Filomat 32:13 (2018), 4485–4502
https://doi.org/10.2298/FIL1813485S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Numerical Method for Solving a Class of Fractional Optimal Control Problems Using Boubaker Polynomial Expansion Scheme

N. Singha[a], C. Nahak[a]

[a] *Indian Institute of Technology Kharagpur, West Bengal-721302, India*

Abstract. We construct a numerical scheme for solving a class of fractional optimal control problems by employing Boubaker polynomials. In the proposed scheme, the state and control variables are approximated by practicing N^{th}-order Boubaker polynomial expansion. With these approximations, the given performance index is transformed to a function of $N + 1$ unknowns. The objective of the present formulation is to convert a fractional optimal control problem with quadratic performance index into an equivalent quadratic programming problem with linear equality constraints. Thus, the latter problem can be handled efficiently in comparison to the original problem. We solve several examples to exhibit the applicability and working mechanism of the presented numerical scheme. Graphical plots are provided to monitor the nature of the state, control variable and the absolute error function. All the numerical computations and graphical representations have been executed with the help of Mathematica software.

1. Introduction

Fractional optimal control problem (FOCP) is an extension of the classical optimal control problem, in which the system dynamical constraints are described with fractional order operators. Nowadays, fractional derivatives have gained the attention of researchers in describing the properties not considered by integer order derivatives. For the historical development and applications of fractional calculus, we refer the reader to [17, 20, 21].

A variety of FOCPs are available in the literature (see [1],[18],[22]), but Agrawal [1] first introduced the simplest FOCP by using Riemann-Liouville fractional derivative in the governing differential equation of the system dynamical constraints. This formulation was further enhanced by Agrawal [3] with Caputo fractional derivatives. Some of the FOCP formulations have been suggested in [2, 22, 25]. The complexity of obtaining analytical solutions of FOCPs leads to explore numerical methods. Previously, different numerical techniques have been applied to solve such problems which include a central difference numerical scheme [5], Legendre multi-wavelet collocation method [30], a discrete numerical method [4], approximation method [29], and Beizer curves method [13].

Recently, special attention has been given to find the numerical solution of these FOCPs (see [7, 10–12, 14, 26]) by using orthogonal polynomials expansion. The approximation of a function by orthogonal

2010 *Mathematics Subject Classification*. 26A33, 49M25, 49M30, 49M37
Keywords. Fractional optimal control problem, Riemann-Liouville fractional operators, Caputo fractional derivatives, Quadratic programming problem, Boubaker polynomials
Received: 25 March 2018; Accepted: 03 June 2018
Communicated by Marko Petković
Email addresses: neelam.singha1990@gmail.com (N. Singha), cnahak@maths.iitkgp.ernet.in (C. Nahak)

polynomials scales down the complexity of the fractional dynamical system by reducing it to a simpler system of algebraic equations. For example, Chebyshev polynomials [14], modified Jacobi polynomials [10], shifted Jacobi polynomials [11], Legendre polynomials [12], Chebyshev-Legendre operational technique [7], Laguerre polynomials [26], have already been used to construct solution schemes for a variety of FOCPs. The widely adopted idea is to obtain the operational matrices of the fractional derivatives and integrals for the corresponding orthogonal polynomials (see [11, 12]). In our recent work [26], we have established a new class of FOCP and obtained its numerical solution by Laguerre orthogonal polynomial expansion method.

In this paper, we intend to adopt Boubaker polynomials to construct a numerical algorithm for solving a class of FOCP. These polynomials are non-orthogonal in nature and have already been used for solving optimal control problems [15] and FOCPs [23]. In [23], authors have obtained Boubaker operational matrices for fractional order operators and applied them to solve FOCPs. Involvement of same non-orthogonal polynomials in [23] and present work leads to a comparison of the results. For fundamental properties and applications of Boubaker polynomials, one can see [8].

Our approach is based on parameterizing the state and control variables with Boubaker polynomial expansion scheme. The parameterized state and control variables directly assist in approximating the performance index of the concerned FOCP. The underlying objective is to convert the given FOCP into an equivalent standard programming problem with linear equality constraints. The solution of the latter problem corresponds to the solution of original control problem. Thus, the original FOCP can be solved directly without using any necessary conditions or Hamiltonian formulas. With a quadratic performance index, an equivalent quadratic programming problem with linear equality constraints can be handled efficiently. Additionally, we have worked out both time-invariant and time-varying FOCPs to follow the working mechanism of the proposed algorithm. To analyze the performance of the solution, we have provided the plot of the state, control variables, and absolute error functions.

This paper is organized as follows: Section 2 discusses the FOCP statement which is considered throughout the work. In Section 3, some basic definitions of fractional operators and Boubaker polynomials are provided with necessary details. Section 4 corresponds to the mathematical formulation of the proposed algorithm, followed by Section 5 discussing the convergence analysis. In Section 6, we present illustrative examples to demonstrate the applicability of solution scheme. The last section provides the concluding remarks and scope of future work.

2. Problem Statement

In this section, we state a class of FOCPs formulated to find the optimal control $u(t)$ that minimizes the given performance index J

$$\textbf{(P)} \qquad \text{Minimize} \quad J = \int_{t_0}^{t_1} F(x, u, t)\, dt, \tag{1}$$

subject to the system dynamic constraints

$$G_1\left(x'(t), {}_{t_0}^{\;c}D_t^{\alpha} x(t)\right) = G_2(x, u, t), \quad 0 < \alpha < 1, \tag{2}$$

and the initial condition

$$x(t_0) = x_0, \tag{3}$$

where $x(t)$ is the state variable known as the optimal trajectory. The functions F and G_2 are continuously differentiable in all the three arguments. Here, G_1 is an arbitrary function of classical and fractional derivative of the state variable $x(t)$. Our aim is to construct a well-organized algorithm by exercising non-orthogonal Boubaker polynomial approximation for solving the FOCP **(P)**. For necessary and sufficient optimality conditions, and a solution scheme for the problem **(P)** by means of Laguerre orthogonal approximation, we refer the reader to [26].

3. Preliminaries

In this section, we enlist some definitions of fractional order operators and Boubaker polynomials that will be required in the sequel. We suggest the reader to [20, 21], for detailed information on existing fractional derivatives and integrals.

Fractional Order Operators. Let $f \in C[a,b]$, where $C[a,b]$ is the space of all continuous real valued functions defined on $[a,b]$.

Definition 3.1. *For all $t \in [a,b]$ and $\alpha > 0$, left Riemann-Liouville fractional integral of order α is defined as*

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau. $$

Let $f \in C^n[a,b]$; $n \in \mathbb{N}$, where $C^n[a,b]$ is the space of all n times continuously differentiable functions defined on the closed interval $[a,b]$.

Definition 3.2. *For all $t \in [a,b]$, $n-1 \leq \alpha < n$, left Riemann-Liouville fractional derivative of order α is defined as*

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau. $$

Definition 3.3. *For all $t \in [a,b]$, $n-1 \leq \alpha < n$, left Caputo fractional derivative of order α is defined as*

$$ {}_a^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^n(\tau)\, d\tau. $$

Boubaker Polynomials. In [8], authors have introduced Boubaker polynomials for solving a one-dimensional heat equation. The expression of Boubaker polynomials, denoted by $B_n(t)$, is described as

$$ B_n(t) = \sum_{p=0}^{\xi(n)} (-1)^p \frac{(n-4p)}{(n-p)} \begin{pmatrix} n-p \\ p \end{pmatrix} t^{n-2p}, \tag{4} $$

where $\xi(n) = \lfloor \frac{n}{2} \rfloor = \frac{2n+((-1)^n-1)}{4}$, and $\lfloor \cdot \rfloor$ is known as the floor function.

• The first few Boubaker polynomials are

$$ B_0(t) = 1, \quad B_1(t) = t, \quad B_2(t) = t^2 + 2, $$

$$ B_3(t) = t^3 + t, \quad B_4(t) = t^4 - 2. $$

These polynomials can also be defined with the recurrence relation given below

$$ B_n(t) = t\, B_{n-1}(t) - B_{n-2}(t), \; n > 2. \tag{5} $$

We may note that the above recurrence formula holds for $n > 2$, as the first three polynomials are defined explicitly.

4. Mathematical Framework: Algorithm

Let $Q \subset C[t_0, t_1]$ be the set of all functions satisfying the initial condition (3), and $P_N \subset Q$ be the class of all Boubaker polynomials of order up to N.

In this section, we use the N^{th}-order Boubaker polynomials to approximate the state variable as a N^{th}-order polynomial in t. The approximated state variable and the given system dynamical constraints assist in representing the control variable with a lesser number of parameters that minimizes J.

Algorithm: Solution scheme.

Step I. Approximate the state variable $x(t)$ as a linear combination of N^{th}-order Boubaker polynomials,

$$x(t) \cong x_N(t) = \sum_{k=0}^{N} a_k\, B_k(t), \quad N = 1, 2, 3, ... \tag{6}$$

where B_k's are the k^{th}-order Boubaker polynomials and a_k's are the unknown coefficients to be determined.

Step II. Apply the given initial condition (3),

$$x(t_0) \cong x_N(t)\Big|_{t=t_0} = \sum_{k=0}^{N} a_k\, B_k(t)\Big|_{t=t_0} = x_0,$$

which results in an algebraic equation of the unknown coefficients a_k's. The above equation corresponds to the required linear equality constraints for the equivalent standard programming problem.

Step III. With the given dynamical constraint (2), find an expression of $u(t)$ ($\cong u_N(t)$) as a function (say ϕ) of t, $x_N(t)$, $x'_N(t)$, and the fractional derivative ${}_{t_0}^{c}D_t^{\alpha} x_N(t)$,

$$\begin{aligned} u(t) &\cong u_N(t), \\ &= \phi\left(t, x_N(t), x'_N(t), {}_{t_0}^{c}D_t^{\alpha} x_N(t)\right), \\ &= \phi\left(t, \sum_{k=0}^{N} a_k B_k(t), \sum_{k=0}^{N} a_k B'_k(t), \sum_{k=0}^{N} a_k\, {}_{t_0}^{c}D_t^{\alpha} B_k(t)\right). \end{aligned}$$

Step IV. Substitute the approximated state variable $x_N(t)$ and the control variable $u_N(t)$ in the given performance index (1),

$$J \cong \hat{J}[a_0, ..., a_N] = \int_{t_0}^{t_1} F\left(t, x_N(t), u_N(t)\right)\, dt,$$

where

$$\begin{aligned} x_N(t) &= \sum_{k=0}^{N} a_k\, B_k(t), \\ u_N(t) &= \phi\left(t, \sum_{k=0}^{N} a_k B_k(t), \sum_{k=0}^{N} a_k B'_k(t), \sum_{k=0}^{N} a_k\, {}_{t_0}^{c}D_t^{\alpha} B_k(t)\right). \end{aligned}$$

The given performance index J is now transformed into $\hat{J}$, which is a function of $N + 1$ unknown coefficients a_k.

Step V. Next, the standard programming problem is to minimize $\hat{J}[a_0, ..., a_N]$ for a_k's, subject to the constraint $x_N(t_0) = \sum_{k=0}^{N} a_k\, B_k(t)\Big|_{t=t_0} = x_0$.

Step VI. With a quadratic performance index in FOCP, the original problem **(P)** is converted into minimizing a quadratic function $\hat{J}$ subject to the linear equality constraint $\sum_{k=0}^{N} a_k B_k(t_0) = x_0$. Finally, we are required to solve the quadratic programming problem described below in equivalent matrix form

$$\textbf{Minimize } \left(\frac{1}{2} a' H a + G' a\right), \quad a \in \mathbb{R}^{N+1}, \tag{7}$$

subject to the linear equality constraint

$$Ba = C, \tag{8}$$

where $a = [a_0\ a_1 \ldots a_N]'$, $B = [B_0(t_0)\ B_1(t_0) \ldots B_N(t_0)]$, and $C = [x_0]$.

Step VII. Find the optimal value a^* to solve the quadratic programming problem (7)-(8) as follows,

$$\begin{aligned} a^* &= -H^{-1}(G + B'\lambda^*), \\ \text{where} \quad \lambda^* &= -(BH^{-1}B')^{-1}(C + BH^{-1}G). \end{aligned}$$

Step VIII. At last, use the optimal value a^* to write the expressions for $x_N(t)$, $u_N(t)$ and the optimal value of $\hat{J}$ that approximates the original performance index J.

We have performed entire numerical and graphical part with Mathematica software. As per the convenience, one may also use quadratic programming problem solver in MATLAB software. In that case, the optimal value a^* can be obtained by providing the matrices H, G, B, C from the problem (7)-(8) as input and extracting the column matrix a as output.

5. Convergence Analysis

In this section, we compute the α^{th}-order Caputo's fractional derivative of Boubaker polynomials for $\alpha \in (0, 1)$. Afterward, we discuss an approximation formula for the fractional derivative of the state variable followed by the convergence analysis of the designed approximation by Boubaker polynomial expansion.

Theorem 5.1. *For $0 < \alpha < 1$, the α^{th}-order Caputo fractional derivative of Boubaker polynomials of is given by*

$$ {}_0^cD_t^\alpha B_n(t) = B_n^\alpha(t), \tag{9}$$

$$\text{where} \quad B_n^\alpha(t) = \sum_{p=\lceil\alpha\rceil}^{\lfloor \frac{n}{2} \rfloor} (-1)^p \frac{(n-4p)\Gamma(n-p)}{\Gamma(p+1)\Gamma(n-2p-\alpha+1)}\, t^{n-2p-\alpha}. \tag{10}$$

Proof. By Eq. (4), we have

$$B_n(t) = \sum_{p=0}^{\xi(n)} (-1)^p \frac{(n-4p)}{(n-p)} \begin{pmatrix} n-p \\ p \end{pmatrix} t^{n-2p},$$

where $\xi(n) = \lfloor \frac{n}{2} \rfloor = \frac{2n+((-1)^n-1)}{4}$.

For Caputo's fractional derivative, we have

$$\begin{aligned} {}_0^cD_t^\alpha K &= 0, \quad K \text{ is a constant}, \\ \text{and} \quad {}_0^cD_t^\alpha t^p &= \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}\, t^{p-\alpha} & \text{for} \quad p > 0, \\ 0 & \text{for} \quad p \le 0. \end{cases} \end{aligned}$$

Next, we compute

$$\begin{aligned} B_n^\alpha(t) &= {}_0^cD_t^\alpha B_n(t), \\ &= {}_0^cD_t^\alpha \sum_{p=0}^{\xi(n)} (-1)^p \frac{(n-4p)}{(n-p)} \binom{n-p}{p} t^{n-2p}, \\ &= \sum_{p=\lceil\alpha\rceil}^{\xi(n)} (-1)^p \frac{(n-4p)\Gamma(n-p)}{\Gamma(p+1)\Gamma(n-2p-\alpha+1)} t^{n-2p-\alpha}, \end{aligned}$$

which completes the proof. □

Theorem 5.2. *For $0 < \alpha < 1$, an approximation formula for the α^{th}-order Caputo fractional derivative of the state variable is given by*

$$ {}_0^cD_t^\alpha x(t) \approxeq {}_0^cD_t^\alpha x_N(t) = \sum_{k=\lceil\alpha\rceil}^{N} a_k B_k^\alpha(t), \quad N = 1, 2, 3... $$

where $B_k^\alpha(t)$ is given by Eq. (9). Or,

$$ {}_0^cD_t^\alpha x(t) \approxeq \sum_{k=0}^{N} a_k \sum_{p=\lceil\alpha\rceil}^{\lfloor\frac{k}{2}\rfloor} \frac{(-1)^p(k-4p)\Gamma(k-p)}{\Gamma(p+1)\Gamma(k-2p-\alpha+1)} t^{k-2p-\alpha}. $$

Proof. Using Eq. (6), we approximate ${}_0^cD_t^\alpha x(t)$ as

$$\begin{aligned} {}_0^cD_t^\alpha x(t) &\approxeq {}_0^cD_t^\alpha x_N(t), \\ &= {}_0^cD_t^\alpha \sum_{k=0}^{N} a_k B_k(t), \\ &= \sum_{k=0}^{N} a_k B_k^\alpha(t), \\ &= \sum_{k=0}^{N} a_k \sum_{p=\lceil\alpha\rceil}^{\lfloor\frac{k}{2}\rfloor} \frac{(-1)^p(k-4p)\Gamma(k-p)}{\Gamma(p+1)\Gamma(k-2p-\alpha+1)} t^{k-2p-\alpha}, \end{aligned}$$

which completes the proof. □

Remark 5.3. *With Boubaker polynomial expansion scheme, one may observe that the powers $\{t^{1-\alpha}, t^{2-\alpha}, ..., t^{n-\alpha}\}$ arising in the fractional derivative of the approximated state variable can be described in terms of Boubaker polynomials as follows*

$$ T^\alpha = K B^\alpha, $$

where $T^\alpha = [t^{1-\alpha}\ t^{2-\alpha}\ t^{2-\alpha}\ ...\ t^{n-\alpha}]$, $B^\alpha = [B_1^\alpha(t)\ B_2^\alpha(t)\ B_2^\alpha(t)\ ...\ B_n^\alpha(t)]$, and K is the corresponding coefficient matrix.

To make the comments clear, let us describe the above matrix structure for $n = 5$. We first compute the α^{th}-order Caputo's fractional derivative of Boubaker polynomials as given below

$$\begin{aligned}
B_1^{\alpha}(t) &= \frac{1!}{\Gamma(2-\alpha)}\,t^{1-\alpha},\\
B_2^{\alpha}(t) &= \frac{2!}{\Gamma(3-\alpha)}\,t^{2-\alpha},\\
B_3^{\alpha}(t) &= \frac{1!}{\Gamma(2-\alpha)}\,t^{1-\alpha} + \frac{3!}{\Gamma(4-\alpha)}\,t^{3-\alpha},\\
B_4^{\alpha}(t) &= \frac{4!}{\Gamma(5-\alpha)}\,t^{4-\alpha},\\
B_5^{\alpha}(t) &= -3 * \frac{1!}{\Gamma(2-\alpha)}\,t^{1-\alpha} - \frac{3!}{\Gamma(4-\alpha)}\,t^{3-\alpha} + \frac{5!}{\Gamma(6-\alpha)}\,t^{5-\alpha}.
\end{aligned}$$

Rearranging the above system of equations to get the desired form $T^{\alpha} = K\,B^{\alpha}$,

$$\begin{aligned}
t^{1-\alpha} &= \frac{\Gamma(2-\alpha)}{1!}B_1^{\alpha}(t),\\
t^{2-\alpha} &= \frac{\Gamma(3-\alpha)}{2!}B_2^{\alpha}(t),\\
t^{3-\alpha} &= \frac{\Gamma(4-\alpha)}{3!}(B_3^{\alpha}(t) - B_1^{\alpha}(t)),\\
t^{4-\alpha} &= \frac{\Gamma(5-\alpha)}{4!}B_4^{\alpha}(t),\\
t^{5-\alpha} &= \frac{\Gamma(6-\alpha)}{5!}(B_5^{\alpha}(t) + B_3^{\alpha}(t) + 2 * B_1^{\alpha}(t)).
\end{aligned}$$

Or,

$$\begin{bmatrix} t^{1-\alpha} \\ t^{2-\alpha} \\ t^{3-\alpha} \\ t^{4-\alpha} \\ t^{5-\alpha} \end{bmatrix} = \begin{bmatrix} \frac{\Gamma(2-\alpha)}{1!} & 0 & 0 & 0 & 0 \\ 0 & \frac{\Gamma(3-\alpha)}{2!} & 0 & 0 & 0 \\ -\frac{\Gamma(4-\alpha)}{3!} & 0 & \frac{\Gamma(4-\alpha)}{3!} & 0 & 0 \\ 0 & 0 & 0 & \frac{\Gamma(5-\alpha)}{4!} & 0 \\ 2 * \frac{\Gamma(6-\alpha)}{5!} & 0 & \frac{\Gamma(6-\alpha)}{5!} & 0 & \frac{\Gamma(6-\alpha)}{5!} \end{bmatrix} \begin{bmatrix} B_1^{\alpha}(t) \\ B_2^{\alpha}(t) \\ B_3^{\alpha}(t) \\ B_4^{\alpha}(t) \\ B_5^{\alpha}(t) \end{bmatrix}.$$

Next, we discuss the convergence analysis of the proposed algorithm provided by the Weierstrass approximation theorem.

Theorem 5.4. (Wierstrass approximation theorem [24]) *Let $f \in C[a,b]$. Then, there is a sequence of polynomials $P_n(x)$, that converges uniformly to $f(x)$ on $[a,b]$.*

We clearly observe that a continuous function (say the state variable $x(t)$) can be uniformly approximated by a sequence of Boubaker polynomials $\{B_n(t)\}$. Mathematically, if $x_m(t) = \sum_{k=0}^{m} a_k\,B_k(t)$. So, for $m \to \infty$, we have $x_m(t) \to x(t)$.

Theorem 5.5. (See [14]) *If $\xi_n = \inf_{Q_n} J$, $n \in \mathbb{N}$, where Q_n is a subset of Q, consisting of all polynomials of degree at most n. Then, $\lim_{n\to\infty}\xi_n = \xi$, where $\xi = \inf_Q J$.*

Theorem 5.6. *If J has continuous first order derivatives, and for $n \in \mathbb{N}$, $\beta_n = \inf_{Q_n} J$. Then, $\lim_{n\to\infty}\beta_n = \beta$, where $\beta = \inf_Q J$.*

Proof. This theorem is already proved when Q_n is a class of Chebyshev polynomials [14], Laguerre polynomials [26], Boubaker Polynomials [15]. □

6. Computational segment: Examples

In this section, we demonstrate the applicability of the formulated numerical scheme. For this purpose, we consider some examples of time-invariant and time-varying FOCPs. The efficiency and accuracy of the strategy are observed by comparing the solution obtained by the proposed scheme with the solution given in [23]. Furthermore, we analyze the plot of the state and control variable together with their absolute error functions.

Example 6.1. *Find an optimal control $u(t)$ for the time invariant FOCP with quadratic performance index*

$$\text{Minimize} \quad J = \frac{1}{2}\int_0^1 [x^2(t) + u^2(t)]\,dt, \tag{11}$$

subject to the system dynamic constraints

$${}_0^CD_t^\alpha\, x(t) = -x(t) + u(t), \quad \alpha \in (0,1), \tag{12}$$

and the initial condition

$$x(0) = 1. \tag{13}$$

Following the numerical scheme detailed in Section 4, we parameterize the state variable $x(t)$ with Boubaker polynomials (B_k's) of order up to $N = 5$.

$$x(t) \cong x_5(t) = \sum_{k=0}^{5} a_k\, B_k(t), \tag{14}$$

$$= (a_0 + 2a_2 - 2a_4) + (a_1 + a_3 - 3a_5)\,t + a_2\,t^2 + (a_3 - a_5)\,t^3 + a_4\,t^4 + a_5\,t^5, \tag{15}$$

where a_k's ($k = 0, 1, ..., 5$) are the unknown coefficients to be determined.

By given initial condition (13), we have

$$x_5(0) = 1 = a_0 + 2a_2 - 2a_4. \tag{16}$$

Next, we approximate the control variable $u(t)$ ($\cong u_5(t)$) by substituting (15) in (12) as

$$\begin{aligned} u_5(t) &= x_5(t) + {}_0^cD_t^\alpha\, x_5(t), \\ &= (a_0 + 2a_2 - 2a_4) + (a_1 + a_3 - 3a_5)t + a_2t^2 + (a_3 - a_5)\,t^3 + a_4t^4 + a_5t^5 + \frac{1}{\Gamma(2-\alpha)}(a_1 + a_3 - 3a_5)t^{1-\alpha} \\ &\quad + \frac{2}{\Gamma(3-\alpha)}\,a_2\,t^{2-\alpha} + \frac{3!}{\Gamma(4-\alpha)}\,(a_3 - a_5)\,t^{3-\alpha} + \frac{4!}{\Gamma(5-\alpha)}\,a_4\,t^{4-\alpha} + \frac{5!}{\Gamma(6-\alpha)}\,a_5\,t^{5-\alpha}, \end{aligned}$$

where $\alpha \in (0, 1)$.

To make the mechanism evident, choose $\alpha = 0.9$ (refer to Table 1 for optimal values of J corresponding to different values of α and $N = 5$).

$$\begin{aligned} u_5(t) &= (a_0 + 2a_2 - 2a_4) + (a_1 + a_3 - 3a_5)t + a_2t^2 + (a_3 - a_5)t^3 + a_4t^4 + a_5t^5 + 1.05113701(a_1 + a_3 - 3a_5)t^{0.1} \\ &\quad + 1.91115819a_2t^{1.1} + 2.73022599(a_3 - a_5)t^{2.1} + 3.52287224a_4t^{3.1} + 4.29618566a_5t^{4.1}. \end{aligned} \tag{17}$$

Using $x_5(t)$ and $u_5(t)$, we transform the performance index J into $\hat{J}$ as follows

$$\begin{aligned}\hat{J}[a_0, a_1, ..., a_5] \;=\;& a_0^2 + 1.95558\, a_0a_1 + 5.57674\, a_0a_2 + 3.3363\, a_0a_3 - 2.7407\, a_0a_4\\ &-6.07173\, a_0a_5 + 1.29425\, a_1^2 + 6.27987\, a_1a_2 + 4.8076\, a_1a_3\\ &-1.79929\, a_1a_4 - 7.95382\, a_1a_5 + 8.39033\, a_2^2 + 11.5271\, a_2a_3\\ &-6.16371\, a_2a_4 - 19.2868\, a_2a_5 + 4.2053\, a_3^2\\ &-1.87868\, a_3a_4 - 14.5067\, a_3a_5 + 2.88941\, a_4^2\\ &+6.03077\, a_4a_5 + 12.321\, a_5^2,\\ =\;& \frac{\mathbf{1}}{\mathbf{2}}\mathbf{a}'\mathbf{H}\mathbf{a},\end{aligned}$$

where

$$H = \begin{pmatrix} 2 & 1.955579 & 5.576742 & 3.336297 & -2.740763 & -6.071733\\ 1.955579 & 2.588490 & 6.279869 & 4.807599 & -1.799288 & -7.953817\\ 5.576742 & 6.279869 & 16.780654 & 11.527069 & -6.163708 & -19.286782\\ 3.336297 & 4.807599 & 11.527069 & 9.641066 & -1.878678 & -14.506717\\ -2.740763 & -1.799288 & -6.163708 & -1.878678 & 5.778817 & 6.030768\\ -6.071733 & -7.953817 & -19.286782 & -14.506717 & 6.030768 & 24.642043 \end{pmatrix}.$$

Finally, the fractional optimal control problem (11)-(13) is converted into an equivalent quadratic programming problem described below.

$$\begin{aligned}\text{Minimize}\quad J &= \frac{1}{2}a'Ha,\\ \text{subject to}\quad Ba &= C,\end{aligned}$$

where $a = [a_0\ a_1\ a_2\ a_3\ a_4\ a_5]'$, $B = [1\ \ 0\ \ 2\ \ 0\ \ -2\ \ 0]$, and $C = [1]$.

On simplifying, we obtain

$$a = \begin{pmatrix} a_0\\ a_1\\ a_2\\ a_3\\ a_4\\ a_5 \end{pmatrix} = \begin{pmatrix} 2.35720894\\ 0.37306204\\ 3.47757762\\ -6.38926186\\ 4.15618209\\ -1.35545937 \end{pmatrix}.$$

Using the optimal value of a, we write the expression for the approximated state $x_5(t)$ and optimal control $u_5(t)$ variable, as a function of time t.

$$x_5(t) = 1 - 1.949822\, t + 3.477578\, t^2 - 5.033802\, t^3 + 4.156182\, t^4 - 1.355459\, t^5,$$

$$u_5(t) = x_5(t) - 2.049530\, t^{0.1} + 6.646201\, t^{1.1} - 13.743418\, t^{2.1} + 14.641698\, t^{3.1} - 5.823305\, t^{4.1},$$

and thus we arrive at the approximate minimum value of the performance index $J = 0.17996229$.

Note that for $\alpha = 1$, given fractional optimal control problem reduces to a classical optimal control problem. The classical case is widely investigated (see [11]) and the analytic solution for this system is given as

$$\begin{aligned} x(t) &= \beta\, e^{\sqrt{2}\,t} + (1-\beta)\, e^{-\sqrt{2}\,t}, \\ y(t) &= \beta\,(\sqrt{2}\,t+1)e^{\sqrt{2}\,t} - (1-\beta)\,(\sqrt{2}-1)e^{-\sqrt{2}\,t}, \\ \text{and} \quad J &= \frac{e^{-\sqrt{2}\,t}}{2}\Big((\sqrt{2}+1)\,(e^{4\sqrt{2}}-1)\,\beta^2 + (\sqrt{2}-1)\,(e^{2\sqrt{2}}-1)\,(1-\beta)^2\Big), \end{aligned}$$

where $\beta = \frac{2\sqrt{2}-3}{-e^{\sqrt{2}}+2\sqrt{2}-3}$.

The exact solution for the performance index is $J = 0.1929092978$, while $J = 0.19290929$ by the proposed 5^{th}-order Boubaker polynomial expansion method (as shown in Table 1). The graphs for the state and control variables for different values of α are plotted in Figure 1. We have also plotted the exact and approximated state and control variable ($\alpha = 1$) in Figure 2.

Figure 1: The approximated state $x_5(t)$ and control variable $u_5(t)$, as a function of time t (Example 6.1).

Figure 2: The approximated, exact state and control variables, as a function of time t ($\alpha = 1$, Example 6.1).

α	Optimal J
0.6	0.15167258
0.7	0.15940466
0.8	0.16880011
0.9	0.17996229
0.99	0.19153506
1.0	0.19290929

Table 1: Approximate optimal value of performance index J.

Example 6.2. *Find an optimal control $u(t)$ to solve the time-varying FOCP with quadratic performance index given below.*

$$\text{Minimize} \quad K = \frac{1}{2}\int_0^1 [3\,x^2(t) + u^2(t)]\,dt, \tag{18}$$

subject to the system dynamic constraints

$${}_0^C D_t^{\alpha}\, x(t) = -x(t) + u(t), \quad \alpha \in (0,1), \tag{19}$$

and the boundary conditions

$$x(0) = 0, \quad \text{and} \quad x(1) = 2. \tag{20}$$

Following the same way as in previous example, we parameterize the state variable $x(t)$ ($\cong x_5(t)$) by Boubaker polynomials (see equation (15)) with the initial condition given in (20). The approximated control variable $u(t)$ ($\cong u_5(t)$) can be determined by equation (19), that is,

$$u(t) \cong u_5(t) = {}_0^C D_t^{\alpha}\, x_5(t) - x_5(t).$$

Choose $\alpha = 0.99$, and substitute the approximated state and control variable into the performance index K given by equation (18). The FOCP (18)-(20) is now transformed into a quadratic programming problem described below.

$$\begin{aligned} \text{Minimize} \quad \hat{K} &= \frac{1}{2} a' H a, \\ \text{subject to} \quad Ba &= C, \end{aligned}$$

where $a = [a_0\ a_1\ a_2\ a_3\ a_4\ a_5]'$, $B = \begin{bmatrix} 1 & 0 & 2 & 0 & -2 & 0 \\ 1 & 1 & 3 & 2 & -1 & -3 \end{bmatrix}$, $C = [0\ 2]'$, and

$$H = \begin{pmatrix} 4 & 2.995749 & 10.324128 & 4.983252 & -6.214959 & -9.325009 \\ 2.995749 & 3.325650 & 8.978763 & 6.107542 & -3.347468 & -10.213858 \\ 10.324128 & 8.978763 & 28.403046 & 16.083744 & -13.951122 & -27.625773 \\ 4.983252 & 6.107542 & 16.083744 & 12.209985 & -3.884646 & -18.364262 \\ -6.214959 & -3.347468 & -13.951122 & -3.884646 & 12.513145 & 11.090903 \\ -9.325009 & -10.213858 & -27.625773 & -18.364262 & 11.090903 & 31.678212 \end{pmatrix}.$$

On simplifying the above quadratic programming problem, we obtain

$$a = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} -1.97596702 \\ 0.97662053 \\ -0.35168571 \\ 2.56243119 \\ -1.33966922 \\ 0.80337599 \end{pmatrix}.$$

Using the optimal value of a, we write the expressions for the approximated state $x_5(t)$ and optimal control $u_5(t)$ variable, as a function of time t.

$$\begin{aligned} x_5(t) &= 3.552714 * 10^{-15} + 1.128924\,t - 0.351686\,t^2 + 1.759055\,t^3 - 1.339669\,t^4 + 0.803375\,t^5, \\ u_5(t) &= x_5(t) + 1.135366\,t^{0.01} - 0.700381\,t^{1.01} + 5.228590\,t^{2.01} - 5.291711\,t^{3.01} + 3.956790\,t^{4.01}, \end{aligned}$$

and the approximate optimal value of performance index is $K = 6.09978305$ (one may look at Table 2, for approximate optimal value of performance index K corresponding to different values of α).

For $\alpha = 1$, the classical optimal control problem (see [23]) possess an analytical solution given below

$$x_5*(t) = \frac{2\,\mathrm{Sinh}(2\,t)}{\mathrm{Sinh}(2)}$$
$$u_5*(t) = \frac{2\,(\mathrm{Sinh}(2\,t) + \cosh(2\,t))}{\mathrm{Sinh}(2)},$$

and the optimal value of performance index is $K = 6.149258$. On applying our method for 5^{th}-order Boubaker polynomials expansion, $K = 6.14925898$ (see Table 2).

In Figure 3, we have plotted the approximated state and control variable as a function of time t. Figure 4 represents the exact and approximate solution of the state and control variable, and assists us in visualizing the efficiency of the method.

One may observe Figures 5, for the absolute error functions of state ($|x_5(t) - x_5*(t)|$) and control ($|u_5(t) - u_5*(t)|$) variable corresponding to $\alpha = 1$.

Figure 3: The approximated state $x_5(t)$ and control variable $u_5(t)$, as a function of time t (Example 6.2).

Figure 4: The approximated, exact state and control variables, as a function of time t ($\alpha = 1$, Example 6.2).

Figure 5: The absolute error function of the state and control variable, as a function of time t ($\alpha = 1$, Example 6.2).

α	Optimal K
0.6	2.70717907
0.7	3.69418458
0.8	4.69524002
0.9	5.54278048
0.99	6.09978305
1.0	6.14925898

Table 2: Approximate optimal value of performance index K.

Example 6.3. *Find an optimal control $u(t)$ for solving a time-varying FOCP with quadratic performance index*

$$\text{Minimize} \quad L = \frac{1}{2}\int_0^1 [x^2(t) + u^2(t)]\, dt, \tag{21}$$

subject to the system dynamic constraints

$${}_0^C D_t^\alpha\, x(t) = t\, x(t) + u(t), \quad \alpha \in (0,1), \tag{22}$$

and the initial condition

$$x(0) = 1. \tag{23}$$

One may note that the given constraint (22) contains t explicitly, and the above problem is referred as time-varying FOCP.

After applying the proposed numerical algorithm for Boubaker polynomials of order up to $N = 5$ ($\alpha = 0.8$), the FOCP (21)-(23) is converted into an equivalent quadratic programming problem with linear equality constraint described below in matrix form:

$$\begin{aligned} \text{Minimize} \quad \hat{L} &= \frac{1}{2}a'Ha, \\ \text{subject to} \quad Ba &= C, \end{aligned}$$

where $a = [a_0\ a_1\ a_2\ a_3\ a_4\ a_5]'$, $B = [1\ \ 0\ \ 2\ \ 0\ \ -2\ \ 0]$, $C = [1]$, and

$$H = \begin{pmatrix} 1.333333 & 0.254943 & 2.632748 & 0.082257 & -2.918829 & -0.894583 \\ 0.254943 & 0.699910 & 1.058790 & 1.150214 & -0.127056 & -2.216009 \\ 2.632748 & 1.058790 & 5.811469 & 1.334241 & -5.157620 & -3.457544 \\ 0.082257 & 1.150214 & 1.334241 & 2.301541 & 0.960807 & -3.503244 \\ -2.918829 & -0.127056 & -5.157620 & 0.960807 & 7.165086 & 0.773835 \\ -0.894583 & -2.216009 & -3.457544 & -3.503244 & 0.773835 & 7.121949 \end{pmatrix}.$$

On simplifying the above quadratic programming problem, we obtain

$$a = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 4.59504630 \\ 1.57490790 \\ 4.95809701 \\ -9.84913127 \\ 6.75562016 \\ -2.10450060 \end{pmatrix}.$$

Using the optimal value of a, we write the expression of approximated state and control variable as

$$x_5(t) = 1 - 1.96072156\,t + 4.95809701\,t^2 - 7.74463067\,t^3 + 6.75562016\,t^4 - 2.10450060\,t^5,$$

$$\begin{aligned} u_5(t) = & -2.13546973\,t^{0.2} - 1\,t + 8.99997423\,t^{1.2} + 1.96072156t^2 - 19.17015089t^{2.2} - 4.95809701t^3 \\ & +20.90258778\,t^{3.2} + 7.74463067\,t^4 - 7.751183645t^{4.2} - 6.75562016t^5 + 2.10450060t^6, \end{aligned}$$

and the approximate optimal value of given performance index is $L = 0.46782331$ (The optimal value of L is enlisted in Table 3 for distinct values of α).

The graphs of approximated state $x_5(t)$ and control $u_5(t)$ variables are presented as a function of time t, see Figure 6.

Figure 6: The approximated state $x_5(t)$ and control variable $u_5(t)$, as a function of time t (Example 6.3).

α	Optimal L
0.6	0.45990240
0.7	0.45984642
0.8	0.46782331
0.9	0.47609956
0.99	0.48346526
1.0	0.48426776

Table 3: The approximate optimal value of L.

Example 6.4. *Find the optimal control $u(t)$ that minimizes the time varying fractional optimal control problem with quadratic performance index*

$$M = \frac{1}{2}\int_0^1 [t\,u(t) - (\alpha+2)\,x(t)]^2\,dt, \tag{24}$$

subject to the system dynamic constraints

$$x'(t) + {}_0^C D_t^\alpha\, x(t) = u(t) + t^2, \quad \alpha \in (0,1), \tag{25}$$

and the boundary conditions

$$x(0) = 0, \quad \text{and} \quad x(1) = \frac{2}{\Gamma(3+\alpha)}. \tag{26}$$

Following the numerical algorithm for $\alpha = 0.99$, the FOCP (24)-(26) is transformed into an equivalent quadratic programming problem described as

$$\begin{aligned} \text{Minimize} \quad \hat{M} &= \frac{1}{2}a'Ha + G'a + k, \\ \text{subject to} \quad Ba &= C, \end{aligned}$$

where $a = [a_0\ a_1\ a_2\ a_3\ a_4\ a_5]'$, $B = \begin{bmatrix} 1 & 0 & 2 & 0 & -2 & 0 \\ 1 & 1 & 3 & 2 & -1 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 0 & \frac{2}{\Gamma(3.99)} \end{bmatrix}$, $k = 0.142857$,
$G = [1.495\ 0.39452\ 2.657272\ -0.456378\ -4.22877\ -1.872604]$, and

$$H = \begin{pmatrix} 2*8.9401 & 2.957998 & 33.777269 & -1.489586 & -41.683147 & -11.330328 \\ 2.957998 & 2*0.325144 & 5.424313 & -0.523977 & -7.543968 & -2.727826 \\ 33.777269 & 5.424313 & 2*31.993151 & -2.480172 & -77.986391 & -20.445735 \\ -1.489586 & -0.523977 & -2.480172 & 2*0.417930 & 5.041098 & 2.848491 \\ -41.683147 & -7.543968 & -77.986391 & 5.041098 & 2*50.334591 & 30.724071 \\ -11.330328 & -2.727826 & -20.445735 & 2.848491 & 30.724071 & 2*6.303430 \end{pmatrix}.$$

On simplifying the above quadratic problem with linear equality constraints, we obtain

$$a = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} -0.00631042 \\ -0.33687207 \\ 0.00314584 \\ 0.33483620 \\ -0.00001873 \\ -0.00053194 \end{pmatrix}.$$

Using the optimal value of a, we express the approximated state and optimal control variable as a function of time t.

$$x_5(t) = 0.00001873 - 0.00044004\,t + 0.00314584\,t^2 + 0.33536814\,t^3 - 0.00001873\,t^4 - 0.00053194\,t^5,$$

$$\begin{aligned} u_5(t) = {} & -0.00044004 - 0.00044255\,t^{0.01} + 0.00629168\,t + 0.00626494\,t^{1.01} + 0.00610443\,t^2 + 0.99684331\,t^{2.01} \\ & -0.00007493\,t^3 - 0.00007399\,t^{3.01} - 0.00265971\,t^4 - 0.00261993\,t^{4.01}, \end{aligned}$$

and thus the approximate minimum value of given performance index is $M = 4.73327022 * 10^{-10}$.

For $\alpha \in (0,1)$, the FOCP (24)-(26) possess an analytical solution given below

$$(x^*(t), u^*(t)) = \left(\frac{2\,t^{\alpha+2}}{\Gamma(\alpha+3)}, \frac{2\,t^{\alpha+1}}{\Gamma(\alpha+2)} \right),$$

and the optimal value of the performance index is $M = 0$ (One may look at Table 4 for approximate minimum value of M, corresponding to different values of α).

One may observe Figures 7,8 for approximated and exact state, control variables followed by the absolute error function in Figure 9.

6.1. *Discussion on graphical representations*

The graphs of every example presented here are essential to analyze the action of the approximated state and control variable for different values of $\alpha \in (0,1)$. We have provided the approximate value of optimal performance index J, K, L, M, in Tables 1, 2, 3, 4, respectively. To review the less error in computation, one

Figure 7: The approximated state $x_5(t)$ and control variable $u_5(t)$, as a function of time t (Example 6.4).

Figure 8: The approximated, exact state and control variables, as a function of time t ($\alpha = 1$, Example 6.4).

Figure 9: The absolute error function of the state and control variable, as a function of time t (Example 6.4).

α	Optimal M
0.6	$4.37156092 * 10^{-7}$
0.7	$2.89930096 * 10^{-7}$
0.8	$1.48802417 * 10^{-7}$
0.9	$4.24179444 * 10^{-8}$
0.99	$4.73327022 * 10^{-10}$
1.0	$1.91463271 * 10^{-19}$

Table 4: The approximate optimal value of M.

can look at the absolute error function of the state and control variable (i.e. $|x(t) - x_5^*(t)|$ and $|u(t) - u_5^*(t)|$) in Figure 5 (Example 6.2), and Figure 9 (Example 6.4). We conclude that for approximated state and control variable, the edges come closer as α approaches 1 and meet the exact solution for $\alpha = 1$. For instance, dotted line in Figure 6 corresponds to the exact solution whereas pink and orange lines correspond to $\alpha = 1.0$ and $\alpha = 0.99$, respectively. To have a nice interpretation, one may observe Figure 9 for the absolute error of the control function corresponding to $\alpha = 0.6, 0.7, 0.8, 0.9, 0.99, 1.0$.

7. Conclusion

We have formulated a computational technique to find an approximate optimal control for solving a class of FOCPs. The non-orthogonal Boubaker polynomials have been utilized to approximate the state and control variable. The presented algorithm is advantageous as it does not require the necessary optimality conditions or Hamiltonian equations. In addition to this, we have easily converted the original FOCP into a quadratic programming problem that can be handled conveniently. We work out some examples of time-invariant and time-varying FOCPs to demonstrate the applicability of the method. The graphs of the approximated state, control variable, and their absolute error functions are provided to validate the efficiency and accuracy of the presented numerical scheme.

References

[1] O.P. Agrawal, A general formulation and solution scheme for fractional optimal control problems, Nonlinear Dynam. 38 (2004), 323–337.
[2] O.P. Agrawal and D. Baleanu, A Hamiltonian formulation and a direct numerical scheme for fractional optimal control problems, J. Vib. Cont. 13 (2007), 1269–1281.
[3] O.P. Agrawal, A formulation and numerical scheme for fractional optimal control problems, J. Vib. Control 14 (2008), 1291–1299.
[4] R. Almeida and D.F.M. Torres, A discrete method to solve fractional optimal control problems, Nonlinear Dyn. 80 (2015), 1811–1816.
[5] D. Baleanu, O. Defterli and O.P. Agrawal, A central difference numerical scheme for fractional optimal control problems, J. Vib. Cont. 15 (2009), 583–597.
[6] S.K. Bhatt, An existence theorem for a fractional control problem, J. Optim. Theory Appl. 11 (1973), 379–385.
[7] A.H. Bhrawy, S.S. Ezz-Eldien, E.H. Doha, M.A. Abdelkawy and D. Baleanu, Solving fractional optimal control problems within a Chebyshev-Legendre operational technique, Internat. J. Control 90 (2017), 1230–1244.
[8] K. Boubaker, On modified Boubaker polynomials: some differential and analytical properties of the new polynomial issued from an attempt for solving bi-varied heat equation, Trends Appl. Sci. Res. 2 (2007), 540–544.
[9] A.E. Jr. Bryson and Y.C. Ho, Applied Optimal Control. Optimization, Estimation, and Control, Hemisphere Publishing Corp. Washington, 1975
[10] M. Dehghan, E.A. Hamedi and H. Khosravian-Arab, A numerical scheme for the solution of a class of fractional variational and optimal control problems using the modified Jacobi polynomials, J. Vib. Cont. 22 (2016), 1547–1559.
[11] E.H. Doha, A.H. Bhrawy, D. Baleanu, S.S. Ezz-Eldien and R.M. Hafez, An efficient numerical scheme based on the shifted orthonormal Jacobi polynomials for solving fractional optimal control problems, Adv. Difference Equ. (2015), 1–17.
[12] S.S. Ezz-Eldien, E.H. Doha, D. Baleanu and A.H. Bhrawy, A numerical approach based on Legendre orthonormal polynomials for numerical solutions of fractional optimal control problems, J. of Vib. and Cont. 20 (2015), 16–30.
[13] F. Ghomanjani, A numerical technique for solving fractional optimal control problems and fractional Riccati differential equations, J. Egyptian Math. Soc. 24 (2016), 638–643.
[14] B. Kafash, A. Delavarkhalafi and S.M. Karbassi, Application of Chebyshev polynomials to derive efficient algorithms for the solution of optimal control problems, Sci. Iranica 19 (2012), 795–805.
[15] B. Kafash, A. Delavarkhalafi, S.M. Karbassi and K. Boubaker, A numerical approach for solving optimal control problems using the Boubaker polynomials expansion scheme, J. Interpolat. Approx. Sci. Comput. (2014), 1–18.
[16] E. Keshavarz, Y. Ordokhani and M. Razzaghi, A numerical solution for fractional optimal control problems via Bernoulli polynomials, J. Vib. Control 22 (2016), 3889–3903.
[17] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, Amsterdam, 2006.
[18] A. Lotfi, M. Dehghan and S.A. Yousefi, A numerical technique for solving fractional optimal control problems, Comput. Math. Appl. 62 (2011), 1055–1067.
[19] A. Lotfi, S.A. Yousefi and M. Dehghan, Numerical solution of a class of fractional optimal control problems via the Legendre orthonormal basis combined with the operational matrix and the Gauss quadrature rule, J. Comput. Appl. Math. 250 (2013), 143–160.
[20] K.S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, New York, 1993.
[21] K. Oldham and J. Spanier, The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order, Academic Press, New York, 1974.
[22] S. Pooseh, R. Almieda and D.F.M. Torres, Fractional order optimal control problems with free terminal time, J. Ind. Manag. Optim. 10 (2014), 363–381.
[23] K. Rabiei, Y. Ordokhani and E. Babolian, The Boubaker polynomials and their application to solve fractional optimal control problems, Nonlinear Dynam. 88 (2017), 1013–1026.
[24] W. Rudin, Real and Complex Analysis, (3rd edition), McGraw-Hill, New York, 1986.
[25] N. Singha and C. Nahak, A numerical Scheme for generalized fractional optimal control problems, Appl. Appl. Math. 11 (2016), 798–814.

[26] N. Singha and C. Nahak, An Efficient Approximation Technique for Solving a Class of Fractional Optimal Control Problems, J. Optim. Theory Appl. 174 (2017), 785–802.

[27] H.M. Srivastava and M. Saigo, Multiplication of fractional calculus operators and boundary value problems involving the Euler-Darboux equation, J. Math. Anal. Appl. 121 (1987), 325–369.

[28] H.M. Srivastava and R.K. Saxena, Operators of fractional integration and their applications, Appl. Math. Comput. 118 (2001), 1–52.

[29] C. Tricaud and Y.Q. Chen, An approximation method for numerically solving fractional order optimal control problems of general form, Comput. Math. Appl. 59 (2010), 1644–1655.

[30] S.A. Yousefi, A. Lotfi and M. Dehghan, The use of Legendre multiwavelet collocation method for solving the fractional optimal control problems, J. Vib. Control 13 (2011), 2059–2065.