Filomat 32:14 (2018), 4993–5002
https://doi.org/10.2298/FIL1814993B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Arens Regularity and Weakly Compact Operators

Janko Bračič[a]

[a] *University of Ljubljana*

Abstract. We explore the relation between Arens regularity of a bilinear operator and the weak compactness of the related linear operators. Since every bilinear operator has natural factorization through the projective tensor product a special attention is given to Arens regularity of the tensor operator. We consider topological centers of a bilinear operator and we present a few results related to bilinear operators which can be approximated by linear operators.

1. Introduction

There are two natural ways how to extend a bilinear operator $m : X \times Y \to Z$, where X, Y, Z are complex Banach spaces, to a bilinear operator from $X^{**} \times Y^{**}$ to Z^{**}. However, in general, those natural extensions are not equal. Arens [1] was the first who considered this phenomena. He characterized those bilinear operators, called (Arens) regular, which have a unique natural extension to second duals (see [1, Theorems 2.3, 3.3]). Arens regularity is intimately connected with weakly compact linear operators. For instance, Hennefeld [5, Theorem 2.1] proved that multiplication in a Banach algebra $\mathcal{A}$ is Arens regular if and only if every operator $L_\xi : \mathcal{A} \to \mathcal{A}^*$ ($\xi \in \mathcal{A}^*$), which is defined by $L_\xi a = \xi \cdot a$ ($a \in \mathcal{A}$), is weakly compact. Here $\xi \cdot a \in \mathcal{A}^*$ is defined by $\langle \xi \cdot a, b\rangle = \langle \xi, ab\rangle$ ($b \in \mathcal{A}$). Recently, see [12], Arens ideas have been extended to locally convex algebras.

Arens proved that a bilinear operator $m : X \times Y \to Z$ is Arens regular if and only if, for every $\zeta \in Z^*$, the bilinear form $\zeta \circ m : X \times Y \to \mathbb{C}$ is Arens regular. By [11, Theorem 2.2], $\zeta \circ m$ is Arens regular if and only if it is represented by a weakly compact operator from X to Y^*. If m is not Arens regular, then those $\zeta \in Z^*$ for which $\zeta \circ m$ is Arens regular form a proper weakly closed subspace of Z^*, which we call Arens space of m and denote it by $Ar(m)$. In Section 3 we consider Arens spaces of bilinear operators which are compositions of m with linear operators. Since every bilinear operator $m : X \times Y \to Z$ has canonical factorization $m = M \circ \tau$, where M is a linear operator from $X \widehat{\otimes} Y$ to Z and $\tau : X \times Y \to X \widehat{\otimes} Y$ is the tensor operator, it turns out that m is Arens regular if and only if $M^*(Z^*) \subseteq Ar(\tau)$. Section 4 is devoted to the Arens space of the tensor operator. It is proven that $\zeta \in (X \widehat{\otimes} Y)^*$ is in $Ar(\tau)$ if and only if the operator from X to Y^* which naturally represents ζ is weakly compact. It follows that τ is Arens regular if and only if every operator from X to Y^* is weakly compact. In Section 5 we consider topological centers of a bilinear operator and in the last section we prove a few results for a special class of bilinear operators which can be approximated by linear operators.

2010 *Mathematics Subject Classification*. Primary 46H99; Secondary 47B10
Keywords. Arens regular bilinear operator, weakly compact operator
Received: 28 February 2018; Accepted: 12 August 2018
Communicated by Dragan S. Djordjević
Research supported by the Slovenian Research Agency through the research program P2-0268.
Email address: janko.bracic@fmf.uni-lj.si (Janko Bračič)

2. Preliminaries

Let X be a complex Banach space. We denote by X^*, X^{**} and X^{***} its topological first, second and third dual, respectively. The pairing between a Banach space and its dual is denoted by $\langle\cdot,\cdot\rangle$. The canonical embedding of X into X^{**} is denoted by ι_X. Usually we write $\widehat{x}$ instead of $\iota_X(x)$ and $\widehat{X}$ denotes the image of ι_X in X^{**}. By $B(X)$ we denote the Banach algebra of all bounded linear operators on X and by I, or by I_X, we denote the identity operator. If Y is another complex Banach space, then $B(X,Y)$ denotes the Banach space of all bounded linear operators from X to Y.

Let X, Y and Z be complex Banach spaces. By $Bil(X\times Y,Z)$ we denote the Banach space of all bounded bilinear mappings from $X\times Y$ to Z. Now on, we will call elements of $Bil(X\times Y,Z)$ bilinear operators and elements of $Bil(X\times Y,\mathbb{C})$ bilinear forms.

Let $m\in Bil(X\times Y,Z)$. Then $m^*\in Bil(Z^*\times X,Y^*)$ is defined by $\langle m^*(\zeta,x),y\rangle=\langle\zeta,m(x,y)\rangle$, where $x\in X,y\in Y,\zeta\in Z^*$ are arbitrary. It is obvious that $\|m^*\|=\|m\|$. Similarly one defines $m^{**}\in Bil(Y^{**}\times Z^*,X^*)$ and $m^{***}\in Bil(X^{**}\times Y^{**},Z^{**})$ by $\langle m^{**}(\Gamma,\zeta),x\rangle=\langle\Gamma,m^*(\zeta,x)\rangle$ and $\langle m^{***}(\Phi,\Gamma),\zeta\rangle=\langle\Phi,m^{**}(\Gamma,\zeta)\rangle$, where $x\in X,\Phi\in X^{**},\Gamma\in Y^{**}$, and $\zeta\in Z^*$ are arbitrary (see [1]). It is easily seen that m^{***} is an extension of m, that is, $m^{***}(\widehat{x},\widehat{y})=\widehat{m(x,y)}$ for all $x\in X,y\in Y$. However this extension is not necessary unique. Namely, let $m^t:Y\times X\to Z$ be the transpose of m, given by $m^t(y,x)=m(x,y)$ for all $x\in X,y\in Y$. It is easily seen that $m^{t***t}(\widehat{x},\widehat{y})=\widehat{m(x,y)}$ for all $x\in X,y\in Y$. In general, m^{***} and m^{t***t} do not coincide on the whole space $X^{**}\times Y^{**}$. If they do coincide, then m is said to be Arens regular.

If $m\in Bil(X\times Y,Z)$, then for every $x\in X$ one has a bounded linear operator $m(x,\cdot):Y\to Z$ which maps $y\in Y$ to $m(x,y)\in Z$. Let $\rho:Bil(X\times Y,Z)\to B(X,B(Y,Z))$ be given by $\rho(m)x=m(x,\cdot)$, for all $x\in X$. It is not hard to see that ρ is an isometric isomorphism. Similarly, mapping λ, which is given by $\lambda(m)y=m(\cdot,y)$ for $m\in Bil(X\times Y,Z)$ and $y\in Y$, is an isometric isomorphism from $Bil(X\times Y,Z)$ to $B(Y,B(X,Z))$. We point out the particular case when $Z=\mathbb{C}$. Since $B(X,\mathbb{C})=X^*$ and $B(Y,\mathbb{C})=Y^*$ we see that the Banach space $Bil(X\times Y,\mathbb{C})$ of bilinear forms can be identified with Banach spaces of bounded operators $B(X,Y^*)$, respectively $B(Y,X^*)$, through the isometric isomorphisms ρ, respectively λ. More precisely, for a bilinear form $\omega\in Bil(X\times Y,\mathbb{C})$, operators $\rho(\omega)\in B(X,Y^*)$ and $\lambda(\omega)\in B(Y,X^*)$ are given by $\langle\rho(\omega)x,y\rangle=\omega(x,y)=\langle\lambda(\omega)y,x\rangle$ $(x\in X,y\in Y)$.

We will use another representation of bilinear forms. Let $X\widehat{\otimes}Y$ be the projective tensor product of Banach spaces X and Y (see [10]). The dual space $(X\widehat{\otimes}Y)^*$ can be identified with $Bil(X\times Y,\mathbb{C})$ through isometric isomorphism μ, which is defined as follows. For every $\zeta\in(X\widehat{\otimes}Y)^*$, the bilinear form $\mu(\zeta)$ is given by $\mu(\zeta)(x,y)=\langle\zeta,x\otimes y\rangle$, where $x\in X,y\in Y$ are arbitrary (see [10, §2.2] for details). Hence, there are isometric isomorphisms between Banach spaces $Bil(X\times Y,\mathbb{C})$, $(X\widehat{\otimes}Y)^*$, $B(X,Y^*)$ and $B(Y,X^*)$. We will denote by $\varphi=\rho\circ\mu$ the isometric isomorphism between $(X\widehat{\otimes}Y)^*$ and $B(X,Y^*)$, similarly, $\psi=\lambda\circ\mu$ denotes the isometric isomorphism between $(X\widehat{\otimes}Y)^*$ and $B(Y,X^*)$. Hence, if $\zeta\in(X\widehat{\otimes}Y)^*$, then $\varphi(\zeta)\in B(X,Y^*)$ and $\psi(\zeta)\in B(Y,X^*)$ are operators such that

$$\langle\varphi(\zeta)x,y\rangle=\langle\zeta,x\otimes y\rangle=\langle\psi(\zeta)y,x\rangle\qquad\text{for all } x\in X,y\in Y. \tag{1}$$

Consider $\mathbb{C}$ as a one-dimensional Banach space spanned by 1. Let $1^*:\mathbb{C}\to\mathbb{C}$ be the identity map. Then $\mathbb{C}^*=\mathbb{C}1^*$ and for arbitrary $\alpha,\beta\in\mathbb{C}$ one has $\langle\alpha1^*,\beta\rangle=\alpha\beta$. The second dual of $\mathbb{C}$ is $\mathbb{C}^{**}=\mathbb{C}1^{**}$, where $1^{**}:\mathbb{C}^*\to\mathbb{C}$ is a linear map determined by $\langle1^{**},1^*\rangle=1$. Let now Z be an arbitrary complex Banach space and let $\zeta\in Z^*$. The adjoint ζ^* is given by $\langle\zeta^*(\alpha1^*),z\rangle=\langle\alpha1^*,\zeta(z)\rangle=\alpha\zeta(z)=\langle\alpha\zeta,z\rangle$, where $\alpha1^*\in\mathbb{C}^*$ and $z\in Z$ are arbitrary. Hence $\zeta^*(\alpha1^*)=\alpha\zeta$. Similarly we see, that for all $\Theta\in Z^{**}$,

$$\zeta^{**}(\Theta)=\langle\Theta,\zeta\rangle1^{**}. \tag{2}$$

Let X, Y, Z be complex Banach spaces and $m\in Bil(X\times Y,Z)$. Compositions of m with linear operators give bilinear operators. Let X', Y' and Z' be complex Banach spaces and let $Q\in B(X',X)$, $R\in B(Y',Y)$, $T\in B(Z,Z')$. Then $m\circ(Q,R)\in Bil(X'\times Y',Z)$ and $T\circ m\in Bil(X\times Y,Z')$ are given by

$$m\circ(Q,R)(x',y')=m(Qx',Ry')\qquad(x'\in X',\ y'\in Y')$$

and

$$T \circ m(x, y) = T(m(x, y)) \qquad (x \in X, y \in Y).$$

Since the statements in the following lemmas are easy to check the proofs are omitted.

Lemma 2.1. *Let $m \in Bil(X \times Y, Z)$, $Q \in B(X', X)$ and $R \in B(Y', Y)$ be arbitrary. Then*
(i) $(m \circ (Q, R))^t = m^t \circ (R, Q)$;
(ii) $\big(m \circ (Q, R)\big)^* = R^* \circ m^* \circ (I_{Z^*}, Q)$;
(iii) $\big(m \circ (Q, R)\big)^{**} = Q^* \circ m^{**} \circ (R^{**}, I_{Z^*})$;
(iv) $\big(m \circ (Q, R)\big)^{***} = m^{***} \circ (Q^{**}, R^{**})$ and $\big(m \circ (Q, R)\big)^{t***t} = m^{t***t} \circ (Q^{**}, R^{**})$.

Lemma 2.2. *Let $m \in Bil(X \times Y, Z)$ and $T \in B(Z, Z')$ be arbitrary. Then*
(i) $(T \circ m)^t = T \circ m^t$;
(ii) $(T \circ m)^* = m^* \circ (T^*, I_X)$;
(iii) $(T \circ m)^{**} = m^{**} \circ (I_{Y^{**}}, T^*)$;
(iv) $(T \circ m)^{***} = T^{**} \circ m^{***}$ and $(T \circ m)^{t***t} = T^{**} \circ m^{t***t}$.

3. Arens functionals of a bilinear operator

Let $m \in Bil(X \times Y, Z)$. We say that $\zeta \in Z^*$ is Arens functional of m if the bilinear form $\zeta \circ m : X \times Y \to \mathbb{C}$ is Arens regular. Of course, 0 is Arens functional of every m. Let $Ar(m) \subseteq Z^*$ be the set of all Arens functionals of m. As a special example we mention that for a bilinear form $\omega \in Bil(X \times Y, \mathbb{C})$ we have $Ar(\omega) = \mathbb{C}1^*$ if and only if ω is Arens regular and $Ar(\omega) = \{0\}$ otherwise. Arens [1] proved that $m \in Bil(X \times Y, Z)$ is Arens regular if and only if $Ar(m) = Z^*$. We include a slightly more general result. To prove it, we need the following proposition.

Proposition 3.1. *$Ar(m)$ is a weakly closed subspace of Z^*.*

Proof. It is obvious that $Ar(m)$ is a linear subspace of Z^*. Let $(\zeta_i)_{i \in \mathbb{I}} \subseteq Ar(m)$ be a net which converges to $\zeta \in Z^*$ in the weak topology. Then

$$\langle m^{***}(\Phi, \Gamma) - m^{t***t}(\Phi, \Gamma), \zeta\rangle = \lim_{i \in \mathbb{I}} \langle m^{***}(\Phi, \Gamma) - m^{t***t}(\Phi, \Gamma), \zeta_i\rangle = 0$$

for arbitrary $\Phi \in X^{**}$, $\Gamma \in Y^{**}$. It follows, by (2), that $\zeta^{**}\Big(m^{***}(\Phi, \Gamma) - m^{t***t}(\Phi, \Gamma)\Big) = 0$ and therefore $(\zeta \circ m)^{***}(\Phi, \Gamma) = (\zeta \circ m)^{t***t}(\Phi, \Gamma)$, by Lemma 2.2 (iv). □

We will call $Ar(m)$ the Arens space of a bilinear operator m.

Theorem 3.2 (cf. Theorem 2.3 [1]). *The following are equivalent for $m \in Bil(X \times Y, Z)$:*
(i) m is Arens regular;
(ii) for every complex Banach space Z' and every $T \in B(Z, Z')$, the bilinear mapping $T \circ m$ is Arens regular;
(iii) there exists a subset $\mathcal{D} \subseteq Z^*$ whose linear span is weakly dense in Z^* and $\zeta \circ m$ is Arens regular for every $\zeta \in \mathcal{D}$.

Proof. (i)⇒(ii). Let Z' be a complex Banach space and $T \in B(Z, Z')$. Since m is Arens regular we have $(T \circ m)^{***} = (T \circ m)^{t***t}$, by Lemma 2.2 (iv). To prove (ii)⇒(iii), take $Z' = \mathbb{C}$. To see (iii)⇒(i), assume that m is not Arens regular. Then there exist $\Phi \in X^{**}$ and $\Gamma \in Y^{**}$ such that $m^{***}(\Phi, \Gamma) \neq m^{t***t}(\Phi, \Gamma)$. Hence, there exists $\zeta \in Z^*$ such that $\langle m^{***}(\Phi, \Gamma), \zeta\rangle \neq \langle m^{t***t}(\Phi, \Gamma), \zeta\rangle$. By (2), $\zeta^{**} \circ m^{***}(\Phi, \Gamma) \neq \zeta^{**} \circ m^{t***t}(\Phi, \Gamma)$ which gives, by Lemma 2.2 (iv), $(\zeta \circ m)^{***} \neq (\zeta \circ m)^{t***t}$. It follows that $\zeta \notin Ar(m)$. Since, by Proposition 3.1, $Ar(m)$ is a weakly closed subspace of Z^* we conclude that there does not exist $\mathcal{D} \subseteq Z^*$ such that (iii) holds. □

In the following theorem the inclusion relations between the Arens space of $m \in Bil(X \times Y, Z)$ and the Arens spaces of compositions of m with linear operators and their consequences for Arens regularity are presented.

Theorem 3.3. *Let X, X', Y, Y', Z, Z' be complex Banach spaces and $m \in Bil(X \times Y, Z)$. Then, for arbitrary $Q \in B(X', X)$, $R \in B(Y', Y)$, $T \in B(Z, Z')$, the following hold.*

(i) $Ar(m) \subseteq Ar(m \circ (Q, R))$.

(ii) If m is Arens regular, then $m \circ (Q, R)$ is Arens regular.

(iii) If $m \circ (Q, R)$ is Arens regular and Q^{**}, R^{**} are surjective, then m is Arens regular.

(iv) $Ar(T \circ m) = \{\zeta' \in Z'^*;\ T^*\zeta' \in Ar(m)\}$; in particular, $T \circ m$ is Arens regular if and only if $T^*(Z'^*) \subseteq Ar(m)$.

(v) If m is Arens regular, then $T \circ m$ is Arens regular.

(vi) If $T \circ m$ is Arens regular and T^{**} is injective, then m is Arens regular.

Proof. (i) Let $\zeta \in Ar(m)$. Then $(\zeta \circ m)^{***} = (\zeta \circ m)^{t***t}$. By using Lemma 2.1 it is not hard to see that $\left(\zeta \circ m \circ (Q, R)\right)^{***} = (\zeta \circ m)^{***} \circ (Q^{**}, R^{**}) = (\zeta \circ m)^{t***t} \circ (Q^{**}, R^{**}) = \left(\zeta \circ m \circ (Q, R)\right)^{t***t}$. Hence $\zeta \in Ar(m \circ (Q, R))$.

(ii) If m is Arens regular, then $Ar(m) = Z^*$. It follows, by (i), that $Ar(m \circ (Q, R)) = Z^*$.

(iii) Let $\Phi \in X^{**}$ and $\Gamma \in Y^{**}$ be arbitrary. By the surjectivity, there exist $\Phi' \in X'^{**}$ and $\Gamma' \in Y'^{**}$ such that $\Phi = Q^{**}\Phi'$ and $\Gamma = R^{**}\Gamma'$. Hence

$$\begin{aligned} m^{t***t}(\Phi, \Gamma) &= m^{t***t}(Q^{**}\Phi', R^{**}\Gamma') = \left(m \circ (Q, R)\right)^{t***t}(\Phi', \Gamma') \\ &= \left(m \circ (Q, R)\right)^{***}(\Phi', \Gamma') = m^{***}(\Phi, \Gamma). \end{aligned}$$

(iv) Let $\zeta' \in Z'^*$ be arbitrary. Since $(T^*\zeta') \circ m(x, y) = \langle T^*\zeta', m(x, y)\rangle = \zeta' \circ T \circ m(x, y)$, for all $x \in X, y \in Y$, we have $\zeta' \circ T \circ m = (T^*\zeta') \circ m$. Hence, $\zeta' \in Ar(T \circ m)$ if and only if $T^*\zeta' \in Ar(m)$.

(v) If m is Arens regular, then $Ar(m) = Z^*$ and therefore $T^*\zeta' \in Ar(m)$ for every $\zeta' \in Z'^*$. It follows, by (iv), $Ar(T \circ m) = Z'^*$.

(vi) Since $T \circ m$ is Arens regular $Ar(m)$ contains the image of T^*, by (iv). By injectivity of T^{**}, the image of T^* is a weakly dense subset of Z^*. We conclude, by Proposition 3.1, that $Ar(m) = Z^*$. □

Let $U \subseteq X$, $V \subseteq Y$ be a closed subspace and $Q : U \hookrightarrow X$, $R : V \hookrightarrow Y$ be the embeddings. For $m \in Bil(X \times Y, Z)$, let $m|_{(U,V)} = m \circ (Q, R)$ be the restriction of m to $U \times V$. It follows easily from Theorem 3.3 that $m|_{(U,V)}$ is Arens regular whenever m is Arens regular. In particular, if $\mathcal{A}$ is an Arens regular Banach algebra, i.e., the multiplication in $\mathcal{A}$ is Arens regular, then every closed subalgebra of $\mathcal{A}$ is Arens regular.

4. Arens regularity of the tensor operator

Let X and Y be complex Banach spaces and let $\tau : X \times Y \to X \widehat{\otimes} Y$ be given by $\tau(x, y) = x \otimes y$ ($x \in X, y \in Y$). It is clear that τ is a bilinear operator. Now on we call this mapping the tensor operator and if we want to point out the underlying spaces, then we denote it by $\tau_{X,Y}$. If Z is a complex Banach space and $m \in Bil(X \times Y, Z)$, then there exists a unique linear operator $M : X \widehat{\otimes} Y \to Z$ such that $m(x, y) = M(x \otimes y)$ for all $x \in X, y \in Y$, that is, $m = M \circ \tau$ (see [10, Theorem 2.9]). We call $m = M \circ \tau$ the canonical factorization of m. Note that, by Theorem 3.3 (iv), $Ar(m) = \{\zeta \in Z^*;\ M^*\zeta \in Ar(\tau)\}$ and m is Arens regular if and only if $M^*(Z^*) \subseteq Ar(\tau)$.

In order to describe $Ar(\tau)$, recall that an operator $T \in B(X, Y)$ is said to be weakly compact if $T(\{x \in X;\ \|x\| \leq 1\})$ is relatively weakly compact in Y. The set $W(X, Y)$ of all weakly compact operators from X to Y is an operator ideal (see [9, Corollary 4.1]). In the proof of the following theorem we will use the fact that $T \in B(X, Y)$ is weakly compact if and only if $T^{**}(X^{**}) \subseteq \iota_Y(Y)$ (see [9, Theorem 4.5]).

Theorem 4.1. *Let X and Y be complex Banach spaces. Then $\zeta \in (X \widehat{\otimes} Y)^*$ is in $Ar(\tau)$ if and only if $\varphi(\zeta)$ is weakly compact, that is, $\varphi(Ar(\tau)) = W(X, Y^*)$. Similarly, $\psi(Ar(\tau)) = W(Y, X^*)$.*

Proof. Let $\zeta \in (X\widehat{\otimes}Y)^*$. Denote by τ_ζ the bilinear form $\zeta \circ \tau : X \times Y \to \mathbb{C}$. Hence $\tau_\zeta(x,y) = \langle \varphi(\zeta)x, y\rangle$ for all $x \in X$ and $y \in Y$. Straightforward computations show that

$$\tau_\zeta^*(\alpha 1^*, x) = \alpha\varphi(\zeta)x, \quad \tau_\zeta^{**}(\Gamma, \alpha 1^*) = \alpha\varphi(\zeta)^*\Gamma, \quad \tau_\zeta^{***}(\Phi,\Gamma) = \langle \varphi(\zeta)^{**}\Phi, \Gamma\rangle 1^{**},$$
$$\tau_\zeta^t(y,x) = \langle \varphi(\zeta)^*\widehat{y}, x\rangle, \quad \tau_\zeta^{t*}(\alpha 1^*, y) = \alpha\varphi(\zeta)^*\widehat{y}, \quad \tau_\zeta^{t**}(\Phi, \alpha 1^*) = \alpha\iota_Y^*(\varphi(\zeta)^{**}\Phi),$$
$$\tau_\zeta^{t***}(\Gamma,\Phi) = \langle \Gamma, \iota_Y^*(\varphi(\zeta)^{**}\Phi)\rangle 1^{**}, \quad \text{and} \quad \tau_\zeta^{t***t}(\Phi,\Gamma) = \langle \iota_{Y^*}\Big(\iota_Y^*(\varphi(\zeta)^{**}\Phi)\Big), \Gamma\rangle 1^{**},$$

where $\alpha \in \mathbb{C}$, $x \in X$, $y \in Y$, $\Phi \in X^{**}$ and $\Gamma \in Y^{**}$ are arbitrary. Hence, τ_ζ is Arens regular if and only if $\langle \iota_{Y^*}\Big(\iota_Y^*(\varphi(\zeta)^{**}\Phi)\Big), \Gamma\rangle = \langle \varphi(\zeta)^{**}\Phi, \Gamma\rangle$ holds for all $\Phi \in X^{**}$ and $\Gamma \in Y^{**}$.

Assume that $\varphi(\zeta)$ is weakly compact. Let $\Phi \in X^{**}$ be arbitrary. By [9, Theorem 4.5], there exists $\eta \in Y^*$ such that $\varphi(\zeta)^{**}\Phi = \widehat{\eta}$. It follows that $\langle \varphi(\zeta)^{**}\Phi, \Gamma\rangle = \langle \Gamma, \eta\rangle$ and $\langle \iota_{Y^*}\Big(\iota_Y^*(\varphi(\zeta)^{**}\Phi)\Big), \Gamma\rangle = \langle \Gamma, \iota_Y^*(\widehat{\eta})\rangle = \langle \Gamma, \eta\rangle$ for every $\Gamma \in Y^{**}$. This proves that τ_ζ is Arens regular. The proof of the opposite implication is shorter: if, for $\Phi \in X^{**}$, equality $\langle \iota_{Y^*}\Big(\iota_Y^*(\varphi(\zeta)^{**}\Phi)\Big), \Gamma\rangle = \langle \varphi(\zeta)^{**}\Phi, \Gamma\rangle$ holds for every $\Gamma \in Y^{**}$, then $\varphi(\zeta)^{**}\Phi = \iota_{Y^*}\Big(\iota_Y^*(\varphi(\zeta)^{**}\Phi)\Big)$. Hence, $\varphi(\zeta)^{**}(X^{**}) \subseteq \iota_{Y^*}(Y^*)$ which is equivalent to the weak compactness of $\varphi(\zeta)$. It is obvious that the equality $\psi(Ar(\tau)) = W(Y, X^*)$ can be proven similarly. $\square$

The following corollary states the equivalence of items (i) and (iv) of [7, Theorem 2.1].

Corollary 4.2. *For $m \in Bil(X \times Y, Z)$ with the canonical factorization $m = M \circ \tau$, the following assertions are equivalent:*
(i) m is Arens regular,
(ii) $\varphi(M^*\zeta) \in B(X, Y^*)$ is weakly compact for every $\zeta \in Z^*$,
(iii) $\psi(M^*\zeta) \in B(Y, X^*)$ is weakly compact for every $\zeta \in Z^*$.

Proof. The equivalences hold by Theorems 3.3 (iv) and 4.1. $\square$

Corollary 4.3. *The bilinear mapping $\tau : X \times Y \to X\widehat{\otimes}Y$ is Arens regular if and only if every operator in $B(X, Y^*)$, or every operator in $B(Y, X^*)$, is weakly compact.*

If X or Y is reflexive, then every $T \in B(X, Y)$ is weakly compact. However there exist pairs of non-reflexive Banach spaces X and Y such that $B(X, Y) = W(X, Y)$. Recall from [9, §4.3] that a Banach space X is said to be a Grothendieck space if every sequence $(\xi_n)_{n=1}^\infty \subseteq X^*$ which converges to 0 in the w^*-topology converges to 0 in the weak topology, as well. A Banach space X is WCG (weakly compactly generated) if there exists a weakly compact subset $K \subseteq X$ such that X is the closed linear span of K. By [9, Theorem 4.9], $B(X, Y) = W(X, Y)$ if X is a Grothendieck space and Y is WCG.

Corollary 4.4. *Let X be a Grothendieck space, Y^* a WCG space, and Z an arbitrary Banach space. Then every $m \in Bil(X \times Y, Z)$ is Arens regular.*

Proof. By the assumptions, $B(X, Y^*) = W(X, Y^*)$ which implies, by Corollary 4.3, that τ is Arens regular. Now the assertion follows by Theorem 3.3 (iv). $\square$

Corollary 4.5 (Ülger, Theorem 2.2 [11]). *Let X and Y be complex Banach spaces. A bilinear form $\mu \in Bil(X\times Y, \mathbb{C})$ is Arens regular if and only if $\rho(\mu) \in B(X, Y^*)$ (or $\lambda(\mu) \in B(Y, X^*)$) is weakly compact.*

Example 4.6. *Let X be a complex Banach space. Then $\gamma(\xi, x) = \langle \xi, x\rangle$ is a bounded bilinear form on $X^* \times X$. Let X', Y' be complex Banach spaces and let $Q \in B(X', X^*)$, $R \in B(Y', X)$ be arbitrary operators. Let $\gamma_{Q,R}(x', y') = \langle Qx', Ry'\rangle$. This is a bilinear form on $X' \times Y'$. Since $\langle \rho(\gamma_{Q,R})x', y'\rangle = \langle Qx', Ry'\rangle = \langle R^*Qx', y'\rangle$ for all $x' \in X'$ and $y' \in Y'$ we have $\rho(\gamma_{Q,R}) = R^*Q$. Hence, $\gamma_{Q,R}$ is Arens regular if and only if R^*Q is a weakly compact operator from X' to Y'^*. In particular, the bilinear form γ is Arens regular if and only if $I_X^* I_{X^*} = I_{X^*}$ is weakly compact. It is well known that the identity operator is weakly compact if and only if the underlying space is reflexive. Hence, γ is Arens regular if and only if X is a reflexive space.*

Proposition 4.7. *Let X, X', Y, Y' and Z be complex Banach spaces. Let $m \in Bil(X \times Y, Z)$ and $Q \in B(X', X)$, $R \in B(Y', Y)$ be arbitrary. Let $m = M \circ \tau$ be the canonical factorization of m. Then $Ar(m \circ (Q, R)) = \{\zeta \in Z^*;\ R^*\varphi(M^*\zeta)Q \in W(X', Y'^*)\}$.*

Proof. Denote $\tau' = \tau_{X',Y'}$ and let $m \circ (Q, R) = M' \circ \tau'$ be the canonical factorization of $m \circ (Q, R)$. For arbitrary $x' \in X'$, $y' \in Y'$ we have $m \circ (Q, R)(x', y') = M \circ \tau(Qx', Ry') = M(Q \otimes R)(x' \otimes y') = M(Q \otimes R)\tau'(x', y')$. Hence, $M' = M(Q \otimes R)$. It follows, by Corollary 4.2, that $\zeta \in Z^*$ is in $Ar(m \circ (Q, R))$ if and only if $\varphi'\big((Q \otimes R)^* M^*\zeta\big)$ is a weakly compact operator, where by φ' we have denoted the natural isometric isomorphism from $(X'\widehat{\otimes}Y')^*$ to $B(X', Y'^*)$. It is easily seen that $\varphi'\big((Q \otimes R)^* M^*\zeta\big) = R^*\varphi(M^*\zeta)Q$. □

Let $\mathcal{A}$ be a complex Banach algebra and let a complex Banach space X be a left Banach module over $\mathcal{A}$ through the multiplication $m(a, x) = a \cdot x$ ($a \in \mathcal{A}, x \in X$). It is assumed that $\|m\| \le 1$, that is $\|a \cdot x\| \le \|a\|\|x\|$ for all $a \in \mathcal{A}$, $x \in X$. Through the mapping $m^* : X^* \times \mathcal{A} \to X^*$ the dual space X^* is equipped with the structure of a right Banach $\mathcal{A}$-module. For every $\xi \in X^*$, we have a bounded linear operator $L_\xi : \mathcal{A} \to X^*$ given by $L_\xi a = \xi \cdot a$. The following proposition is an extension of [5, Theorem 2.1], see also [4, Theorem 3.4].

Proposition 4.8. *Let X be a left Banach $\mathcal{A}$-module through the multiplication m. Then m is Arens regular if and only if $L_\xi \in B(\mathcal{A}, X^*)$ is a weakly compact operator for every $\xi \in X^*$.*

Proof. Let $m = M \circ \tau$ be the canonical factorization. Then, for every $\xi \in X^*$, we have $\langle \varphi(M^*\xi)a, x\rangle = \langle M^*\xi, a \otimes x\rangle = \langle \xi, m(a, x)\rangle = \langle L_\xi a, x\rangle$ for all $a \in \mathcal{A}$, $x \in X$. Hence $\varphi(M^*\xi) = L_\xi$. Now the assertion follows by Corollary 4.2. □

Let $\mathcal{A}$ be a complex Banach algebra and let m denotes the multiplication in $\mathcal{A}$. A functional $\zeta \in \mathcal{A}^*$ is said to be weakly almost periodic on $\mathcal{A}$ if $L_\zeta \in B(\mathcal{A}, \mathcal{A}^*)$ is a weakly compact operator. Hence in this case Arens functionals of m are precisely the weakly almost periodic functionals.

5. Topological centers

The left topological center of $m \in Bil(X \times Y, Z)$ is

$$\mathcal{Z}^\ell(m) = \{\Phi \in X^{**};\quad m^{***}(\Phi, \ \cdot\ ) : Y^{**} \to Z^{**} \quad \text{is } w^*\text{-}w^* \text{ continuous}\}$$

and the right topological center of m is $\mathcal{Z}^r(m) = \mathcal{Z}^\ell(m^t)$ (see [2, 3, 6]). It is not hard to see that $\mathcal{Z}^\ell(m) = \{\Phi \in X^{**};\ m^{***}(\Phi, \Gamma) = m^{t***t}(\Phi, \Gamma) \text{ for all } \Gamma \in Y^{**}\}$, and therefore $\mathcal{Z}^r(m) = \{\Gamma \in Y^{**};\ m^{***}(\Phi, \Gamma) = m^{t***t}(\Phi, \Gamma) \text{ for all } \Phi \in X^{**}\}$. Hence, a bilinear operator $m \in Bil(X \times Y, Z)$ is Arens regular if and only if $\mathcal{Z}^\ell(m) = X^{**}$ or $\mathcal{Z}^r(m) = Y^{**}$. It is clear that $\widehat{X} \subseteq \mathcal{Z}^\ell(m)$ and $\widehat{Y} \subseteq \mathcal{Z}^r(m)$. If m is such that $\mathcal{Z}^\ell(m) = \widehat{X}$, then m is said to be left strongly Arens irregular. Similarly, if $\mathcal{Z}^r(m) = \widehat{Y}$, then m is right strongly Arens irregular. There exist bilinear operators which are left strongly Arens irregular but not right strongly Arens irregular (and vice versa), see [3].

Lemma 5.1. *Let $m \in Bil(X \times Y, Z)$ and let $m = M \circ \tau$ be its canonical factorization. If $\zeta \in Z^*$, then*

$$(\zeta \circ m)^{***}(\Phi, \Gamma) = \langle \varphi(M^*\zeta)^{**}\Phi, \Gamma\rangle 1^{**} = \langle \Phi, \iota_X^*\psi(M^*\zeta)^{**}\Gamma\rangle 1^{**} \tag{3}$$

and

$$(\zeta \circ m)^{t***t}(\Phi, \Gamma) = \langle \psi(M^*\zeta)^{**}\Gamma, \Phi\rangle 1^{**} = \langle \Gamma, \iota_Y^*\big(\varphi(M^*\zeta)^{**}\Phi\big)\rangle 1^{**} \tag{4}$$

for every $\Phi \in X^{**}$, $\Gamma \in Y^{**}$.

Proof. We will prove the first equality in (3) and the second equality in (4). By (1), we have $\zeta\circ m(x,y) = \langle \zeta, M\circ\tau(x,y)\rangle = \langle M^*\zeta, x\otimes y\rangle = \langle \varphi(M^*\zeta)x, y\rangle$ for every $x\in X, y\in Y$. Hence $\langle(\zeta\circ m)^*(1^*,x),y\rangle = \langle 1^*, \zeta\circ m(x,y)\rangle = \langle 1^*, \langle\varphi(M^*\zeta)x,y\rangle\rangle = \langle\varphi(M^*\zeta)x,y\rangle$ for every $x\in X, y\in Y$ which gives $(\zeta\circ m)^*(1^*,x) = \varphi(M^*\zeta)x$ for every $x\in X$. Let $\Gamma\in Y^{**}$ be arbitrary. Then $\langle(\zeta\circ m)^{**}(\Gamma,1^*),x\rangle = \langle\Gamma,(\zeta\circ m)^*(1^*,x)\rangle = \langle\Gamma,\varphi(M^*\zeta)x\rangle = \langle\varphi(M^*\zeta)^*\Gamma,x\rangle$ for every $x\in X$ and therefore $(\zeta\circ m)^{**}(\Gamma,1^*) = \varphi(M^*\zeta)^*\Gamma$. Let $\Phi\in X^{**}, \Gamma\in Y^{**}$ be arbitrary. We have $\langle(\zeta\circ m)^{***}(\Phi,\Gamma),1^*\rangle = \langle\Phi,(\zeta\circ)^{**}(\Gamma,1^*)\rangle = \langle\Phi,\varphi(M^*\zeta)^*\Gamma\rangle = \langle\varphi(M^*\zeta)^{**}\Phi,\Gamma\rangle = \langle\langle\varphi(M^*\zeta)^{**}\Phi,\Gamma\rangle 1^{**},1^*\rangle$ which shows that the first equality in (3) holds.

Since $(\zeta\circ m)^t(y,x) = \langle\varphi(M^*\zeta)x,y\rangle$ we have $\langle(\zeta\circ m)^{t*}(1^*,y),x\rangle = \langle 1^*,\langle\varphi(M^*\zeta)x,y\rangle\rangle = \langle\varphi(M^*\zeta)x,y\rangle = \langle\varphi(M^*\zeta)^*\iota_Y(y),x\rangle$ for every $x\in X, y\in Y$ which gives $(\zeta\circ m)^{t*}(1^*,y) = \varphi(M^*\zeta)^*\iota_Y(y)$. Let $\Phi\in X^{**}$ be arbitrary. Then $\langle(\zeta\circ m)^{t**}(\Phi,1^*),y\rangle = \langle\Phi,\varphi(M^*\zeta)^*\iota_Y(y)\rangle = \langle\iota_Y^*\varphi(M^*\zeta)^{**}\Phi,y\rangle$ $(y\in Y)$ and therefore $(\zeta\circ m)^{t**}(\Phi,1^*) = \iota_Y^*\varphi(M^*\zeta)^{**}\Phi$. Now, for arbitrary $\Phi\in X^{**}$ and $\Gamma\in Y^{**}$, it follows from $\langle(\zeta\circ m)^{t***}(\Gamma,\Phi),1^*\rangle = \langle\Gamma,\iota_Y^*\varphi(M^*\zeta)^{**}\Phi\rangle = \langle\langle\Gamma,\iota_Y^*\varphi(M^*\zeta)^{**}\Phi\rangle 1^{**},1^*\rangle$ that the second equality in (4) holds. □

Proposition 5.2. *Let $m\in Bil(X\times Y,Z)$ and let $m = M\circ\tau$ be its canonical factorization. If $\zeta\in Z^*$, then $\mathcal{Z}^\ell(\zeta\circ m) = \{\Phi\in X^{**};\ \varphi(M^*\zeta)^{**}\Phi\in\widehat{Y^*}\}$ and $\mathcal{Z}^r(\zeta\circ m) = \{\Gamma\in Y^{**};\ \psi(M^*\zeta)^{**}\Gamma\in\widehat{X^*}\}$.*

Proof. Assume that $\Phi\in X^{**}$ is such that $\varphi(M^*\zeta)^{**}\Phi\in\widehat{Y^*}$, say $\varphi(M^*\zeta)^{**}\Phi = \widehat{\eta}$, where $\eta\in Y^*$. Note that $\langle\eta,y\rangle = \langle\widehat{\eta},\widehat{y}\rangle = \langle\iota_Y^*\varphi(M^*\zeta)^{**}\Phi,y\rangle$ for every $y\in Y$ which gives $\eta = \iota_Y^*\varphi(M^*\zeta)^{**}\Phi$. Hence $\langle\varphi(M^*\zeta)^{**}\Phi,\Gamma\rangle = \langle\Gamma,\eta\rangle = \langle\Gamma,\iota_Y^*\big(\varphi(M^*\zeta)^{**}\Phi\big)\rangle$ holds for every $\Gamma\in Y^{**}$. It follows, by Lemma 5.1, that $(\zeta\circ m)^{***}(\Phi,\Gamma) = (\zeta\circ m)^{t***t}(\Phi,\Gamma)$ for every $\Gamma\in Y^{**}$, that is, $\Phi\in\mathcal{Z}^\ell(m)$. To see the opposite inclusion, suppose that $\Phi\in\mathcal{Z}^\ell(m)$. Then $(\zeta\circ m)^{***}(\Phi,\Gamma) = (\zeta\circ m)^{t***t}(\Phi,\Gamma)$ for every $\Gamma\in Y^{**}$ which gives, by Lemma 5.1, $\langle\varphi(M^*\zeta)^{**}\Phi,\Gamma\rangle = \langle\Gamma,\iota_Y^*\big(\varphi(M^*\zeta)^{**}\Phi\big)\rangle$ for every $\Gamma\in Y^{**}$. We conclude that $\varphi(M^*\zeta)^{**}\Phi = \iota_{Y^*}\Big(\iota_Y^*\big(\varphi(M^*\zeta)^{**}\Phi\big)\Big)\in\widehat{Y^*}$.

The second equality follows now from the first equality. Indeed, note that $\mathcal{Z}^r(\zeta\circ m) = \mathcal{Z}^\ell((\zeta\circ m)^t)$ which means that the second equality is actually the first one with X and Y interchanged. □

By [9, Theorem 4.5], it follows from Proposition 5.2 that $\varphi(M^*\zeta)$ is a weakly compact operator if and only if $\mathcal{Z}^\ell(\zeta\circ m) = X^{**}$, that is, if and only if $\zeta\in Ar(m)$. Similarly, $\psi(M^*\zeta)$ is a weakly compact operator if and only if $\mathcal{Z}^r(\zeta\circ m) = Y^{**}$.

For an arbitrary $m\in Bil(X\times Y,Z)$, we have the following characterization of the left and the right topological center of m.

Proposition 5.3. *The topological centers of $m\in Bil(X\times Y,Z)$ are*

$$\mathcal{Z}^\ell(m) = \bigcap_{\zeta\in Z^*}\mathcal{Z}^\ell(\zeta\circ m)\quad \textit{and}\quad \mathcal{Z}^r(m) = \bigcap_{\zeta\in Z^*}\mathcal{Z}^r(\zeta\circ m).$$

Proof. Let $m\in Bil(X\times Y,Z)$ and let $m = M\circ\tau$ be its canonical factorization. If $\zeta\in Z^*$, then $\varphi(M^*\zeta)x = m^*(\zeta,x)$ for all $x\in X$. It follows that $m^{**}(\Gamma,\zeta) = \varphi(M^*\zeta)^*\Gamma$ for all $\Gamma\in Y^{**}$ and finally we have $\langle m^{***}(\Phi,\Gamma),\zeta\rangle = \langle\varphi(M^*\zeta)^{**}\Phi,\Gamma\rangle$ for all $\Phi\in X^{**}$ and $\Gamma\in Y^{**}$. Similarly, it is straightforward that $m^{t*}(\zeta,y) = \varphi(M^*\zeta)^*\widehat{y}$ for all $y\in Y$. This gives $m^{t**}(\Phi,\zeta) = \iota_Y^*\big(\varphi(M^*\zeta)^{**}\Phi\big)$ for all $\Phi\in X^{**}$. Hence, $\langle m^{t***t}(\Phi,\Gamma),\zeta\rangle = \langle\Gamma,\iota_Y^*\big(\varphi(M^*\zeta)^{**}\Phi\big)\rangle = \langle\iota_{Y^*}\Big(\iota_Y^*\big(\varphi(M^*\zeta)^{**}\Phi\big)\Big),\Gamma\rangle$ for all $\Phi\in X^{**}$ and $\Gamma\in Y^{**}$.

Assume that $\Phi\in Z^\ell(m)$. Let $\zeta\in Z^*$ be arbitrary. Since $m^{***}(\Phi,\Gamma) = m^{t***t}(\Phi,\Gamma)$ for all $\Gamma\in Y^{**}$ we conclude that $\varphi(M^*\zeta)^{**}\Phi$ is equal to $\iota_{Y^*}\Big(\iota_Y^*\big(\varphi(M^*\zeta)^{**}\Phi\big)\Big)\in\widehat{Y^*}$, which shows that $\Phi\in\mathcal{Z}^\ell(\zeta\circ m)$.

Suppose now that $\Phi\in X^{**}$ is not in $\mathcal{Z}^\ell(m)$. Then there exists $\Gamma\in Y^{**}$ such that $m^{***}(\Phi,\Gamma)\neq m^{t***t}(\Phi,\Gamma)$. It follows that there exists $\zeta\in Z^*$ such that $\langle m^{***}(\Phi,\Gamma),\zeta\rangle\neq\langle m^{t***t}(\Phi,\Gamma),\zeta\rangle$. By the previous paragraph, $\varphi(M^*\zeta)^{**}\Phi\neq\iota_{Y^*}\Big(\iota_Y^*\big(\varphi(M^*\zeta)^{**}\Phi\big)\Big)$. If there were $\eta\in Y^*$ such that $\varphi(M^*\zeta)^{**}\Phi = \widehat{\eta}$, then we would have $\langle\eta,y\rangle = \langle\widehat{\eta},\widehat{y}\rangle = \langle\varphi(M^*\zeta)^{**}\Phi,\iota_Y(y)\rangle = \langle\iota_Y^*\big(\varphi(M^*\zeta)^{**}\Phi\big),y\rangle$ for every $y\in Y$. Hence, we would have $\eta = \iota_Y^*\big(\varphi(M^*\zeta)^{**}\Phi\big)$ and consequently $\varphi(M^*\zeta)^{**}\Phi = \iota_{Y^*}(\eta) = \iota_{Y^*}\Big(\iota_Y^*\big(\varphi(M^*\zeta)^{**}\Phi\big)\Big)$ which is a contradiction. We conclude that $\varphi(M^*\zeta)^{**}\Phi\notin\widehat{Y^*}$, that is $\Phi\notin\mathcal{Z}^\ell(\zeta\circ m)$.

The second equality is proven similarly. □

Assume that $m \in Bil(X \times Y, Z)$ is a left strongly Arens irregular bilinear operator, that is, $\mathcal{Z}^{\ell}(m) = \widehat{X}$. By Proposition 5.3, for every $\Phi \in X^{**} \setminus \widehat{X}$, there exists $\zeta_\Phi \in Z^*$ such that $\Phi \notin \mathcal{Z}^{\ell}(\zeta_\Phi \circ m)$. It would be interesting to know, for which left strongly Arens irregular bilinear operators m, there exists $\zeta_0 \in Z^*$ such that $\zeta_0 \circ m$ is left strongly Arens irregular, that is, $\mathcal{Z}^{\ell}(\zeta_0 \circ m) = \widehat{X}$.

6. Bilinear operators approximable by linear operators

If $m \in Bil(X \times Y, Z)$, then $m^* \in Bil(Z^* \times X, Y^*)$. Let $m^* = \tilde{M} \circ \tilde{\tau}$ be the canonical factorization (hence $\tilde{\tau} = \tau_{Z^*,X}$) and let $\tilde{\varphi} : (Z^* \widehat{\otimes} X)^* \to B(Z^*, X^*)$ be the natural isometric isomorphism (see (1)). It is not hard to see that, for arbitrary $\zeta \in Z^*$ and $\Gamma \in Y^{**}$, we have $m^{**}(\Gamma, \zeta) = \tilde{\varphi}(\tilde{M}^*\Gamma)\zeta$. It follows that the adjoint of $\tilde{\varphi}(\tilde{M}^*\Gamma) \in B(Z^*, X^*)$ satisfies $m^{***}(\Phi, \Gamma) = \tilde{\varphi}(\tilde{M}^*\Gamma)^*\Phi$ for every $\Phi \in X^{**}$. If $\Gamma = \widehat{y}$, where $y \in Y$, let $A_{\widehat{y}} \in B(X, Z)$ be defined by $A_{\widehat{y}}x = m(x, y)$ $(x \in X)$. Then $\langle m^{**}(\widehat{y}, \zeta), x\rangle = \langle \zeta, m(x, y)\rangle = \langle A^*_{\widehat{y}}\zeta, x\rangle$ for every $x \in X$. Hence $\tilde{\varphi}(\tilde{M}^*\widehat{y}) = A^*_{\widehat{y}}$ and therefore $\tilde{\varphi}(\tilde{M}^*\widehat{y})^* = A^{**}_{\widehat{y}}$. However, if $\Gamma \in Y^{**}$ is arbitrary, then it is not necessary that $\tilde{\varphi}(\tilde{M}^*\Gamma)^*$ is the second adjoint of an operator in $B(X, Z)$.

Example 6.1. *Let X, Y be complex Banach spaces, Y non-reflexive. Let $\xi \in X^*$ be non-zero. Then $m(x, y) = \langle \xi, x\rangle y$, where $x \in X, y \in Y$ are arbitrary, defines $m \in Bil(X \times Y, Y)$. It is easily seen that m^{***} is given by $m^{***}(\Phi, \Gamma) = \langle \Phi, \xi\rangle\Gamma$ for every $\Phi \in X^{**}, \Gamma \in Y^{**}$. Assume that $\Gamma \in Y^{**} \setminus \widehat{Y}$. If $\tilde{\varphi}(\tilde{M}^*\Gamma)^*$ were the second adjoint of $A_\Gamma \in B(X, Y)$, then we would have $\langle \Phi, A^*_\Gamma\eta\rangle = \langle \tilde{\varphi}(\tilde{M}^*\Gamma)^*\Phi, \eta\rangle = \langle \Phi, m^{**}(\Gamma, \eta)\rangle$ for all $\Phi \in X^{**}, \eta \in Y^*$ which would give $A^*_\Gamma\eta = m^{**}(\Gamma, \eta)$ for all $\eta \in Y^*$. It would follow that for $x \in X$ we have $\langle m^{***}(\widehat{x}, \Gamma), \eta\rangle = \langle m^{**}(\Gamma, \eta), x\rangle = \langle \widehat{A_\Gamma x}, \eta\rangle$ for all $\eta \in Y^*$. Hence $\widehat{A_\Gamma x} = m^{***}(\widehat{x}, \Gamma) = \langle \xi, x\rangle\Gamma$ for every $x \in X$. But this is impossible since $\xi \neq 0$ and $\Gamma \notin \widehat{Y}$. Note that this holds even in the case when X is reflexive and therefore m is Arens regular, which means that $\Gamma \in \mathcal{Z}^r(m) = Y^{**}$.*

Proposition 6.2. *Let $m \in Bil(X \times Y, Z)$. As in the paragraph before Example 6.1, let $\tilde{\varphi}(\tilde{M}^*\Gamma) \in B(Z^*, X^*)$ be such that $m^{**}(\Gamma, \zeta) = \tilde{\varphi}(\tilde{M}^*\Gamma)\zeta$ $(\zeta \in Z^*, \Gamma \in Y^{**})$. For $\Gamma \in Y^{**}$, there exists $A_\Gamma \in B(X, Z)$ such that $\tilde{\varphi}(\tilde{M}^*\Gamma) = A^*_\Gamma$ if and only if $\Gamma \in \mathcal{Z}^r(m)$ and $\tilde{\varphi}(\tilde{M}^*\Gamma)^*(\widehat{X}) \subseteq \widehat{Z}$.*

Proof. Assume that for $\Gamma \in Y^{**}$ there exists $A_\Gamma \in B(X, Z)$ such that $\tilde{\varphi}(\tilde{M}^*\Gamma) = A^*_\Gamma$. Then $\tilde{\varphi}(\tilde{M}^*\Gamma)^* = m^{***}(\cdot, \Gamma)$ is w^*-continuous, by [9, Proposition 4.6], which means that $\Gamma \in \mathcal{Z}^r(m)$, by the definition of the right topological center. For an arbitrary $x \in X$, we have $\langle \tilde{\varphi}(\tilde{M}^*\Gamma)^*\widehat{x}, \zeta\rangle = \langle \widehat{x}, A^*_\Gamma\zeta\rangle = \langle \widehat{A_\Gamma x}, \zeta\rangle$ for every $\zeta \in Z^*$. It follows that $\tilde{\varphi}(\tilde{M}^*\Gamma)\widehat{x} = \widehat{A_\Gamma x} \in \widehat{Z}$.

Suppose now that $\Gamma \in \mathcal{Z}^r(m)$ and $\tilde{\varphi}(\tilde{M}^*\Gamma)^*(\widehat{X}) \subseteq \widehat{Z}$. Then for every $x \in X$ there exists a unique $A_\Gamma x \in Z$ such that $\tilde{\varphi}(\tilde{M}^*\Gamma)^*\widehat{x} = \widehat{A_\Gamma x}$. It is not hard to see that $x \mapsto A_\Gamma x$ defines an operator $A_\Gamma \in B(X, Z)$. Since $\widehat{A_\Gamma x} = A^{**}_\Gamma\widehat{x}$ for every $x \in X$ and $\tilde{\varphi}(\tilde{M}^*\Gamma)^*$ is w^*-continuous, by the definition of the right topological center, we conclude that $\tilde{\varphi}(\tilde{M}^*\Gamma)^*\Phi = A^{**}_\Gamma\Phi$ for every $\Phi \in X^{**}$ which gives $\tilde{\varphi}(\tilde{M}^*\Gamma) = A^*_\Gamma$. $\square$

Let $m \in Bil(X \times Y, Z)$ and $\Gamma \in Y^{**}$ be such that there exists $A_\Gamma \in B(X, Z)$ satisfying $A^*_\Gamma\zeta = m^{**}(\Gamma, \zeta)$ for every $\zeta \in Z^*$. By the Goldstine's Theorem (see [8, Theorem 2.6.26]), there exists a net $(y_j)_{j\in\mathbb{J}} \subseteq Y$, bounded by $\|\Gamma\|$, such that Γ is the w^*-limit of $(\widehat{y_j})_{j\in\mathbb{J}}$. Hence, if $x \in X$, then $\langle \zeta, A_\Gamma x\rangle = \langle m^{**}(\Gamma, \zeta), x\rangle = \lim_{j\in\mathbb{J}}\langle \widehat{y_j}, m^*(\zeta, x)\rangle = \lim_{j\in\mathbb{J}}\langle \zeta, m(x, y_j)\rangle$ for every $\zeta \in Z^*$ which means that $(m(x, y_j))_{j\in\mathbb{J}}$ converges to $A_\Gamma x$ in the weak topology.

Definition 6.3. *A bilinear operator $m \in Bil(X \times Y, Z)$ is approximable by a linear operator $A \in B(X, Z)$ at a bounded net $(y_j)_{j\in\mathbb{J}} \subseteq Y$ if $w{-}\lim_{j\in\mathbb{J}} m(x, y_j) = Ax$ for every $x \in X$.*

Our definition is inspired by the notion of approximately unital bilinear mappings which are considered in [6]. In the following proposition we state a few conditions on $m \in Bil(X \times Y, Z)$ which are equivalent to the condition formulated in Definition 6.3.

Proposition 6.4. *Let $m \in Bil(X \times Y, Z)$ and $\Gamma \in Y^{**}$. Let $(y_j)_{j\in\mathbb{J}} \subseteq Y$ be a bounded net such that $\Gamma = w^*-\lim_{j\in\mathbb{J}} \widehat{y_j}$. Let $A_\Gamma \in B(X, Z)$. The following assertions are equivalent:*

(i) m is approximable by A_Γ at $(y_j)_{j\in\mathbb{J}}$;

(ii) $m^{**}(\Gamma, \zeta) = A_\Gamma^*\zeta$ for every $\zeta \in Z^*$;

(ii') $m^{*t*}(\Gamma, x) = A_\Gamma^{**}\widehat{x}$ for every $x \in X$;

(iii) $m^{***}(\Phi, \Gamma) = A_\Gamma^{**}\Phi$ for every $\Phi \in X^{**}$;

(iii') $m^{*t**}(\Omega, \Gamma) = \iota_X^*(A_\Gamma^{***}\Omega)$ for every $\Omega \in Z^{***}$.

Proof. (i)⇒(ii). Let $\zeta \in Z^*$ be arbitrary. For every $x \in X$, we have $|\langle m^{**}(\Gamma, \zeta) - A_\Gamma^*\zeta, x\rangle| \leq |\langle \Gamma, m^*(\zeta, x)\rangle - \langle \widehat{y_j}, m^*(\zeta, x)\rangle| + |\langle \zeta, m(x, y_j)\rangle - \langle \zeta, A_\Gamma x\rangle|$. Since Γ is a w^*-limit of $(\widehat{y_j})_{j\in\mathbb{J}}$ and $A_\Gamma x$ is w-limit of $(m(x, y_j))_{j\in\mathbb{J}}$ we conclude that $m^{**}(\Gamma, \zeta) = A_\Gamma^*\zeta$.

(ii)⇒(i). Let $x \in X$ be arbitrary. For every $\zeta \in Z^*$, we have $\langle \zeta, m(x, y_j) - A_\Gamma x\rangle = \langle m^{**}(\widehat{y_j}, \zeta) - A_\Gamma^*\zeta, x\rangle = \langle m^{**}(\widehat{y_j}, \zeta) - m^{**}(\Gamma, \zeta), x\rangle = \langle \widehat{y_j} - \Gamma, m^*(\zeta, x)\rangle$. Since $(\widehat{y_j})_{j\in\mathbb{J}}$ converges to Γ in the w^*-topology we conclude that $(m(x, y_j))_{j\in\mathbb{J}}$ converges to $A_\Gamma x$ in the weak topology.

(ii) $\iff$ (ii'). Let $x \in X$ and $\zeta \in Z^*$ be arbitrary. Then $\langle m^{**}(\Gamma, \zeta) - A_\Gamma^*\zeta, x\rangle = \langle \Gamma, m^{*t}(x, \zeta)\rangle - \langle A_\Gamma^{**}\widehat{x}, \zeta\rangle = \langle m^{*t*}(\Gamma, x) - A_\Gamma^{**}\widehat{x}, \zeta\rangle$. Hence, $m^{**}(\Gamma, \zeta) = A_\Gamma^*\zeta$ for every $\zeta \in X^*$ if and only if $m^{*t*}(\Gamma, x) = A_\Gamma^{**}\widehat{x}$ for every $x \in X$.

(ii) $\iff$ (iii). Let $\Phi \in X^{**}$ and $\zeta \in Z^*$ be arbitrary. Then $\langle m^{***}(\Phi, \Gamma_0) - A_\Gamma^{**}\Phi, \zeta\rangle = \langle \Phi, m^{**}(\Gamma, \zeta) - A_\Gamma^*\zeta\rangle$. Hence, $m^{***}(\Phi, \Gamma) = A_\Gamma^{**}\Phi$ for every $\Phi \in X^{**}$ if and only if $m^{**}(\Gamma, \zeta) = A_\Gamma^*\zeta$ for every $\zeta \in Z^*$.

(ii') $\iff$ (iii'). Let $x \in X$ and $\Omega \in Z^{***}$ be arbitrary. Then $\langle m^{*t**}(\Omega, \Gamma) - \iota_X^*(A_\Gamma^{***}\Omega), x\rangle = \langle \Omega, m^{*t*}(\Gamma, x)\rangle - \langle A_\Gamma^{***}\Omega, \widehat{x}\rangle = \langle \Omega, m^{*t*}(\Gamma, x) - A_\Gamma^{**}\widehat{x}\rangle$. Hence, $m^{*t**}(\Omega, \Gamma) = \iota_X^*(A_\Gamma^{***}\Omega)$ for every $\Omega \in Z^{***}$ if and only if $m^{*t*}(\Gamma, x) = A_\Gamma^{**}\widehat{x}$ for every $x \in X$. □

Let $m \in Bil(X \times Y, Z)$. If for $\Gamma \in Y^{**}$ there exists $A_\Gamma \in B(X, Y)$ such that $m^{***}(\Phi, \Gamma) = A_\Gamma^{**}\Phi$ for all $\Phi \in X^{**}$, then A_Γ is uniquely determined. Let $App(m)$ be a subset of Y^{**} of all those Γ such that there exists $A_\Gamma \in B(X, Z)$ satisfying $m^{***}(\Phi, \Gamma) = A_\Gamma^{**}\Phi$ for every $\Phi \in X^{**}$. It is clear that $App(m)$ is a linear subspace of Y^{**}. We have already observed that $\widehat{Y} \subseteq App(m)$ and, by Proposition 6.2, $App(m) \subseteq \mathcal{Z}^r(m)$. This last inclusion can be proper as we have seen in Example 6.1. Hence, in general, the condition $App(m) = Y^{**}$ is stronger than the condition $\mathcal{Z}^r(m) = Y^{**}$, which is equivalent to the Arens regularity of m. Of course, every bilinear operator satisfying the former condition satisfies the latter condition, as well. The converse does not hold in general as shows the bilinear operator m from Example 6.1 — if in that example X is reflexive and Y is non-reflexive, then $App(m) = \widehat{Y}$ and $\mathcal{Z}^r(m) = Y^{**}$ which means that $App(m)$ is a proper subspace of $\mathcal{Z}^r(m)$.

Theorem 6.5. *Let $m \in Bil(X \times Y, Z)$. Assume that for $\Gamma \in Y^{**}$ there exists $A_\Gamma \in B(X, Z)$ such that $m^{***}(\Phi, \Gamma) = A_\Gamma^{**}\Phi$ for every $\Phi \in X^{**}$. Then $A_\Gamma^{***}\big(\mathcal{Z}^\ell(m^*)\big) \subseteq \widehat{X^*}$ and $A_\Gamma^{**}\big(\mathcal{Z}^r(m^*)\big) \subseteq \widehat{Z}$. In particular, if m^* is Arens regular, then A_Γ is weakly compact.*

Proof. Let $\Omega \in \mathcal{Z}^\ell(m^*)$ be arbitrary. For every $\Phi \in X^{**}$ we have

$$\begin{aligned}\langle A_\Gamma^{***}\Omega, \Phi\rangle &= \langle \Omega, m^{***}(\Phi, \Gamma)\rangle = \langle m^{****}(\Omega, \Phi), \Gamma)\rangle = \langle m^{*t***t}(\Omega, \Phi), \Gamma)\rangle \\ &= \langle \Phi, m^{*t**}(\Omega, \Gamma)\rangle = \langle \Phi, \iota_X^*(A_\Gamma^{***}\Omega)\rangle = \langle \iota_{X^*}\big(\iota_X^*(A_\Gamma^{***}\Omega)\big), \Phi\rangle,\end{aligned}$$

where we used equivalence (iii) $\iff$ (iii') of Proposition 6.4. It follows that $A_\Gamma^{***}\Omega = \iota_{X^*}\big(\iota_X^*(A_\Gamma^{***}\Omega)\big) \in \widehat{X^*}$.

Let $(y_j)_{j\in\mathbb{J}} \subseteq Y$ be a bounded net such that $\Gamma = w^*-\lim_{j\in\mathbb{J}} \widehat{y_j}$. For an arbitrary $\Phi \in \mathcal{Z}^r(m^*)$, let $(x_i)_{i\in\mathbb{I}} \subseteq X$ be a bounded net such that $\Phi = w^*-\lim_{i\in\mathbb{I}} \widehat{x_i}$. We want to show that $A_\Gamma^{**}\Phi$ is the weak limit of $(\widehat{A_\Gamma x_i})_{i\in\mathbb{I}}$. Let

$\Omega \in Z^{***}$ be arbitrary and let $(\zeta_k)_{k\in\mathbb{K}} \subseteq Z^*$ be a bounded net such that $\Omega = w^*-\lim_{k\in\mathbb{K}} \widehat{\zeta_k}$. Then we have

$$\begin{aligned}
\langle \Omega, A_\Gamma^{**}\Phi\rangle &= \langle \Omega, m^{***}(\Phi,\Gamma)\rangle = \langle m^{****}(\Omega,\Phi),\Gamma\rangle = \langle m^{*t***t}(\Omega,\Phi),\Gamma\rangle \\
&= \langle \Phi, m^{*t**}(\Omega,\Gamma)\rangle = \lim_{i\in\mathbb{I}}\langle \widehat{x_i}, m^{*t**}(\Omega,\Gamma)\rangle = \lim_{i\in\mathbb{I}}\langle m^{*t**}(\Omega,\Gamma), x_i\rangle \\
&= \lim_{i\in\mathbb{I}}\langle \Omega, m^{*t*}(\Gamma,x_i)\rangle = \lim_{i\in\mathbb{I}}\lim_{k\in\mathbb{K}}\langle \widehat{\zeta_k}, m^{*t*}(\Gamma,x_i)\rangle = \lim_{i\in\mathbb{I}}\lim_{k\in\mathbb{K}}\langle m^{*t*}(\Gamma,x_i),\zeta_k\rangle \\
&= \lim_{i\in\mathbb{I}}\lim_{k\in\mathbb{K}}\langle \Gamma, m^{*t}(x_i,\zeta_k)\rangle = \lim_{i\in\mathbb{I}}\lim_{k\in\mathbb{K}}\lim_{j\in\mathbb{J}}\langle \widehat{y_j}, m^*(\zeta_k,x_i)\rangle = \lim_{i\in\mathbb{I}}\lim_{k\in\mathbb{K}}\lim_{j\in\mathbb{J}}\langle \zeta_k, m(x_i,y_j)\rangle \\
&= \lim_{i\in\mathbb{I}}\lim_{k\in\mathbb{K}}\langle \zeta_k, A_\Gamma x_i\rangle,
\end{aligned}$$

where the last equality follows by the definition of A_Γ — see the paragraph before Definition 6.3 and the equivalence of (i) and (ii) in Proposition 6.4. Hence, $\langle \Omega, A_\Gamma^{**}\Phi\rangle = \lim_{i\in\mathbb{I}}\lim_{k\in\mathbb{K}}\langle \widehat{\zeta_k}, \widehat{A_\Gamma x_i}\rangle = \lim_{i\in\mathbb{I}}\langle \Omega, \widehat{A_\Gamma x_i}\rangle$, that is, $A_\Gamma^{**}\Phi$ is the weak limit of $(\widehat{A_\Gamma x_i})_{i\in\mathbb{I}}$. Since $\widehat{Z}$ is a weakly closed subspace of Z^{**} we conclude that $A_\Gamma^{**}\Phi \in \widehat{Z}$.

If m^* is Arens regular, then $\mathcal{Z}^\ell(m^*) = Z^{***}$. Hence, A_Γ^{***} maps Z^{***} into $\widehat{X^*}$ which is equivalent, by [9, Theorem 4.5], to the weak compactness of A_Γ^*. Since A_Γ^* is weakly compact if and only if A_Γ is weakly compact (see [9, Theorem 4.7]), we conclude that A_Γ is weakly compact. □

Corollary 6.6. *Let $m \in Bil(X\times Y, Z)$ and for each $y \in Y$ let $A_{\widehat{y}} \in B(X,Z)$ be given by $A_{\widehat{y}}x = m(x,y)$ $(x \in X)$. If m^* is Arens regular, then every operator $A_{\widehat{y}}$ $(y \in Y)$ is weakly compact.*

Proof. It is clear that $m^{***}(\Phi,\widehat{y}) = A_{\widehat{y}}^{**}\Phi$ for every $\Phi \in X^{**}$. Hence the assumption of Theorem 6.5 is satisfied. □

Next corollary generalizes Theorem 4.1 in [6] as well as some results in [3, Section 2].

Corollary 6.7. *Let $m \in Bil(X\times Y, X)$. Assume that for $\Gamma \in Y^{**}$ there exists an invertible $A_\Gamma \in B(X)$ such that $m^{***}(\Phi,\Gamma) = A_\Gamma^{**}\Phi$ for every $\Phi \in X^{**}$. Then $m^* \in Bil(X^*\times X, Y^*)$ is left and right strongly Arens irregular. In particular, m^* is Arens regular if and only if X is reflexive.*

Proof. By Theorem 6.5, for every $\Omega \in \mathcal{Z}^\ell(m^*)$ there exists $\xi \in X^*$ such that $A_\Gamma^{***}\Omega = \widehat{\xi}$. Since A_Γ is invertible we have $\Omega = (A_\Gamma^{***})^{-1}\widehat{\xi} = \widehat{(A_\Gamma^*)^{-1}\xi} \in \widehat{X^*}$. Hence, $\mathcal{Z}^\ell(m^*) = \widehat{X^*}$. Similarly, $\mathcal{Z}^r(m^*) = \widehat{Z}$. □

References

[1] R. Arens, The adjoint of a bilinear operation, Proc. Amer. Math. Soc. 2 (1951) 839–848.
[2] S. Barootkoob, S. Mohammadzadeh, H. R. E. Vishki, Topological centers of certain Banach module actions, Bull. Iran. Math. Soc. 35 (2009) 25–36.
[3] M. Eshaghi Gordji, M. Filali, Arens regularity of module actions, Studia Math. 181(3) (2007) 237–254.
[4] S. L. Gulick, Commutativity and ideals in the biduals of topological algebras, Pac. J. Math. 18 (1966) 121–137.
[5] J. Hennefeld, A note on the Arens products, Pac. J. Math. 26 (1968) 115–119.
[6] A. A. Khadem-Maboudi, H. R. E. Vishki, Strong Arens irregularity of bilinear mappings and reflexivity, Banach J. Math. Anal. 6 (2012) 155–160.
[7] S. Mohammadzadeh, H. R. E. Vishki, Arens regularity of module actions and the second adjoint of a derivation, Bull. Austral. Math. Soc. 77 (2008) 465–476.
[8] R. E. Megginson, An introduction to Banach space theory, GTM 183, Springer, 1998.
[9] T. J. Morrison, Functional analysis, John Wiley& Sons, 2001.
[10] R. A. Ryan, Introduction to tensor products of Banach spaces, Springer, 2002.
[11] A. Ülger, Weakly compact bilinear forms and Arens regularity, Proc. Amer. Math. Soc. 101 (1987) 697–704.
[12] A. Zivari-Kazempour, On the Arens Product and Approximate Identity in Locally Convex Algebras, Filomat 30:6 (2016) 1493–1496.