac{\partial u(x)}{\partial x_j}U(x-\xi) - u(x)\frac{\partial U(x-\xi)}{\partial x_j}\right)\cos(\nu_x, x_j)dx +$$

$$+ \int_D u(x)\sum_{j=1}^{3}\frac{\partial^2 U(x-\xi)}{\partial x_j^2}dx + \int_D a^2 u(x))U(x-\xi)dx =$$

$$= \sum_{j=1}^{3}\int_\Gamma \left(\frac{\partial u(x)}{\partial x_j}U(x-\xi) - u(x)\frac{\partial U(x-\xi)}{\partial x_j}\right)\cos(\nu_x, x_j)dx + \int_D u(x)\delta(x-\xi)dx\,. \tag{6}$$

As $U(x-\xi)$ is a fundamental solution of Helmholtz equation then

$$\sum_{j=1}^{3}\frac{\partial^2 U(x-\xi)}{\partial x_j^2} + a^2 U(x-\xi) = (\Delta_x + a^2 I)U(x-\xi) = \delta(x-\xi)$$

is Dirac's δ-function. Taking into account this and substituting (6) into (5), we have the relationship

$$-\int_D f(x)U(x-\xi)dx = \int_D \left\{\sum_{j=1}^{3}\frac{\partial^2 u(x)}{\partial x_j^2} + a^2 u(x)\right\}U(x-\xi)dx =$$

$$= \sum_{j=1}^{3} \int_D \frac{\partial^2 u(x)}{\partial x_j^2} U(x-\xi)dx + \int_D a^2 u(x)U(x-\xi)dx =$$

$$= \sum_{j=1}^{3} \left(\int_\Gamma \frac{\partial u(x)}{\partial x_j} U(x-\xi)\cos(\nu_x, x_j)dx - \int_D \frac{\partial u(x)}{\partial x_j} \frac{\partial U(x-\xi)}{\partial x_j} dx \right) + \int_D a^2 u(x)U(x-\xi)dx =$$

$$= \sum_{j=1}^{3} \int_\Gamma \frac{\partial u(x)}{\partial x_j} U(x-\xi)\cos(\nu_x, x_j)dx -$$

$$- \sum_{j=1}^{3} \left(\int_\Gamma u(x) \frac{\partial U(x-\xi)}{\partial x_j} \cos(\nu_x, x_j)dx - \int_D u(x) \frac{\partial^2 U(x-\xi)}{\partial x_j^2} dx \right) + \int_D a^2 u(x)U(x-\xi)dx$$

$$= \sum_{j=1}^{3} \int_\Gamma \left(\frac{\partial u(x)}{\partial x_j} U(x-\xi) - u(x) \frac{\partial U(x-\xi)}{\partial x_j} \right) \cos(\nu_x, x_j)dx + \int_D u(x)\delta(x-\xi)dx$$

whence we obtain the 1st basic relationship

$$- \sum_{j=1}^{3} \int_\Gamma \left(\frac{\partial u(x)}{\partial x_j} U(x-\xi) - u(x) \frac{\partial U(x-\xi)}{\partial x_j} \right) \cos(\nu_x, x_j)dx - \int_D f(x)U(x-\xi)dx =$$

$$= \int_D u(x)\delta(x-\xi)dx = \begin{cases} u(\xi), & \xi \in D, \\ \frac{1}{2}u(\xi), & \xi \in \Gamma. \end{cases} \tag{7}$$

The 2nd of relationships (7) is called **the 1st necessary condition** of solvability of problem (1)-(2):

$$\frac{1}{2}u(\xi) = - \sum_{j=1}^{3} \int_\Gamma \left(\frac{\partial u(x)}{\partial x_j} U(x-\xi) - u(x) \frac{\partial U(x-\xi)}{\partial x_j} \right) \cos(\nu_x, x_j)dx - \int_D f(x)U(x-\xi)dx, \quad \xi \in \Gamma. \tag{8}$$

The necessary condition (8) can be rewritten as follows:

$$\frac{1}{2}u(\xi) = - \int_\Gamma \left(\frac{\partial u(x)}{\partial \nu_x} U(x-\xi) - u(x) \frac{\partial U(x-\xi)}{\partial \nu_x} \right) dx + \int_D f(x)U(x-\xi)dx, \quad \xi \in \Gamma. \tag{9}$$

Thus we have proved

Theorem 3.1. *Let a convex along the direction x_3 domain $D \subset R^3$ be bounded with the boundary Γ which is a Lyapunov surface. Then the obtained first necessary condition* (9) *is regular.*

Multiplying (1) by $\frac{\partial U(x-\xi)}{\partial x_i}$, $i = \overline{1,3}$, , integrating it over the domain D we obtain the rest of **three basic relationships**:

$$\int_\Gamma \frac{\partial u(x)}{\partial x_i}\frac{\partial U(x-\xi)}{\partial \nu_x}dx + \int_\Gamma \frac{\partial u(x)}{\partial x_m}\left[\frac{\partial U(x-\xi)}{\partial x_i}\cos(\nu_x, x_m) - \frac{\partial U(x-\xi)}{\partial x_m}\cos(\nu_x, x_i)\right]dx+$$

$$+\int_\Gamma \frac{\partial u(x)}{\partial x_l}\left[\frac{\partial U(x-\xi)}{\partial x_i}\cos(\nu_x, x_l) - \frac{\partial U(x-\xi)}{\partial x_l}\cos(\nu_x, x_i)\right]dx+$$

$$+\int_\Gamma a^2 u(x)U(x-\xi)\cos(\nu_x, x_i)dx + \int_D f(x)\frac{\partial U(x-\xi)}{\partial x_i}dx = \begin{cases} \frac{\partial u(\xi)}{\partial \xi_i}, & \xi \in D, \\ \frac{1}{2}\frac{\partial u(\xi)}{\partial \xi_i}, & \xi \in \Gamma, \end{cases} \quad i = \overline{1,3}\,, \tag{10}$$

where the numbers i, m, l make a permutation of numbers 1,2,3.

The second expressions in (10) are the other three **necessary conditions (** $\xi \in \Gamma$**,**$i = \overline{1,3}$):

$$\frac{1}{2}\frac{\partial u(\xi)}{\partial \xi_i} = \int_\Gamma \frac{\partial u(x)}{\partial x_i}\frac{\partial U(x-\xi)}{\partial \nu_x}dx+$$

$$+\int_\Gamma \frac{\partial u(x)}{\partial x_m}\left[\frac{\partial U(x-\xi)}{\partial x_i}\cos(\nu_x, x_m) - \frac{\partial U(x-\xi)}{\partial x_m}\cos(\nu_x, x_i)\right]dx+$$

$$+\int_\Gamma \frac{\partial u(x)}{\partial x_l}\left[\frac{\partial U(x-\xi)}{\partial x_i}\cos(\nu_x, x_l) - \frac{\partial U(x-\xi)}{\partial x_l}\cos(\nu_x, x_i)\right]dx +$$

$$+\int_\Gamma a^2 u(x)U(x-\xi)\cos(\nu_x, x_i)dx + \int_D f(x)\frac{\partial U(x-\xi)}{\partial x_i}dx, \tag{11}$$

where the numbers i, m, l make a permutation of numbers 1,2,3.

As

$$\frac{\partial U(x-\xi)}{\partial x_j} = -\frac{\partial}{\partial x_j}\frac{e^{ia|x-\xi|}}{4\pi\,|x-\xi|} = -\frac{1}{4\pi}\left(\frac{\frac{i2a(x_j-\xi_j)}{2|x-\xi|}e^{ia|x-\xi|}\,|x-\xi| - \frac{2(x_j-\xi_j)}{2|x-\xi|}e^{ia|x-\xi|}}{|x-\xi|^2}\right) =$$

$$= \frac{1}{4\pi}\left(\frac{\frac{x_j-\xi_j}{|x-\xi|}e^{ia|x-\xi|} - ia(x_j-\xi_j)e^{ia|x-\xi|}}{|x-\xi|^2}\right) = \frac{1}{4\pi}\frac{\frac{x_j-\xi_j}{|x-\xi|}e^{ia|x-\xi|}\left(1 - ia\,|x-\xi|\right)}{|x-\xi|^2} =$$

$$= \frac{e^{ia|x-\xi|}\cos(x-\xi, x_j)}{4\pi\,|x-\xi|^2}\left(1 - ia\,|x-\xi|\right)$$

then

$$\frac{\partial U(x-\xi)}{\partial x_j} = \frac{\cos(x-\xi, x_j)}{4\pi\,|x-\xi|^2}e^{ia|x-\xi|}\left(1 - ia\,|x-\xi|\right). \tag{12}$$

Substituting (12) into (11), we'll obtain that

$$\frac{1}{2}u(\xi) = \frac{1}{4\pi}\int_\Gamma u(x)\frac{\cos(x-\xi,\nu_x)}{|x-\xi|^2}e^{ia|x-\xi|}(1-ia|x-\xi|)\,dx+$$

$$+\frac{1}{4\pi}\int_\Gamma \frac{e^{ia|x-\xi|}}{|x-\xi|}\frac{\partial u(x)}{\partial\nu_x}dx + \frac{1}{4\pi}\int_D f(x)\frac{e^{ia|x-\xi|}}{|x-\xi|}dx, \quad \xi\in\Gamma. \tag{13}$$

Introducing the designations:

$$K_{ij}(x,\xi) = \Big(\cos(x-\xi,x_i)\cos(\nu_x,x_j) - \cos(x-\xi,x_j)\cos(\nu_x,x_i)\Big)e^{ia|x-\xi|}(1-ia|x-\xi|) \tag{14}$$

and representing

$$\frac{\partial U(x-\xi)}{\partial x_i}\cos(\nu_x,x_m) - \frac{\partial U(x-\xi)}{\partial x_m}\cos(\nu_x,x_i) =$$

$$= \frac{e^{ia|x-\xi|}(1-ia|x-\xi|)}{4\pi|x-\xi|^2}(\cos(x-\xi,x_i)\cos(\nu_x,x_m) - \cos(x-\xi,x_m)\cos(\nu_x,x_i)) =$$

$$= \frac{K_{im}(x,\xi)}{4\pi|x-\xi|^2},$$

we can rewrite the **2nd , the 3rd and 4th necessary conditions** (11) in the form of

$$\frac{1}{2}\frac{\partial u(\xi)}{\partial\xi_i} = \int_\Gamma \frac{\partial u(x)}{\partial x_m}\frac{K_{im}(x,\xi)}{4\pi|x-\xi|^2}dx + \int_\Gamma \frac{\partial u(x)}{\partial x_l}\frac{K_{il}(x,\xi)}{4\pi|x-\xi|^2}dx+$$

$$+\int_\Gamma \frac{\partial u(x)}{\partial x_i}\frac{\partial U(x-\xi)}{\partial\nu_x}dx - \int_\Gamma k^2u(x)\frac{e^{ia|x-\xi|}\cos(\nu_x,x_i)}{4\pi|x-\xi|}dx+$$

$$+\int_D f(x)\frac{\cos(x-\xi,x_i)}{4\pi|x-\xi|^2}e^{ia|x-\xi|}(1-ia|x-\xi|)\,dx, \tag{15}$$

where the numbers i, m, l make a permutation of numbers 1,2,3.

As the normal derivative of the fundamental solution has no singularity at point $x=\xi$ if Γ is Lyapunov's surface, the order of singularity in the 4th integral in the right hand side is lower than the multiplicity of the surface integral then we have singularity only in the 1st and 2nd integrals in RHS of (5).

To distinguish only singular terms in the 2nd, 3rd, 4th necessary conditions we'll disclose two first surface integrals in the $(i+1)$-th relationship (15) $(i=1,2,3)$ over the lower and the upper half surfaces Γ_k, $k=1,2$:

$$\frac{1}{2}\frac{\partial u}{\partial\xi_i}\Big|_{\xi_3=\gamma_k(\xi')} = \sum_{j=1}^{2}(-1)^{j+1}\int_S \frac{\partial u(x)}{\partial x_m}\Big|_{x_3=\gamma_j(x')}\frac{K_{im}(x,\xi)}{4\pi|x-\xi|^2}\Bigg|_{\substack{x_3=\gamma_j(x')\\ \xi_3=\gamma_k(\xi')}}\frac{dx'}{\cos(\nu_x,x_3)}+$$

$$+\sum_{j=1}^{2}(-1)^{j+1}\int_S \frac{\partial u(x)}{\partial x_l}\Big|_{x_3=\gamma_j(x')}\frac{K_{il}(x,\xi)}{4\pi\,|x-\xi|^2}\Bigg|_{\substack{x_3=\gamma_j(x')\\ \xi_3=\gamma_k(\xi')}}\frac{dx'}{\cos(\nu_x,x_3)}+...,$$

whence discarding nonsingular terms $(k\neq j)$, we'll obtain the 2nd, 3rd, 4th necessary conditions for $k=1,2$ in the form:

$$\frac{1}{2}\frac{\partial u}{\partial \xi_i}\Big|_{\xi_3=\gamma_k(\xi')}=(-1)^{k+1}\int_S \frac{\partial u(x)}{\partial x_m}\Big|_{x_3=\gamma_k(x')}\frac{K_{im}(x,\xi)}{4\pi\,|x-\xi|^2}\Bigg|_{\substack{x_3=\gamma_k(x')\\ \xi_3=\gamma_k(\xi')}}\frac{dx'}{\cos(\nu_x,x_3)}+$$

$$+(-1)^{k+1}\int_S \frac{\partial u(x)}{\partial x_l}\Big|_{x_3=\gamma_k(x')}\frac{K_{il}(x,\xi)}{4\pi\,|x-\xi|^2}\Bigg|_{\substack{x_3=\gamma_k(x')\\ \xi_3=\gamma_k(\xi')}}\frac{dx'}{\cos(\nu_x,x_3)}+..., \tag{16}$$

where three dots designate the sum of nonsingular terms.

Let ua introduce the designations:

$$K_{ij}^{(k)}(x',\xi')=K_{ij}(x,\xi)\Bigg|_{\substack{x_3=\gamma_k(x')\\ \xi_3=\gamma_k(\xi')}},\ k=1,2. \tag{17}$$

Now we consider $|x-\xi|^2\Bigg|_{\substack{x_3=\gamma_k(x')\\ \xi_3=\gamma_k(\xi')}}$, k=1,2, in the denominator of the integrands (16):

$$|x-\xi|^2\Bigg|_{\substack{x_3=\gamma_k(x')\\ \xi_3=\gamma_k(\xi')}}=|x'-\xi'|^2+(\gamma_k(x')-\gamma_k(\xi'))^2=$$

$$=|x'-\xi'|^2\left[1+\sum_{m=1}^{2}\left(\frac{\partial\gamma_k(x')}{\partial x_m}\right)^2\cos^2(x'-\xi',x_m)+\right.$$

$$\left.2\frac{\partial\gamma_k(x')}{\partial x_1}\frac{\partial\gamma_k(x')}{\partial x_2}\cos(x'-\xi',x_1)\cos(x'-\xi',x_2)+O(|x'-\xi'|)\right]. \tag{18}$$

Designating:

$$P_k(x',\xi')=1+\sum_{m=1}^{2}\left(\frac{\partial\gamma_k(x')}{\partial x_m}\right)^2\cos^2(x'-\xi',x_m)+$$

$$+2\frac{\partial\gamma_k(x')}{\partial x_1}\frac{\partial\gamma_k(x')}{\partial x_2}\cos(x'-\xi',x_1)\cos(x'-\xi',x_2)+O(|x'-\xi'|)$$

we can rewrite (18) as follows

$$|x-\xi|^2 \Big|_{\substack{x_3=\gamma_k(x') \\ \xi_3=\gamma_k(\xi')}} = |x'-\xi'|^2 P_k(x',\xi'). \tag{19}$$

R e m a r k 3.1. It should be noted that for $\xi' = x'$ we have:

$$P_k(x',x') = 1 + \left(\frac{\partial\gamma_k}{\partial x_1}\right)^2 + \left(\frac{\partial\gamma_k}{\partial x_2}\right)^2 + 2\frac{\partial\gamma_k}{\partial x_1}\frac{\partial\gamma_k}{\partial x_2} \neq 0, k = 1,2.$$

By means of designations (17), (19) we can rewrite necessary conditions (15) for k=1, 2 as follows:

$$\frac{1}{2}\frac{\partial u}{\partial \xi_i}\Big|_{\xi_3=\gamma_k(\xi')} = (-1)^{k+1}\int_S \frac{\partial u(x)}{\partial x_m}\Big|_{x_3=\gamma_k(x')} \frac{1}{4\pi\,|x'-\xi'|^2}\,\frac{K_{im}^{(k)}(x',\xi')}{P_k(x',\xi')}\,\frac{dx'}{\cos(\nu_x,x_3)}+$$

$$+(-1)^{k+1}\int_S \frac{\partial u(x)}{\partial x_l}\Big|_{x_3=\gamma_k(x')}\,\frac{1}{4\pi\,|x'-\xi|'^2}\,\frac{K_{il}^{(k)}(x',\xi')}{P_k(x',\xi')}\,\frac{dx'}{\cos(\nu_x,x_3)} + \ldots,\quad i=1,2,3, \tag{20}$$

where three dots designate the sum of nonsingular terms.

Theorem 3.2. *Under assumptions of Theorem 3.1 necessary conditions* (20) *are singular.*

Let us return to the 1st necessary condition (13) and disclose each surface integral over the upper and lower semi-surfaces $\Gamma_k, k = 1,2,$ of the boundary Γ taking into account that $\Gamma_k = \{\xi = (\xi_1,\xi_2,\xi_3) : \xi_3 = \gamma_k(\xi'),\ \xi' = (\xi_1,\xi_2) \in S\}$, $k = 1$, 2:

$$\frac{1}{2}u(\xi)\Big|_{\xi_3=\gamma_k(\xi')} = \sum_{m=1}^{2}(-1)^m\frac{1}{4\pi}\int_S u(x)\Big|_{x_3=\gamma_m(x')} \times$$

$$\times\left(\frac{\cos(x-\xi,\nu_x)}{|x-\xi|^2}e^{ia|x-\xi|}\left(1-ia\,|x-\xi|\right)\right)\Bigg|_{\substack{\xi_3=\gamma_k(\xi') \\ x_3=\gamma_m(x')}}\ \frac{dx'}{\cos(\nu_x,x_3)}+$$

$$+\sum_{m=1}^{2}(-1)^i\frac{1}{4\pi}\int_S\left(\frac{e^{ia|x-\xi|}}{|x-\xi|}\frac{\partial u(x)}{\partial\nu_x}\right)\Bigg|_{\substack{\xi_3=\gamma_k(\xi') \\ x_3=\gamma_m(x')}}\ \frac{dx'}{\cos(\nu_x,x_3)}+$$

$$+\int_D f(x)\frac{e^{ia|x-\xi|}}{|x-\xi|}dx,\ \xi\in\Gamma_k. \tag{21}$$

Evidently, when $k \neq m$ in (21), the corresponding integral is nonsingular. When $k = m$ in the first sum of (21), the corresponding integral has a removable singularity at $x \to \xi$; in the second sum of (21) the integral has a weak singularity as the order of singularity is less than the multiplicity of the integral, and in the 3rd integral the Hölder condition for $f(x)$ gives us a removable singularity as well. So designating nonsingular terms in (21) with three dots, introducing a new designation

$$Q_k(x',\xi') = \left(\cos(x-\xi,\nu_x)e^{ia|x-\xi|}\left(1 - ia\,|x-\xi|\right)\right)\Bigg|_{\substack{\xi_3=\gamma_k(\xi')\\ x_3=\gamma_k(x')}}$$

and taking into account (21), we obtain the 1st necessary condition in the form of (for k=1, 2):

$$\frac{1}{2}u(\xi)\left|_{\xi_3=\gamma_k(\xi')}\right. = (-1)^k\frac{1}{4\pi}\int_S u(x)\left|_{x_3=\gamma_k(x')}\right. \frac{Q_k(x',\xi')}{P_k(x',\xi')\,|x'-\xi'|^2}\frac{dx'}{\cos(\nu_x,x_3)} + \dots . \tag{22}$$

4. Regularization of the necessary conditions

Let us build a linear combination of necessary conditions (20) for k=1,2 (j=1,2,3) with unknown yet coefficients $\beta_{ij}^k(\xi')$:

$$\beta_{ij}^{(1)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\left|_{\xi_3=\gamma_1(\xi')}\right. + \beta_{ij}^{(2)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\left|_{\xi_3=\gamma_2(\xi')}\right. = \int_S\sum_{k=1}^{2}\beta_{ij}^{(k)}(\xi')(-1)^k\times$$

$$\times\left(\frac{\partial u(x)}{\partial x_m}\left|_{x_3=\gamma_k(x')}\right. \frac{K_{jm}^{(k)}(x,\xi)}{P_k(x',\xi')} + \frac{\partial u(x)}{\partial x_l}\left|_{x_3=\gamma_k(x')}\right. \frac{K_{jl}^{(k)}(x,\xi)}{P_k(x',\xi')}\right)\frac{1}{2\pi\,|x'-\xi'|^2}\frac{dx'}{\cos(\nu_x,x_3)} + \dots, \tag{23}$$

where the numbers j, m ,l make a permutation of numbers 1,2,3.

Form a sum of (23) for j=1,2,3 and bracket the common factor $\frac{1}{2\pi|x'-\xi'|^2}$ under the sign of integral ($i = 1,2$):

$$\sum_{j=1}^{3}\left(\beta_{ij}^{(1)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\left|_{\xi_3=\gamma_1(\xi')}\right. + \beta_{ij}^{(2)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\left|_{\xi_3=\gamma_2(\xi')}\right.\right) =$$

$$= \int_S\frac{1}{2\pi\,|x'-\xi'|^2}\frac{dx'}{\cos(\nu_x,x_3)}\sum_{k=1}^{2}(-1)^k\sum_{j=1}^{3}\beta_{ij}^{(k)}(\xi')\times$$

$$\times\left(\frac{\partial u(x)}{\partial x_2}\left|_{x_3=\gamma_k(x')}\right. \frac{K_{jm}^{(k)}(x',\xi')}{P_k(x',\xi')} + \frac{\partial u(x)}{\partial x_3}\left|_{x_3=\gamma_k(x')}\right. \frac{K_{jl}^{(k)}(x',\xi')}{P_k(x',\xi')}\right) + \dots . \tag{24}$$

Adding and subtracting $\beta_{ij}^{(k)}(x')$ from $\beta_{ij}^{(k)}(\xi')$, $k = 1,2$ in (24) we obtain:

$$\sum_{j=1}^{3}\left(\beta_{ij}^{(1)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\left|_{\xi_3=\gamma_1(\xi')}\right. + \beta_{ij}^{(2)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\left|_{\xi_3=\gamma_2(\xi')}\right.\right) =$$

$$= \int_S\frac{1}{2\pi\,|x'-\xi'|^2}\frac{dx'}{\cos(\nu_x,x_3)}\sum_{k=1}^{2}(-1)^k\times$$

$$\times \sum_{j=1}^{3} \beta_{ij}^{(k)}(x') \left(\frac{\partial u(x)}{\partial x_m} \Big|_{x_3=\gamma_k(x')} \frac{K_{jm}^{(k)}(x',\xi')}{P_k(x',\xi')} + \frac{\partial u(x)}{\partial x_l} \Big|_{x_3=\gamma_k(x')} \frac{K_{jl}^{(k)}(x',\xi')}{P_k(x',\xi')} \right) +$$

$$+ \int_S \frac{1}{2\pi |x'-\xi'|^2} \frac{dx'}{\cos(\nu_x, x_3)} \sum_{k=1}^{2} (-1)^k \sum_{j=1}^{3} \left[\beta_{ij}^{(k)}(\xi') - \beta_{ij}^{(k)}(x') \right] \times$$

$$\times \left(\frac{\partial u(x)}{\partial x_m} \Big|_{x_3=\gamma_k(x')} \frac{K_{jm}^{(k)}(x',\xi')}{P_k(x',\xi')} + \frac{\partial u(x)}{\partial x_l} \Big|_{x_3=\gamma_k(x')} \frac{K_{jl}^{(k)}(x',\xi')}{P_k(x',\xi')} \right) + \ldots \ . \tag{25}$$

Suggesting that functions $\beta_{ij}^{(k)}(\xi')$ satisfy Hölder condition we get a week singularity under the second integral in the right-hand side of (25). Intending to group the derivatives in the first integral we at first expand all the coefficients at the derivatives by Taylor's formula at point $\xi' = x'$:

$$\frac{K_{ij}^{(k)}(x',\xi')}{P_k(x',\xi')} = \frac{K_{ij}^{(k)}(x',x')}{P_k(x',x')} + \frac{\partial}{\partial x_1} \left(\frac{K_{ij}^{(k)}(x',x')}{P_k(x',x')} \right) (x_1 - \xi_1) + \frac{\partial}{\partial x_2} \left(\frac{K_{ij}^{(k)}(x',x')}{P_k(x',x')} \right) (x_2 - \xi_2) + \ldots \ .$$

All the terms except the first one reduce the order of singularity and make it weaker for a double integral over the surface S. That's why we'll only consider the first term from each such expansion:

$$\sum_{j=1}^{3} \left(\beta_{ij}^{(1)}(\xi') \frac{\partial u(\xi)}{\partial \xi_j} \Big|_{\xi_3=\gamma_1(\xi')} + \beta_{ij}^{(2)}(\xi') \frac{\partial u(\xi)}{\partial \xi_j} \Big|_{\xi_3=\gamma_2(\xi')} \right) =$$

$$= \int_S \frac{1}{2\pi |x'-\xi'|^2} \frac{dx'}{\cos(\nu_x, x_3)} \sum_{k=1}^{2} (-1)^k \sum_{j=1}^{3} \beta_{ij}^{(k)}(x') \times$$

$$\times \left(\frac{\partial u(x)}{\partial x_m} \Big|_{x_3=\gamma_k(x')} \frac{K_{jm}^{(k)}(x',\xi')}{P_k(x',\xi')} + \frac{\partial u(x)}{\partial x_l} \Big|_{x_3=\gamma_k(x')} \frac{K_{jl}^{(k)}(x',\xi')}{P_k(x',\xi')} \right) + \ldots . =$$

$$= \int_S \frac{1}{2\pi |x'-\xi'|^2} \frac{dx'}{\cos(\nu_x, x_3)} \sum_{k=1}^{2} (-1)^k \sum_{j=1}^{3} \frac{\partial u(x)}{\partial x_j} \Big|_{x_3=\gamma_k(x')} \times$$

$$\times \left(\beta_{il}^{(k)}(x') \frac{K_{lj}^{(k)}(x',x')}{P_k(x',x')} + \beta_{im}^{(k)}(x') \frac{K_{mj}^{(k)}(x',x')}{P_k(x',x')} \right) + \ldots \tag{26}$$

where the numbers j, l, m form a permutation of numbers 1,2,3; i=1,2.

To regularize the integral in the right hand side of (26) let us impose conditions on the coefficients $\beta_{ij}^{(k)}(\xi')$, i.e. let the coefficients at the derivatives under the sign of integral (26) be equal to the coefficients $\alpha_{ij}^{(k)}(\xi')$ from the boundary conditions (2). Then we get a system of 6 equations for each i=1,2:

$$(-1)^k\beta_{il}^{(k)}(x')\frac{K_{lj}^{(k)}(x',x')}{P_k(x',x')}+(-1)^k\beta_{im}^{(k)}(x')\frac{K_{mj}^{(k)}(x',x')}{P_k(x',x')}=\alpha_{ij}^{(k)}(x'), \tag{27}$$

where k=1,2; j=1,2,3, and as we mentioned above the numbers j, l, m form a permutation of numbers 1,2,3.

We suppose that systems (27) have the solution $\beta_{ij}^{(k)}$ for i,k=1,2; j=1,2,3, respectively.

Remark 4.1. The obtained functions $\beta_{ij}^{(k)}$, $i,k=1,2;\ j=1,2,3$, are linear functions of the given functions $\alpha_{ij}^{(k)}$, $i,k=1,2;\ j=1,2,3$, and, therefore, indeed satisfy Hölder condition.

Then for the further regularization we replace the expression under the integral sign in the RHS of (26) using boundary conditions (2):

$$\sum_{j=1}^{3}\left(\beta_{ij}^{(1)}(\xi')\frac{\partial u(\xi)}{\partial\xi_j}\left|_{\xi_3=\gamma_1(\xi')}\right.+\beta_{ij}^{(2)}(\xi')\frac{\partial u(\xi)}{\partial\xi_j}\left|_{\xi_3=\gamma_2(\xi')}\right.\right)=\int_S\frac{f_i(x')}{2\pi\,|x'-\xi'|^2}\frac{dx'}{\cos(\nu_x,x_3)}-$$

$$-\int_S\frac{1}{2\pi\,|x'-\xi'|^2}\left[\sum_{k=1}^{2}\alpha_i^{(k)}(x')u(x',\gamma_k(x'))\right]\frac{dx'}{\cos(\nu_x,x_3)}..... \tag{28}$$

Substituting the 1st necessary condition (22) for $u(\xi)$ on Γ_k, $k=1$, 2, into (28), we have:

$$\sum_{j=1}^{3}\left(\beta_{ij}^{(1)}(\xi')\frac{\partial u(\xi)}{\partial\xi_j}\left|_{\xi_3=\gamma_1(\xi')}\right.+\beta_{ij}^{(2)}(\xi')\frac{\partial u(\xi)}{\partial\xi_j}\left|_{\xi_3=\gamma_2(\xi')}\right.\right)=$$

$$=-\int_S\frac{dx'}{2\pi\cos(\nu_x,x_3)\,|x'-\xi'|^2}\sum_{m=1}^{2}\alpha_i^{(m)}(x')\int_S u(\zeta)\left|_{\zeta_3=\gamma_m(\zeta')}\right.\times$$

$$\times\frac{Q_m(\zeta',x')}{2\pi\,|x'-\zeta'|^2\,P_m(x',\zeta')}\frac{d\zeta'}{\cos(\nu_\zeta,\zeta_3)}+\dots .$$

Changing the order of integration we get two regular relationships (k=1,2):

$$\sum_{j=1}^{3}\left(\beta_{ij}^{(1)}(\xi')\frac{\partial u(\xi)}{\partial\xi_j}\left|_{\xi_3=\gamma_1(\xi')}\right.+\beta_{ij}^{(2)}(\xi')\frac{\partial u(\xi)}{\partial\xi_j}\left|_{\xi_3=\gamma_2(\xi')}\right.\right)=\int_S\frac{u(\zeta)\left|_{\zeta_3=\gamma_1(\zeta')}\right.d\zeta'}{2\pi\cos(\nu_\zeta,\zeta_3)}\times$$

$$\times\int_S\sum_{m=1}^{2}\left(\frac{\alpha_i^{(m)}(x')Q_m(\zeta',x')}{2\pi\,|x'-\xi'|^2\,|x'-\zeta'|^2\,P_m(x',\zeta')}\right)\frac{dx'}{\cos(\nu_x,x_3)}+\dots . \tag{29}$$

The internal integrals in the RHS of (29) are singular but they don't contain the unknown function $u(\xi)=u(\xi_1,\ \xi_2,\ \xi_3)$ and converge in Cauchy's sense. Thus, we have regularized relationships (**??**) and , therefore, we have established the following

Theorem 4.1. *Let the conditions of Theorem 3.1 hold true. If system* (27) *is uniquely resolved, the conditions* (2) *are linear independent, the coefficients* $\alpha_{ij}^{(k)}(x')$ *for* $i=1,2;\ j=\overline{1,3};\ k=1,2$, *belong to some Hölder class and the rest of the coefficients and kernels are continuous functions, functions* $f_i(x')$, $i=1,2$, *are continuously differentiable and vanish on the boundary* $\partial S=\bar{S}\backslash S$ *then the relationships* (29) *are regular.*

5. Fredholm property of the problem

It is well-known that

$$\frac{\partial}{\partial x_p}u(x_1,x_2,\gamma_k(x_1,x_2))=\frac{\partial u(x)}{\partial x_p}\Big|_{x_3=\gamma_k(x')}+\frac{\partial u(x)}{\partial x_3}\Big|_{x_3=\gamma_k(x')}\frac{\partial \gamma_k(x')}{\partial x_p},\ k=1,2;p=1,2,$$

whence we have

$$\frac{\partial u(x)}{\partial x_p}\Big|_{x_3=\gamma_k(x')}=\frac{\partial u(x',\gamma_k(x'))}{\partial x_p}-\frac{\partial u(x)}{\partial x_3}\Big|_{x_3=\gamma_k(x')}\frac{\partial \gamma_k(x')}{\partial x_p},\ p=1,2;k=1,2. \tag{30}$$

So, the derivatives $\frac{\partial u(x)}{\partial x_1}\Big|_{x_3=\gamma_k(x')}$ and $\frac{\partial u(x)}{\partial x_2}\Big|_{x_3=\gamma_k(x')}$ are defined through the derivative $\frac{\partial u(x)}{\partial x_3}\Big|_{x_3=\gamma_k(x')}$. Then we have two unknown quantities: the boundary values of the unknown function $u(x',\gamma_1(x'))$ and $u(x',\gamma_2(x'))$.

We substitute the expressions (30) for $\frac{\partial u(x)}{\partial x_1}\Big|_{x_3=\gamma_k(x')}$ and $\frac{\partial u(x)}{\partial x_2}\Big|_{x_3=\gamma_k(x')}$ into boundary conditions (2):

$$l_iu=\sum_{j=1}^{2}\left[\sum_{m=1}^{2}\left[\alpha_{ij}^{(m)}(x')\left(\frac{\partial u(x',\gamma_m(x'))}{\partial x_j}-\frac{\partial u(x)}{\partial x_3}\Big|_{x_3=\gamma_m(x')}\frac{\partial \gamma_m(x')}{\partial x_j}\right)+\right.\right.$$

$$\left.\left.+\alpha_{i3}^{(m)}(x')\frac{\partial u(x)}{\partial x_3}\Big|_{x_3=\gamma_m(x')}+\alpha_{i}^{(m)}(x')u(x',\gamma_m(x'))\right]\right]=-$$

$$=-\sum_{k=1}^{2}\frac{\partial u(x)}{\partial x_3}\Big|_{x_3=\gamma_k(x')}\left[\sum_{m=1}^{2}\alpha_{im}^{(k)}(x')\frac{\partial \gamma_k(x')}{\partial x_m}-\alpha_{i3}^{(k)}(x')\right]+$$

$$+\sum_{j=1}^{2}\left[\alpha_{ij}^{(1)}(x')\frac{\partial u(x',\gamma_1(x'))}{\partial x_j}+\ \alpha_{ij}^{(2)}(x')\frac{\partial u(x',\gamma_2(x'))}{\partial x_j}\right]+$$

$$+\alpha_{i}^{(1)}(x')u(x',\gamma_1(x'))+\alpha_{i}^{(2)}(x')u(x',\gamma_2(x'))=0\,,x'\in S,\ i=1,\,2. \tag{31}$$

Let us introduce the designations:

$$A_{ij}(x')=-\left[\sum_{m=1}^{2}\alpha_{im}^{(1)}(x')\frac{\partial \gamma_1(x')}{\partial x_m}-\alpha_{i3}^{(1)}(x')\right],\ i,j=1,2\,.$$

Then system (31) will be rewritten in the form:

$$A_{i1}(x')\frac{\partial u(x)}{\partial x_3}\Big|_{x_3=\gamma_1(x')}+A_{i2}(x')\frac{\partial u(x)}{\partial x_3}\Big|_{x_3=\gamma_2(x')}=F_i(x')\,,\ i=1,2, \tag{32}$$

where the right-hand sides of system (32) have the form:

$$\mathrm{F_i}(x')=f_i(x')-\sum_{j=1}^{2}\sum_{m=1}^{2}\alpha_{ij}^{(m)}(x')\frac{\partial u(x',\gamma_m(x'))}{\partial x_j}+\sum_{k=1}^{2}\alpha_{i}^{(k)}(x')u(x',\gamma_k(x')),\ x'\in S,\ i=1,\,2. \tag{33}$$

Remark 5.1. Note that the right hand sides $\mathrm{F_i}(x')$ of system (32) are functionals of the boundary values of the desired function and partial derivatives of these boundary values what follows from (33):

$$\mathrm{F_i}(x') = \mathrm{F_i}(x', u\big|_{\Gamma_1}, u\big|_{\Gamma_2}, \frac{\partial u\big|_{\Gamma_1}}{\partial x_1}, \frac{\partial u\big|_{\Gamma_1}}{\partial x_2}, \frac{\partial u\big|_{\Gamma_2}}{\partial x_1}, \frac{\partial u\big|_{\Gamma_2}}{\partial x_2}), i = 1, 2.$$

We'll reduce system (32) to a normal form. For this purpose we require the determinant of the system to be nonzero:

$$\Delta(x') = \begin{vmatrix} A_{11}(x') & A_{12}(x') \\ A_{21}(x') & A_{22}(x') \end{vmatrix} \neq 0. \tag{34}$$

If there holds true condition (34) then by Cramer's formulas we have:

$$\begin{aligned} \frac{\partial u(x)}{\partial x_3}\Big|_{x_3=\gamma_1(x')} &= \frac{1}{\Delta(x')}\begin{vmatrix} F_1(x') & A_{12}(x') \\ F_2(x') & A_{22}(x') \end{vmatrix}, \\ \frac{\partial u(x)}{\partial x_3}\Big|_{x_3=\gamma_2(x')} &= \frac{1}{\Delta(x')}\begin{vmatrix} A_{11}(x') & F_1(x') \\ A_{21}(x') & F_2(x') \end{vmatrix}. \end{aligned} \tag{35}$$

System (35) is a system of integral Fredholm equations of 2-nd kind with a regular kernel. Therefore, it has a unique solution which in in the virtue of (34) has the form:

$$\frac{\partial u(x)}{\partial x_3}\Big|_{x_3=\gamma_k(x')} = \Phi_k(u\big|_{\Gamma_1}, u\big|_{\Gamma_2}, \frac{\partial u\big|_{\Gamma_1}}{\partial x_1}, \frac{\partial u\big|_{\Gamma_1}}{\partial x_2}, \frac{\partial u\big|_{\Gamma_2}}{\partial x_1}, \frac{\partial u\big|_{\Gamma_2}}{\partial x_2}), k = 1, 2. \tag{36}$$

Let us substitute expressions (30) for the derivatives $\frac{\partial u(x)}{\partial x_j}\big|_{x_3=\gamma_k(x')}$, $j, k = 1, 2,$ into regular relationships (29):

$$\sum_{j=1}^{3}\left(\beta_{ij}^{(1)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\Big|_{\xi_3=\gamma_1(\xi')} + \beta_{ij}^{(2)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\Big|_{\xi_3=\gamma_2(\xi')}\right) =$$

$$= \sum_{j=1}^{2}\sum_{m=1}^{2}\beta_{ij}^{(m)}(\xi')\left(\frac{\partial u(\xi', \gamma_m(\xi'))}{\partial \xi_j} - \frac{\partial u(\xi)}{\partial \xi_3}\Big|_{\xi_3=\gamma_m(\xi')}\frac{\partial \gamma_m(\xi')}{\partial \xi_j}\right) +$$

$$+\beta_{i3}^{(1)}(\xi')\frac{\partial u(\xi)}{\partial \xi_3}\Big|_{\xi_3=\gamma_1(\xi')} + \beta_{i3}^{(2)}(\xi')\frac{\partial u(\xi)}{\partial \xi_3}\Big|_{\xi_3=\gamma_2(\xi')} =$$

$$= -\sum_{k=1}^{2}\int_S \frac{u(\zeta)\big|_{\zeta_3=\gamma_k(\zeta')}\, d\zeta'}{\cos(\nu_\zeta, \zeta_3)}\int_S \alpha_i^{(k)}(x')\frac{\cos(\zeta-\xi, \nu_\zeta)\Big|_{\substack{\zeta_3=\gamma_k(\zeta') \\ x_3=\gamma_k(x')}}}{2\pi|x'-\zeta'|^2 P_k(x', \zeta')}\times$$

$$\times\frac{dx'}{\cos(\nu_x, x_3)} + \int_S \frac{f_i(x')}{2\pi|x'-\xi'|^2}\frac{dx'}{\cos(\nu_x, x_3)} + ... \,. \tag{37}$$

Let us group the terms in (37) as follows:

$$-\sum_{k=1}^{2}\frac{\partial u(\xi)}{\partial \xi_3}\Big|_{\xi_3=\gamma_k(\xi')}\left[\sum_{m=1}^{2}\beta_{im}^{(k)}(\xi')\frac{\partial \gamma_k(\xi')}{\partial \xi_m}-\beta_{i3}^{(k)}(\xi')\right]+$$

$$+\sum_{j=1}^{2}\left[\beta_{ij}^{(1)}(\xi')\frac{\partial u(\xi',\gamma_1(\xi'))}{\partial \xi_j}+\beta_{ij}^{(2)}(\xi')\frac{\partial u(\xi',\gamma_2(\xi'))}{\partial \xi_j}\right]=$$

$$=-\sum_{k=1}^{2}\int_S\frac{u(\zeta)\big|_{\zeta_3=\gamma_k(\zeta')}\,d\zeta'}{\cos(\nu_\zeta,\zeta_3)}\int_S\alpha_i^{(k)}(x')\frac{\cos(\zeta-\xi,\nu_\zeta)\Big|_{\substack{\zeta_3=\gamma_k(\zeta')\\ x_3=\gamma_k(x')}}}{2\pi\,|x'-\zeta'|^2\,P_k(x',\zeta')}\times$$

$$\times\frac{dx'}{\cos(\nu_x,x_3)}+\int_S\frac{f_i(x')}{2\pi\,|x'-\xi'|^2}\frac{dx'}{\cos(\nu_x,x_3)}+\dots\ . \tag{38}$$

The terms in (38) have either weakly singular kernels or regular ones (inside three dots).

If we introduce the designations:

$$C_{ij}(\xi')=-\left[\sum_{m=1}^{2}\beta_{im}^{(j)}(\xi')\frac{\partial \gamma_j(\xi')}{\partial \xi_m}-\beta_{i3}^{(j)}(\xi')\right],\ i,j=1,2,$$

$$B_i(\xi')=-\sum_{j=1}^{2}\sum_{m=1}^{2}\beta_{ij}^{(m)}(\xi')\frac{\partial u(\xi',\gamma_m(\xi'))}{\partial \xi_j}-\sum_{k=1}^{2}\int_S\frac{u(\zeta)\big|_{\zeta_3=\gamma_k(\zeta')}\,d\zeta'}{\cos(\nu_\zeta,\zeta_3)}\times$$

$$\times\int_S\alpha_i^{(k)}(x')\frac{\cos(\zeta-\xi,\nu_\zeta)\Big|_{\substack{\zeta_3=\gamma_k(\zeta')\\ x_3=\gamma_k(x')}}}{2\pi\,|x'-\zeta'|^2\,P_k(x',\zeta')}\frac{dx'}{\cos(\nu_x,x_3)}+$$

$$+\int_S\frac{f_i(x')}{2\pi\,|x'-\xi'|^2}\frac{dx'}{\cos(\nu_x,x_3)}+\dots,\ \xi'\in S,\ i=1,2,$$

then system (38) can be rewritten in the form:

$$\sum_{m=1}^{2}C_{im}(\xi')\frac{\partial u(\xi)}{\partial \xi_3}\Big|_{\xi_3=\gamma_m(\xi')}=B_i(\xi')\ ,\ i=1,2. \tag{39}$$

In the virtue of remark 3.1 system (39), or (38), is a system of integral Fredholm equations of the second kind with respect to $\frac{\partial u(\xi)}{\partial \xi_3}\big|_{\xi_3=\gamma_k(\xi')}$, $k=1,2$. Consequently, the system has the unique solution. As the right hand sides $B_i(\xi')$ are linear functionals of the boundary values of the desired function and partial derivatives of these boundary values then we have that

$$\frac{\partial u(\xi)}{\partial \xi_3}\Big|_{\xi_3=\gamma_k(\xi')} = \Psi_k\Big(u\Big|_{\Gamma_1},\ u\Big|_{\Gamma_2},\frac{\partial u\big|_{\Gamma_1}}{\partial \xi_1},\ \frac{\partial u\big|_{\Gamma_2}}{\partial \xi_1},\ \frac{\partial u\big|_{\Gamma_1}}{\partial \xi_2},\ \frac{\partial u\big|_{\Gamma_2}}{\partial \xi_2}\Big), \tag{40}$$

where $u\Big|_{\Gamma_k} = u(\xi', \gamma_k(\xi'))$, $\frac{\partial u|_{\Gamma_k}}{\partial \xi_j} = \frac{\partial u(\xi', \gamma_k(\xi'))}{\partial \xi_j}$, $j = 1, 2, k = 1, 2$, are the boundary values of the desired solution $u(x)$ on the surfaces Γ_k, $k = 1, 2$, and the derivatives of its boundary values correspondingly.

The functionals Φ_k, Ψ_k, $k = 1, 2$, from (36) and (40) are linear with respect to the unknown values $u\Big|_{\Gamma_1}$, $u\Big|_{\Gamma_2}$, $\frac{\partial u|_{\Gamma_1}}{\partial \xi_j}$, $\frac{\partial u|_{\Gamma_2}}{\partial \xi_j}$, $j = 1, 2$:

$$\Phi_k\Big(u\Big|_{\Gamma_1},\ u\Big|_{\Gamma_2},\frac{\partial u\big|_{\Gamma_1}}{\partial \xi_1},\ \frac{\partial u\big|_{\Gamma_2}}{\partial \xi_1},\ \frac{\partial u\big|_{\Gamma_1}}{\partial \xi_2},\ \frac{\partial u\big|_{\Gamma_2}}{\partial \xi_2}\Big) =$$

$$= \sum_{i=1}^{2} a_i^{(k)}(\xi')u\Big|_{\Gamma_i} + \sum_{i,j=1}^{2} b_{ij}^{(k)}(\xi')\frac{\partial u\big|_{\Gamma_i}}{\partial \xi_j} + \sum_{i=1}^{2}\int_S c_i^{(k)}(\zeta')u\Big|_{\Gamma_i}\,d\zeta +$$

$$+ \sum_{i,j=1}^{2}\int_S d_{ij}^{(k)}(\zeta')\frac{\partial u\big|_{\Gamma_i}}{\partial \zeta_j}d\zeta + \varphi_k(\xi'),\ k = 1,\ 2, \tag{41}$$

$$\Psi_k\Big(u\Big|_{\Gamma_1},\ u\Big|_{\Gamma_2},\frac{\partial u\big|_{\Gamma_1}}{\partial \xi_1},\ \frac{\partial u\big|_{\Gamma_2}}{\partial \xi_1},\ \frac{\partial u\big|_{\Gamma_1}}{\partial \xi_2},\ \frac{\partial u\big|_{\Gamma_2}}{\partial \xi_2}\Big) =$$

$$= \sum_{i=1}^{2} a_i^{(l)}(\xi')u\Big|_{\Gamma_i} + \sum_{i,j=1}^{2} b_{ij}^{(l)}(\xi')\frac{\partial u\big|_{\Gamma_i}}{\partial \xi_j} + \sum_{i=1}^{2}\int_S c_i^{(l)}(\zeta')u\Big|_{\Gamma_i}\,d\zeta +$$

$$+ \sum_{i,j=1}^{2}\int_S d_{ij}^{(l)}(\zeta')\frac{\partial u\big|_{\Gamma_i}}{\partial \zeta_j}d\zeta + \varphi_l(\xi'),\ l = 3, 4;\ k = 1,\ 2. \tag{42}$$

Excluding $\frac{\partial u(\xi)}{\partial \xi_3}\Big|_{\xi_3=\gamma_k(\xi')}$, $k = 1, 2$, from system (41), (42) we'll obtain a system of linear integro-differential Fredholm equations of the second kind with respect to $u(\xi', \gamma_k(\xi'))$, $k = 1, 2$:

$$\sum_{i=1}^{2} A_i^{(k)}(\xi')u\Big|_{\Gamma_i} + \sum_{i,j=1}^{2} B_{ij}^{(k)}(\xi')\frac{\partial u\big|_{\Gamma_i}}{\partial \xi_j} + \sum_{i=1}^{2}\int_S C_i^{(k)}(\zeta')u\Big|_{\Gamma_i}\,d\zeta +$$

$$+ \sum_{i,j=1}^{2}\int_S D_{ij}^{(k)}(\zeta')\frac{\partial u\big|_{\Gamma_i}}{\partial \zeta_j}d\zeta + g_k(\xi') = 0, \quad k = 1, 2, \tag{43}$$

where

$$A_i^{(k)}(\xi') = a_i^{(k)}(\xi') - a_i^{(k+2)}(\xi'),\ B_{ij}^{(k)}(\xi') = b_{ij}^{(k)}(\xi') - b_{ij}^{(k+2)}(\xi'),$$

$$C_i^{(k)}(\zeta') = c_i^{(k)}(\zeta') - c_i^{(k+2)}(\zeta'), D_{ij}^{(k)}(\zeta') = d_{ij}^{(k)}(\zeta') - d_{ij}^{(k+2)}(\zeta'),$$

$$g_k(\xi') = \varphi_k(\xi') - \varphi_{k+2}(\xi'), k = 1,\ 2.$$

Thus, we have come to a two-dimensional system of linear integro-differential equations of the first order for which Dirichlet's conditions (3) are given on the boundary $\partial S = \overline{S}\backslash S = \bar{\Gamma}_1 \bigcap \bar{\Gamma}_2$ of a two-dimensional domain S. As this boundary is one-dimensional then this Dirichlet's condition doesn't restrict the generality because its dimension is two units less than the dimension of the domain D.

Thus, we have established the following

Theorem 5.1. *If the assumptions of Theorem 4.1 and conditions* (34) *hold true and system* (39) *is uniquely resolved then boundary-value problem* (1)-(2) *is reduced to a two-dimensional system of linear integro-differential equations* (43) *with Dirichlet's condition* (3) *on the boundary $\partial S = \overline{S}\backslash S$.*

Finally, there has been established

Theorem 5.2. *If the assumptions of Theorem 5.1 hold true then boundary value problem* (1), (2), (3) *has Fredholm property.*

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