Filomat 32:13 (2018), 4619–4625
https://doi.org/10.2298/FIL1813619C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Family of Multivalent Analytic Functions Associated with Srivastava-Tomovski Generalization of the Mittag-Leffler Function

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Abstract. In this paper we introduce an operator associated with Srivastava-Tomovski generalization of the Mittag-Leffler function. By using this operator and the virtue of differential subordination, we define a new family of multivalent analytic functions. Some novel properties such as inclusion relation, Hadamard product and the Fekete-Szegö inequality of this new family are discussed.

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=2}^{\infty} a_{n+p-1} z^{n+p-1} \quad (p \in \mathbb{N}) \tag{1.1}$$

which are analytic in $\mathbb{U} = \{z : z \in \mathbb{C} \quad \textit{and} \quad |z| < 1\}$. For $p = 1$, we write $\mathcal{A} := \mathcal{A}(1)$. The Hadamard product (or convolution) of two functions

$$f_j(z) = z^p + \sum_{n=2}^{\infty} a_{n+p-1,j} z^{n+p-1} \in \mathcal{A}(p) \quad (j = 1, 2)$$

is given by

$$(f_1 * f_2)(z) = z^p + \sum_{n=2}^{\infty} a_{n+p-1,1} a_{n+p-1,2} z^{n+p-1} = (f_2 * f_1)(z).$$

Throughout this paper, unless otherwise indicated, we assume that

$$\alpha, \beta, \gamma, k \in \mathbb{C}; \quad \mathrm{Re}(\alpha) > \max\{0, \mathrm{Re}(k) - 1\} \quad \text{and} \quad \mathrm{Re}(k) > 0.$$

2010 *Mathematics Subject Classification*. Primary 33E12, 30C45

Keywords. Analytic function; Srivastava-Wright operator; Srivastava-Tomovski generalization of the Mittag-Leffler function; subordination; Hadamard product (convolution); convex univalent; Fekete-Szegö inequality.

Received: 04 January 2018; Accepted: 08 August 2018

Communicated by Hari M. Srivastava

This work was supported by National Natural Science Foundation of China (Grant No.11571299), Natural Science Foundation of Jiangsu Province (Grant No. BK20151304) and Natural Science Foundation of Jiangsu Gaoxiao (Grant No.17KJB110019).

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Recently, Srivastava and Tomovski [18] defined a generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,k}(z)$ as follows:

$$E_{\alpha,\beta}^{\gamma,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk} z^n}{\Gamma(\alpha n + \beta) n!}, \tag{1.2}$$

where $(x)_n$ is the Pochhammer symbol

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\cdots(x+n-1) \quad (n \in \mathbb{N}; \quad x \in \mathbb{C})$$

and $(x_0) = 1$. They proved that the function $E_{\alpha,\beta}^{\gamma,k}(z)$ defined by (1.2) is an entire function in the complex z-plane. The function $E_{\alpha,\beta}^{\gamma,k}(z)$ is called Srivastava-Tomovski generalization of the Mittag-Leffler function.

For $f \in \mathcal{A}(p)$, we introduce the following new operator $H_{\alpha,\beta}^{\gamma,k} : \mathcal{A}(p) \to \mathcal{A}(p)$ associated with the Srivastava-Tomovski generalization of the Mittag-Leffler function by

$$\begin{aligned} H_{\alpha,\beta}^{\gamma,k} f(z) &= \frac{\Gamma(\alpha+\beta)}{(\gamma)_k} z^{p-1} \left(E_{\alpha,\beta}^{\gamma,k}(z) - \frac{1}{\Gamma(\beta)} \right) * f(z) \\ &= z^p + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma+nk)\Gamma(\alpha+\beta)}{\Gamma(\gamma+k)\Gamma(\alpha n+\beta) n!} a_{n+p-1} z^{n+p-1}. \end{aligned} \tag{1.3}$$

Note that $H_{0,\beta}^{1,1} f(z) = f(z)$. From (1.3) we easily have the following identity:

$$z\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)' = \left(\frac{\gamma}{k}+1\right) H_{\alpha,\beta}^{\gamma+1,k} f(z) - \left(\frac{\gamma}{k}+1-p\right) H_{\alpha,\beta}^{\gamma,k} f(z). \tag{1.4}$$

It should be remarked in passing that the Fox-Wright hypergeometric function ${}_q\Psi_s$ is much more general than many of the extensions of the Mittag-Leffler function. The study of the more complicated and general case of the Srivastava-Wright operator (see [18, 4]), defined by the Fox-Wright function ${}_q\Psi_s$, is a recent interesting topic in Geometric Function Theory. Many properties of the Srivastava-Wright operator can be found in a number of recent works [1, 2, 3, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and the references therein.

Suppose that f and g are analytic in $\mathbb{U}$. We say that the function f is subordinate to g and write $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function w, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. If g is univalent in $\mathbb{U}$, then the following equivalence relationship holds true:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\mathcal{P}$ be the class of functions φ with $\varphi(0) = 1$, which are analytic and convex univalent in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\rho)$ if $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \rho$ $(z \in \mathbb{U})$ for some ρ $(\rho < 1)$. When $0 \le \rho < 1$, $\mathcal{S}^*(\rho)$ is the class of starlike functions of order ρ in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be prestarlike of order ρ in $\mathbb{U}$ if $\frac{z}{(1-z)^{2(1-\rho)}} * f(z) \in \mathcal{S}^*(\rho)$ $(\rho < 1)$. We denote this class by $\mathcal{R}(\rho)$ (see [9]). Clearly, a function $f \in \mathcal{A}$ is in the class $\mathcal{R}(0)$ if and only if f is convex univalent in $\mathbb{U}$ and $\mathcal{R}\left(\frac{1}{2}\right) = \mathcal{S}^*\left(\frac{1}{2}\right)$.

Definition. A function $f \in \mathcal{A}(p)$ is said to be in the class $\Omega_{\alpha,\beta}^{\gamma,k}(\lambda;\varphi)$ if it satisfies first-order differential subordination:

$$(1-\lambda) z^{-p} H_{\alpha,\beta}^{\gamma,k} f(z) + \frac{\lambda}{p} z^{-p+1} \left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)' \prec \varphi(z), \tag{1.5}$$

where $\lambda \in \mathbb{C}$ and $\varphi \in \mathcal{P}$.

Lemma 1 ([7]). *Let g be analytic in $\mathbb{U}$ and h be analytic and convex univalent in $\mathbb{U}$ with $h(0) = g(0)$. If*

$$g(z) + \frac{1}{\mu} z g'(z) \prec h(z),$$

where $\mathrm{Re}\mu \geq 0$ *and* $\mu \neq 0$, *then*

$$g(z) \prec \widetilde{h}(z) = \mu z^{-\mu} \int_0^z t^{\mu-1} h(t) dt \prec h(z) \tag{1.6}$$

and $\widetilde{h}$ *is the best dominant of* (1.6).

Lemma 2 ([9]). *Let* $\rho < 1$, $f \in \mathcal{S}^*(\rho)$ *and* $g \in \mathcal{R}(\rho)$. *Then, for any analytic function* F *in* $\mathbb{U}$,

$$\frac{g * (fF)}{g * f}(\mathbb{U}) \subset \overline{co}(F(\mathbb{U})),$$

where $\overline{co}(F(\mathbb{U}))$ *denotes the closed convex hull of* $F(\mathbb{U})$.

Lemma 3 ([6]). *If* $g(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *is an analytic function with positive real part in* $\mathbb{U}$ *and* μ *is a complex number, then* $|c_2 - \mu c_1^2| \leq 2\max\{1; |2\mu - 1|\}$.

2. Properties of the class $\Omega_{\alpha,\beta}^{\gamma,k}(\lambda;\varphi)$

Theorem 1. *Let* $0 \leq \lambda_1 < \lambda_2$. *Then* $\Omega_{\alpha,\beta}^{\gamma,k}(\lambda_2;\varphi) \subset \Omega_{\alpha,\beta}^{\gamma,k}(\lambda_1;\varphi)$.

Proof. Let

$$g(z) = z^{-p} H_{\alpha,\beta}^{\gamma,k} f(z) \tag{2.1}$$

for $f \in \Omega_{\alpha,\beta}^{\gamma,k}(\lambda_2;\varphi)$. Then the function g is analytic in $\mathbb{U}$ and $g(0) = 1$. Differentiating both sides of (2.1), we have

$$\begin{aligned}(1-\lambda_2) z^{-p} H_{\alpha,\beta}^{\gamma,k} f(z) + \frac{\lambda_2}{p} z^{-p+1}\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)' \\ = g(z) + \frac{\lambda_2}{p} z g'(z) \prec \varphi(z).\end{aligned} \tag{2.2}$$

Hence, by an application of Lemma 1, we have $g(z) \prec \varphi(z)$ $(z \in \mathbb{U})$. Now, by noting that $0 \leq \frac{\lambda_1}{\lambda_2} \leq 1$ and that φ is convex univalent in $\mathbb{U}$, it follows that

$$\begin{aligned}&(1-\lambda_1) z^{-p} H_{\alpha,\beta}^{\gamma,k} f(z) + \frac{\lambda_1}{p} z^{-p+1}\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)' \\ &= \frac{\lambda_1}{\lambda_2}\left((1-\lambda_2) z^{-p} H_{\alpha,\beta}^{\gamma,k} f(z) + \frac{\lambda_2}{p} z^{-p+1}\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)'\right) + \left(1 - \frac{\lambda_1}{\lambda_2}\right) g(z) \\ &\prec \varphi(z) \quad (z \in \mathbb{U}).\end{aligned}$$

Thus $f \in \Omega_{\alpha,\beta}^{\gamma,k}(\lambda_1;\varphi)$ and the proof of Theorem 1 is completed.

Theorem 2. *Let* $0 < \gamma_1 < \gamma_2$ *and* $k = 1$. *Then* $\Omega_{\alpha,\beta}^{\gamma_2,1}(\lambda;\varphi) \subset \Omega_{\alpha,\beta}^{\gamma_1,1}(\lambda;\varphi)$.

Proof. Put

$$g(z) = z + \sum_{n=2}^{\infty} \frac{(\gamma_1)_{n-1}}{(\gamma_2)_{n-1}} z^n \quad (z \in \mathbb{U}). \tag{2.3}$$

Then $g \in \mathcal{A}$ and

$$\frac{z}{(1-z)^{\gamma_2}} * g(z) = \frac{z}{(1-z)^{\gamma_1}}. \tag{2.4}$$

We can see from (2.4) that

$$\frac{z}{(1-z)^{\gamma_2}} * g(z) \in \mathcal{S}^*\left(1-\frac{\gamma_1}{2}\right) \subset \mathcal{S}^*\left(1-\frac{\gamma_2}{2}\right)$$

for $0 < \gamma_1 < \gamma_2$, which implies that $g \in \mathcal{R}\left(1-\frac{\gamma_2}{2}\right)$.

Let $f \in \Omega^{\gamma_2,1}_{\alpha,\beta}(\lambda;\varphi)$. Then we have

$$\begin{aligned}&(1-\lambda)z^{-p}H^{\gamma_1,1}_{\alpha,\beta}f(z) + \frac{\lambda}{p}z^{-p+1}\left(H^{\gamma_1,1}_{\alpha,\beta}f(z)\right)' \\ &= \frac{g(z)}{z} * \psi(z) = \frac{g(z) * (z\psi(z))}{g(z) * z},\end{aligned} \tag{2.5}$$

where

$$\psi(z) = (1-\lambda)z^{-p}H^{\gamma_2,1}_{\alpha,\beta}f(z) + \frac{\lambda}{p}z^{-p+1}\left(H^{\gamma_2,1}_{\alpha,\beta}f(z)\right)' \prec \varphi(z). \tag{2.6}$$

Since the function z belongs to $\mathcal{S}^*\left(1-\frac{\gamma_2}{2}\right)$ and φ is convex univalent in $\mathbb{U}$, it follows from (2.5), (2.6) and Lemma 2 that

$$(1-\lambda)z^{-p}H^{\gamma_1,1}_{\alpha,\beta}f(z) + \frac{\lambda}{p}z^{-p+1}\left(H^{\gamma_1,1}_{\alpha,\beta}f(z)\right)' \prec \varphi(z).$$

Thus $f \in \Omega^{\gamma_1,1}_{\alpha,\beta}(\lambda;\varphi)$. The proof of Theorem 2 is completed.

Theorem 3. *Let* $\mathrm{Re}\left(\frac{g(z)}{z}\right) > \frac{1}{2}$ $(z \in \mathbb{U})$, *where the function* g *is given by*

$$g(z) = z + \sum_{n=2}^{\infty} \frac{(\gamma_2)_k(\gamma_1)_{nk}}{(\gamma_1)_k(\gamma_2)_{nk}} z^n \quad (z \in \mathbb{U}).$$

Then $\Omega^{\gamma_2,k}_{\alpha,\beta}(\lambda;\varphi) \subset \Omega^{\gamma_1,k}_{\alpha,\beta}(\lambda;\varphi)$.

Proof. For $f \in \mathcal{A}(p)$, it is easy to verify that

$$z^{-p}H^{\gamma_1,k}_{\alpha,\beta}f(z) = \left(\frac{g(z)}{z}\right) * \left(z^{-p}H^{\gamma_2,k}_{\alpha,\beta}f(z)\right) \tag{2.7}$$

and

$$z^{-p+1}\left(H^{\gamma_1,k}_{\alpha,\beta}f(z)\right)' = \left(\frac{g(z)}{z}\right) * \left(z^{-p+1}\left(H^{\gamma_2,k}_{\alpha,\beta}f(z)\right)'\right). \tag{2.8}$$

Let $f \in \Omega^{\gamma_2,k}_{\alpha,\beta}(\lambda;\varphi)$. Then from (2.7) and (2.8) we deduce that

$$\begin{aligned}&(1-\lambda)z^{-p}H^{\gamma_1,k}_{\alpha,\beta}f(z) + \frac{\lambda}{p}z^{-p+1}\left(H^{\gamma_1,k}_{\alpha,\beta}f(z)\right)' \\ &= \frac{g(z)}{z} * \psi(z) = \frac{g(z) * (z\psi(z))}{g(z) * z},\end{aligned}$$

where

$$\psi(z) = (1-\lambda)z^{-p}H^{\gamma_2,k}_{\alpha,\beta}f(z) + \frac{\lambda}{p}z^{-p+1}\left(H^{\gamma_2,k}_{\alpha,\beta}f(z)\right)' \prec \varphi(z).$$

In view of the assumptions of Theorem 3, the function $\frac{g(z)}{z}$ has the following Herglotz representation:

$$\frac{g(z)}{z} = \int_{|x|=1} \frac{d\mu(x)}{1-xz} \quad (z \in \mathbb{U}), \tag{2.9}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and $\int_{|x|=1} d\mu(x) = 1$. Since φ is convex univalent in $\mathbb{U}$, it follows from (2.7) to (2.9) that

$$\begin{aligned}&(1-\lambda)z^{-p}H_{\alpha,\beta}^{\gamma_1,k}f(z) + \frac{\lambda}{p}z^{-p+1}\left(H_{\alpha,\beta}^{\gamma_1,k}f(z)\right)' \\ &= \int_{|x|=1}\psi(xz)d\mu(x) \prec \varphi(z).\end{aligned}$$

This shows that $f \in \Omega_{\alpha,\beta}^{\gamma_1,k}(\lambda;\varphi)$. The proof of Theorem 3 is completed.

Theorem 4. *Let $-1 \le B < A \le 1$, $\lambda > 0$ and $\delta \ge 1$. If $f \in \Omega_{\alpha,\beta}^{\gamma,k}\left(\lambda; \frac{1+Az}{1+Bz}\right)$, then*

$$\mathrm{Re}\left\{\left(\frac{H_{\alpha,\beta}^{\gamma,k}f(z)}{z^p}\right)^{1/\delta}\right\} > \left(\frac{p}{\lambda}\int_0^1 u^{\frac{p}{\lambda}-1}\left(\frac{1-Au}{1-Bu}\right)du\right)^{1/\delta}. \tag{2.10}$$

The result is sharp.

Proof. Let

$$g(z) = \frac{H_{\alpha,\beta}^{\gamma,k}f(z)}{z^p}$$

for $f \in \Omega_{\alpha,\beta}^{\gamma,k}\left(\lambda; \frac{1+Az}{1+Bz}\right)$. Then the function $g(z) = 1 + b_1z + b_2z^2 + \cdots$ is analytic in $\mathbb{U}$. By a simple calculation we have from (1.5) that

$$g(z) + \frac{\lambda}{p}zg'(z) \prec \frac{1+Az}{1+Bz}.$$

Now an application of Lemma 1 leads to

$$g(z) \prec \frac{p}{\lambda}z^{-\frac{p}{\lambda}}\int_0^z t^{\frac{p}{\lambda}-1}\left(\frac{1+At}{1+Bt}\right)dt$$

or

$$\frac{H_{\alpha,\beta}^{\gamma,k}f(z)}{z^p} = \frac{p}{\lambda}\int_0^1 u^{\frac{p}{\lambda}-1}\left(\frac{1+Auw(z)}{1+Buw(z)}\right)du, \tag{2.11}$$

where $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$.

In view of $-1 \le B < A \le 1$ and $\lambda > 0$, it follows from (2.11) that

$$\mathrm{Re}\left(\frac{H_{\alpha,\beta}^{\gamma,k}f(z)}{z^p}\right) > \frac{p}{\lambda}\int_0^1 u^{\frac{p}{\lambda}-1}\left(\frac{1-Au}{1-Bu}\right)du > 0 \quad (z \in \mathbb{U}). \tag{2.12}$$

Therefore, with the aid of the elementary inequality $\mathrm{Re}\left(w^{1/\delta}\right) \ge (\mathrm{Re}w)^{1/\delta}$ for $\mathrm{Re}w > 0$ and $\delta \ge 1$, the inequality (2.10) follows directly from (2.12).

To show the sharpness of (2.10), we take $f \in \mathcal{A}(p)$ defined by

$$\frac{H_{\alpha,\beta}^{\gamma,k}f(z)}{z^p} = \frac{p}{\lambda}\int_0^1 u^{\frac{p}{\lambda}-1}\left(\frac{1+Auz}{1+Buz}\right)du.$$

For this function, we find that

$$(1-\lambda)z^{-p}H_{\alpha,\beta}^{\gamma,k}f(z) + \frac{\lambda}{p}z^{-p+1}\left(H_{\alpha,\beta}^{\gamma,k}f(z)\right)' = \frac{1+Az}{1+Bz}$$

and

$$\frac{H_{\alpha,\beta}^{\gamma,k}f(z)}{z^p} \to \frac{p}{\lambda}\int_0^1 u^{\frac{p}{\lambda}-1}\left(\frac{1-Au}{1-Bu}\right)du$$

as $z \to -1$. The proof of Theorem 4 is completed.

Theorem 5. *Let α, β, γ, k and λ be positive real numbers. Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots \in \mathcal{P}$ with $B_1 \neq 0$. If f given by* (1.1) *belongs to the class $\Omega_{\alpha,\beta}^{\gamma,k}(\lambda;\varphi)$, then*

$$\begin{aligned}&\left|a_{p+2} - \mu a_{p+1}^2\right| \\ &\leq \frac{6p\Gamma(\gamma+k)\Gamma(3\alpha+\beta)}{(p+2\lambda)\Gamma(\gamma+3k)\Gamma(\alpha+\beta)} \max\left\{|B_1|; \left|B_2 - \frac{2\mu p B_1^2 (p+2\lambda)\Gamma(\gamma+k)\Gamma(\gamma+3k)(\Gamma(2\alpha+\beta))^2}{3(p+\lambda)^2\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta)(\Gamma(\gamma+2k))^2}\right|\right\}\end{aligned} \tag{2.13}$$

for $\mu \in \mathbb{C}$.

Proof. If $f \in \Omega_{\alpha,\beta}^{\gamma,k}(\lambda;\varphi)$, then there is a Schwarz function w, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{U}$ such that

$$(1-\lambda)z^{-p}H_{\alpha,\beta}^{\gamma,k}f(z) + \frac{\lambda}{p}z^{-p+1}\left(H_{\alpha,\beta}^{\gamma,k}f(z)\right)' = \varphi(w(z)). \tag{2.14}$$

Define the function g by

$$g(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{2.15}$$

Since w is a Schwarz function, we see that $\mathrm{Re}(g(z)) > 0$ and $g(0) = 1$. Therefore,

$$\begin{aligned}\varphi(w(z)) &= \varphi\left(\frac{g(z)-1}{g(z)+1}\right) \\ &= 1 + \frac{B_1 c_1}{2}z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right]z^2 + \cdots.\end{aligned} \tag{2.16}$$

By substituting (2.16) in (2.14), we have

$$(1-\lambda)z^{-p}H_{\alpha,\beta}^{\gamma,k}f(z) + \frac{\lambda}{p}z^{-p+1}\left(H_{\alpha,\beta}^{\gamma,k}f(z)\right)' = 1 + \frac{B_1 c_1}{2}z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right]z^2 + \cdots.$$

From this equation, we get

$$a_{p+1} = \frac{pB_1 c_1 \Gamma(\gamma+k)\Gamma(2\alpha+\beta)}{(p+\lambda)\Gamma(\gamma+2k)\Gamma(\alpha+\beta)}$$

and

$$a_{p+2} = \frac{3p\Gamma(\gamma+k)\Gamma(3\alpha+\beta)}{(p+2\lambda)\Gamma(\gamma+3k)\Gamma(\alpha+\beta)}\left(B_1 c_2 - \frac{B_1 c_1^2}{2} + \frac{B_2 c_1^2}{2}\right).$$

Therefore

$$a_{p+2} - \mu a_{p+1}^2 = \frac{3pB_1\Gamma(\gamma+k)\Gamma(3\alpha+\beta)}{(p+2\lambda)\Gamma(\gamma+3k)\Gamma(\alpha+\beta)}\left(c_2 - \zeta c_1^2\right), \tag{2.17}$$

where

$$\zeta = \frac{1}{2}\left(1 - \frac{B_2}{B_1}\right) + \frac{\mu p B_1 (p+2\lambda)\Gamma(\gamma+k)\Gamma(\gamma+3k)(\Gamma(2\alpha+\beta))^2}{3(p+\lambda)^2\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta)(\Gamma(\gamma+2k))^2}. \tag{2.18}$$

Our result now follows by an application of Lemma 3. This completes the proof of Theorem 5.

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