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On Some Generalizations of Horadam's Numbers

Hacène Belbachir[a], **Amine Belkhir**[a]

[a]*Faculty of Mathematics Recits Lab. USTHB, BP 32, El Alia, 16111 Bab Ezzouar, Algiers Algeria*

Abstract. In this paper, we introduce the incomplete Horadam numbers $W_n(k)$, and hyper-Horadam numbers $W_n^{(k)}$, which generalize the Horadam's numbers defined by the recurrence $W_n = pW_{n-1} + qW_{n-2}$, with $W_0 = a$ and $W_1 = b$. We give some combinatorial properties. As an application, we evaluate a lower and upper bounds for the spectral norms of r-circulant matrices associated with these two generalizations. Moreover, we establish a new bounds for the spectral norms of r-circulant matrices associated with Horadam's numbers in terms of incomplete Horadam and hyper-Horadam numbers.

1. Introduction and Preliminaries

The Fibonacci numbers are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for any $n \geq 2$, with initials $F_0 = 0$, $F_1 = 1$. Several generalizations of the Fibonacci sequence have been investigated. One well-know generalization is the Horadam's numbers $W_n(a, b, p, q)$, denoted briefly W_n, and defined by the following recurrence relation

$$W_n = pW_{n-1} + qW_{n-2}, \tag{1}$$

with the initials $W_0 = a$ and $W_1 = b$, where $a, b, p, q \in \mathbb{Z}$. An explicit formula for the sequence (W_n) is

$$W_n = A\left(\frac{p + \sqrt{p^2 + 4q}}{2}\right)^n + B\left(\frac{p - \sqrt{p^2 + 4q}}{2}\right)^n, \tag{2}$$

where

$$A = \frac{b - a\beta}{\sqrt{p^2 + 4q}} \quad \text{and} \quad B = \frac{a\alpha - b}{\sqrt{p^2 + 4q}},$$

and α, β are the distinct roots of characteristic polynomial $x^2 - px - q = 0$. The generating function is given by

$$\sum_{n\geq 0} W_n x^n = \frac{a + (b - pa)x}{1 - px - qx^2}. \tag{3}$$

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Email addresses: hacenebelbachir@gmail.com (Hacène Belbachir), ambelkhir@gmail.com (Amine Belkhir)

Some special cases of Horadam's numbers $W_n(a,b,p,q)$ are Fibonacci numbers F_n, Lucas numbers L_n, Pell numbers P_n, Pell–Lucas numbers Q_n, Jacobsthal numbers J_n, Jacobsthal–Lucas numbers j_n, Bronze Fibonacci numbers B_n, Signed Fibonacci numbers $\mathfrak{F}_n$, Signed Pell numbers $\mathfrak{P}_n$.

$$\begin{array}{rr} W_n(0,1,1,1)=F_n; & W_n(2,1,1,1)=L_n; \\ W_n(0,1,2,1)=P_n; & W_n(2,2,2,1)=Q_n; \\ W_n(0,1,1,2)=J_n; & W_n(2,1,1,2)=j_n; \\ W_n(0,1,-2,1)=\mathfrak{P}_n; & W_n(1,1,-1,1)=\mathfrak{F}_n. \end{array}$$

Let (a_n) and $(a^{(n)})$ be two real initial sequences. Bahşī et *al.* [3], defined the symmetric infinite matrix associated to these sequences by the following recursive formula,

$$\begin{aligned} a_n^{(0)} &= a_n,\ a_0^{(n)} = a^{(n)}, \quad (n \geq 0), \\ a_n^{(k)} &= v a_{n-1}^{(k)} + u a_n^{(k-1)}, \quad (n \geq 1, k \geq 1), \end{aligned}$$

where $a_n^{(k)}$ represents the k-th row and the n-th column entry; i.e.,

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_n^{(k-1)} & \cdot & \cdot & \cdot \\ & & & & \downarrow u & & & \\ \cdot & \cdot & \cdot & a_{n-1}^{(k)} \overset{v}{\rightarrow} & a_n^{(k)} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

The entry $a_n^{(k)}$ can be expressed in terms of the first row's and the first column's as follows, see [3],

$$a_n^{(k)} = \sum_{i=1}^{k} v^n u^{k-i} \binom{n+k-i-1}{n-1} a_0^{(i)} + \sum_{j=1}^{n} v^{n-j} u^k \binom{n+k-j-1}{k-1} a_j^{(0)}. \tag{4}$$

The Horadam's numbers have numerous interesting properties and applications in various areas of mathematics and science (see [13] for a survey). In recent years, many authors have studied the properties of the circulant matrix and r-circulant matrix with Horadam's numbers and the generalized Horadam numbers. For example, Alptekin et al. [1] have established the spectral norms and eigenvalues of circulant matrices with the Horadam's numbers. Bozkurta and Tam gave explicit determinant and inverse of the r-circulant matrices involving Horadam's numbers in [6]. Yazlid and Taskara [18–20] have introduced the generalized k-Horadam numbers and they established the determinant, lower and upper bounds for the spectral norms of r-circulant matrices with these numbers. Further, the authors in [17] proposed a construction of Horadam's numbers in terms of determinant of tridiagonal matrices.

The paper is organized as follows: In section 2, we introduce the incomplete Horadam and hyper-Horadam sequences and we give some properties. In section 3, we study some combinatorial identities of these two generalizations and we establish that Horadam's numbers can be expressed in terms of the incomplete Horadam and hyper-Horadam numbers. In the last section, we give a lower and upper bounds for the spectral norm of the r-circulant matrix with incomplete Horadam and hyper-Horadam numbers, also we derive a new lower and upper bounds for the spectral norm of r-circulant matrix with Horadam's numbers. In the sequel, we give some bounds related to spectral norm of Hadamard product and Kronecker product of these matrices.

2. Definitions and properties

Let a, b, p and q be integers, define the incomplete Horadam numbers $(W_n(k; a, b, p, q))$, denoted briefly $(W_n(k))$, by

$$W_n(k) = \sum_{j=0}^{k} \frac{(n-2j)b + apj}{n-j} \binom{n-j}{j} q^j p^{n-2j-1}, \quad 0 \le k \le \lfloor n/2 \rfloor . \tag{5}$$

The sequence $(W_n(k))$ satisfy the following recurrence relation,

$$W_n(k) = pW_{n-1}(k) + qW_{n-2}(k-1). \tag{6}$$

From recurrence relation (6), we can easily calculate the first few terms of the sequence $(W_n(k))$.

n/k	0	1	2	3
1	b			
2	bp	$bp + aq$		
3	bp^2	$bp^2 + pqa + bq$		
4	bp^3	$bp^3 + p^2qa + 2bpq$	$bp^3 + p^2qa + 2bpq + q^2a$	
5	bp^4	$bp^4 + p^3qa + 3bp^2q$	$bp^4 + p^3qa + 3bp^2q + 2pq^2a + bq^2$	
6	bp^5	$p^5b + p^4qa + 4qp^3b$	$p^5b + p^4qa + 4qp^3b + 3p^2q^2a + 3pq^2b$	$p^5b + p^4qa + 4qp^3b + 3p^2q^2a$ $+3pq^2b + q^3a$

Table 1: The first values of the incomplete Horadam sequence.

The connection between ordinary and incomplete Horadam numbers is

$$W_n(k) = 0 \quad 0 \le n \le 2k+1, \quad W_{2k+1}(k) = W_{2k+1}, \quad W_{2k+2}(k) = W_{2k+2}.$$

Remark 2.1. *Some specializations*

- *For $W_n(k; 1, 1, 0, 1) = F_n(k)$, we get the incomplete Fibonacci numbers, [11].*
- *For $W_n(k; 1, 1, 2, 1) = L_n(k)$, we have the incomplete Lucas numbers, [11].*
- *For $W_n(k; 2, 1, 0, 1) = P_n(k)$, we obtained the incomplete Pell numbers.*
- *For $W_n(k; 2, 1, 2, 2) = Q_n(k)$, we obtained the incomplete Pell-Lucas numbers.*
- *For $W_n(k; 1, 2, 0, 1) = J_n(k)$, we have the incomplete Jacobsthal numbers.*
- *For $W_n(k; 1, 2, 2, 1) = j_n(k)$, we have the incomplete Jacobsthal-Lucas numbers.*

Relation (6) can be transformed into non homogenous recurrence relation as follows,

Proposition 2.2. *For any $n \ge 2k+3$, we have*

$$W_n(k) = pW_{n-1}(k) + qW_{n-2}(k) - \frac{(n-2k-2)b + apk}{n-2k-2} \binom{n-k-3}{k} q^{k+1} p^{n-2k-3}. \tag{7}$$

Proof. It follows from Relations (5) and (6). □

To establish the generating function of the incomplete Horadam numbers we need the following lemma, see [15].

Lemma 2.3. *Let (s_n) be a sequence of complex numbers satisfying the non-homogeneous second order recurrence relation*

$$s_n = ps_{n-1} + qs_{n-2} + r_n, \quad (n > 1),$$

where (r_n) is a sequence of complex numbers. Then the generating function $U(t)$ of (s_n) is given by

$$U(t) = \frac{G(t) + s_0 + r_0 + (s_1 - ps_0 - r_1)}{1 - pt - qt^2},$$

where $G(t)$ is the generating function of (r_n).

Theorem 2.4. *The generating function of the incomplete Horadam numbers $W_n(k)$ is*

$$\sum_{n\geq 0} W_n(k)t^n = \frac{a + (b - ap)t}{1 - pt - qt^2}\left(1 - \left(\frac{qt^2}{1 - pt}\right)^{k+1}\right). \tag{8}$$

Proof. Let k be a fixed positive integer, and

$$s_0 = W_{2k+1}(k), \; s_1 = W_{2k+2}(k), \; s_n = W_{2k+n+1}(k).$$

From the non homogenous recurrence relation (7), we have

$$s_n = pW_{n+2k}(k) + qW_{n+2k-1}(k) - \frac{(n-1)b + apk}{n-1}\binom{n+k-2}{k}q^{k+1}p^{n-2},$$

also

$$r_0 = r_1 = 0 \text{ and } r_n = -\frac{(n-1)b + apk}{n-1}\binom{n+k-2}{k}q^{k+1}p^{n-2}.$$

The generating function of (r_n) is

$$G(t) = \frac{-(a + (b - ap)t)(qt^2)^{k+1}}{(1 - pt)^{k+1}}.$$

Hence, from Lemma 2.3, we get the generating function of (s_n). □

Proposition 2.5. *We have,*

$$\sum_{n,k\geq 0} W_n(k)x^n y^k = \frac{a + (b - ap)x}{(1 - py)(1 - px - qyx^2)}. \tag{9}$$

Now, we define the hyper-Horadam numbers of order k, $(W_n^{(k)}(p, q, a, b))$, denoted briefly $(W_n^{(k)})$.

Definition 2.6. *For any $n \geq 0$ and $k \geq 1$, the hyper-Horadam numbers $W_n^{(k)}$ are defined by the recurrence relation:*

$$W_n^{(k)} = pW_{n-1}^{(k)} + qW_n^{(k-1)}, \tag{10}$$

with initial conditions $W_n^{(0)} = W_n$ and $W_0^{(k)} = aq^k$, where W_n is n-th Horadam's numbers.

The relation (10) can be written as follows :

$$W_n^{(k)} = \sum_{j=0}^{n} qp^{n-j} W_j^{(k-1)}. \tag{11}$$

Let $a_n^{(0)} = W_n^{(0)} = W_n$ and $a_0^{(n)} = W_0^{(n)} = aq^n$. Then the corresponding infinite symmetric matrix is given by

$$\begin{pmatrix} a & b & aq+bp & bp^2+bq+pq & \cdots \\ aq & apq+bq & ap^2q+aq^2+2bpq & ap^3q+2apq^2+3bp^2q+bq^2 & \cdots \\ aq^2 & 2apq^2+bq^2 & 3ap^2q^2+aq^3+3bpq^2 & 4ap^3q^2+3apq^3+6bp^2q^2+bq^3 & \cdots \\ aq^3 & 3apq^3+bq^3 & 6ap^2q^3+aq^4+4bpq^3 & 10ap^3q^3+4apq^4+10bp^2q^3+bq^4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have some classical identities when $r = 1, 2$ and 3.

$$\begin{aligned} W_n^{(1)} &= W_{n+2} - bp^{n+1}, \\ W_n^{(2)} &= W_{n+4} - (n+2)bqp^{n+1} - aqp^{n+2}, \\ W_n^{(3)} &= W_{n+6} - \frac{((n+2)b+2ap)(n+3)q^2p^{n+1}}{2}. \end{aligned}$$

In the next theorem we give an explicit formula for the hyper-Horadam numbers.

Theorem 2.7. *For any $n \geq 0$ and $k \geq 1$, we have*

$$W_n^{(k)} = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(n-2j)b + ap(k+j)}{n+k-j} \binom{n+k-j}{j+k} q^{j+k} p^{n-2j-1}. \tag{12}$$

Proof. We will prove the theorem by double induction. For any $m \geq 0$, let $\mathcal{S}_m := \{W_i^{(j)} | i + j = m\}$. The sum (12) clearly holds for the elements of $\mathcal{S}_0$ and $\mathcal{S}_1$. Now, suppose that the result is true for any elements of the set $\mathcal{S}_m$ with $m < n + k + 1$, we prove it for the elements of the set $\mathcal{S}_{n+k+1}$. Without lost the generality let $i = n + 1$ and $j = k$, then from recurrence relation (10), we have

$$\begin{aligned} W_{n+1}^{(k)} &= pW_n^{(k)} + qW_{n+1}^{(k-1)} \\ &= \sum_{j\geq 0} \frac{(n-2j)b + ap(k+j)}{n+k-j} \binom{n+k-j}{j+k} q^{j+k}p^{n-2j} + \sum_{j\geq 0} \frac{(n+1-2j)b + ap(k+j-1)}{n+k-j} \binom{n+k-j}{j+k-1} q^{j+k}p^{n-2j} \\ &= \sum_{j\geq 0} \frac{q^{k+j}p^{n-2j}}{n+k-j} \left[((n-2j)b + (k+j)ap) \binom{n+k-j}{k+j} + ((n-2j+1)b + (k+j-1)ap) \binom{n+k-j}{k+j-1} \right] \\ &= \sum_{j\geq 0} q^{k+j}p^{n-2j} \left[\left(b + \frac{(k+j)ap}{n-2j}\right) \binom{n+k-j-1}{k+j} + \left(b + \frac{(k+j-1)ap}{n-2j+1}\right) \times \binom{n+k-j-1}{k+j-1} \right] \\ &= \sum_{j\geq 0} q^{k+j}p^{n-2j} \left[b \left(\binom{n+k-j-1}{k+j} + \binom{n+k-j-1}{k+j-1} \right) + ap \left(\frac{k+j}{n-2j} \times \binom{n+k-j-1}{k+j} \right.\right. \\ &\qquad \left.\left. + \frac{k+j-1}{n-2j+1} \binom{n+k-j-1}{k+j-1} \right) \right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\geq 0} q^{k+j}p^{n-2j}\left[b\binom{n+k-j}{k+j} + ap\left(\binom{n+k-j-1}{k+j-1} + \binom{n+k-j-1}{k+j-2}\right)\right]\\
&= \sum_{j\geq 0} q^{k+j}p^{n-2j}\left[b\binom{n+k-j}{k+j} + ap\binom{n+k-j}{k+j-1}\right]\\
&= \sum_{j\geq 0} q^{k+j}p^{n-2j}\left[\frac{(n-2j+1)b}{n+k-j+1}\binom{n+k-j+1}{k+j} + \frac{(k+j)ap}{n+k-j+1}\binom{n+k-j+1}{k+j}\right]\\
&= \sum_{j\geq 0} \frac{(n-2j+1)b+(k+j)ap}{n+k-j+1}\binom{n+k-j+1}{k+j} q^{k+j}p^{n-2j}.
\end{aligned}
$$

Thus, we conclude the proof of Theorem 2.7. □

From Relations (10) and (12), we obtain the following non homogenous recurrence relation,

$$
W_n^{(k)} = pW_{n-1}^{(k)} + qW_{n-2}^{(k)} + \frac{nb+ap(k-1)}{n+k-1}\binom{n+k-1}{k-1}q^k p^{n-1}. \tag{13}
$$

Theorem 2.8. *The generating function of the hyper-Horadam numbers is*

$$
\sum_{n\geq 0} W_n^{(k)} t^n = \frac{a+(b-ap)t}{1-pt-qt^2}\left(\frac{qt^2}{1-pt}\right)^k. \tag{14}
$$

Proof. The result is obtained using Lemma 2.3 and recurrence relation (13). □

3. Some combinatorial identities

In this section, we provide some combinatorial identities involving the incomplete Horadam and hyper-Horadam numbers.

Proposition 3.1. *We have*

$$
\sum_{j=0}^{h}\binom{h}{j}q^j p^{h-j} W_{n+h-j}(k+h-j) = W_{n+2h}(k+h), \quad 0\leq k\leq \frac{n-h}{2}. \tag{15}
$$

Proof. We proceed by induction on h. It is clearly true for $h=0$ and $h=1$. Assuming the result holds for any integer $h\geq 1$, we show it for $h+1$.

$$
\begin{aligned}
\sum_{j=0}^{h+1}\binom{h+1}{j}q^j p^{h-j+1}W_{n+j}(k+j) &= \sum_{j=0}^{h+1}\binom{h}{j}q^j p^{h-j+1}W_{n+h-j+1}(k+h-j+1)\\
&\quad +\sum_{j=0}^{h+1}\binom{h}{j-1}q^j p^{h-j+1}W_{n+h-j+1}(k+h-j+1)\\
&= p\sum_{j=0}^{h}\binom{h}{j}q^j p^{h-j}W_{n+h-j+1}(k+h-j+1)\\
&\quad +q\sum_{j=0}^{h}\binom{h}{j}q^j p^{h-j}W_{n+h-j}(k+h-j)\\
&= pW_{n+2h+1}(k+h+1) + qW_{n+2h}(k+h)\\
&= W_{n+2h+2}(k+h+1),
\end{aligned}
$$

which completes the proof. □

Proposition 3.2. *For any $h \geq 2k+2$, we have*

$$\sum_{j=0}^{h-1} qp^{h-j-1} W_{n+j}(k) = W_{n+h+1}(k+1) - p^h W_{n+1}(k+1). \tag{16}$$

Proof. We proceed by induction on h. It is clearly true for $h = 1$ and $h = 2$. Assuming the result holds for any integer $h \geq 1$, we show it for $h+1$.

$$\begin{aligned}
\sum_{j=0}^{h} qp^{h-j} W_{n+j}(k) &= p \sum_{j=0}^{h-1} qp^{h-j-1} W_{n+j}(k) + qW_{n+h}(k) \\
&= p\left(W_{n+h+1}(k+1) - p^h W_{n+1}(k+1)\right) + qW_{n+h}(k) \\
&= \left(pW_{n+h+1}(k+1) + qW_{n+h}(k)\right) - p^h W_{n+1}(k+1) \\
&= W_{n+h+2}(k+1) - p^{h+1} W_{n+1}(k+1).
\end{aligned}$$

□

Proposition 3.3. *For any $n \geq 0$, $r \geq 1$ and $k \geq 0$, we have*

$$W_n^{(k+r)} = \sum_{j=0}^{n} q^r p^{n-j} \binom{n+r-j-1}{r-1} W_j^{(k)} \tag{17}$$

Proof. Let $a_n^{(0)} = W_n^{(k)}$ and $a_0^{(n)} = W_0^{(k+n)} = aq^{k+n}$, then the corresponding infinite matrix is given by

$$\begin{pmatrix}
W_0^{(k)} & W_1^{(k)} & W_2^{(k)} & W_3^{(k)} & \cdots \\
W_0^{(k+1)} & W_1^{(k+1)} & W_2^{(k+1)} & W_3^{(k+1)} & \cdots \\
W_0^{(k+2)} & W_1^{(k+2)} & W_2^{(k+2)} & W_3^{(k+2)} & \cdots \\
W_0^{(k+3)} & W_1^{(k+3)} & W_2^{(k+3)} & W_3^{(k+3)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \tag{18}$$

From Relation (4), we have

$$\begin{aligned}
a_n^{(r)} &= \sum_{i=1}^{r} p^n q^{r-i} \binom{n+r-i-1}{n-1} aq^{k+i} + \sum_{j=1}^{n} p^{n-j} q^r \binom{n+r-j-1}{r-1} W_j^{(k)} \\
&= ap^n q^{r+k} \sum_{i=0}^{r-1} \binom{n+r-i-2}{n-1} + q^r \sum_{j=0}^{n-1} p^{n-j-1} \binom{n+r-j-2}{r-1} W_{j+1}^{(k)} \\
&= ap^n q^{r+k} \sum_{i=0}^{r-1} \binom{n+i-1}{n-1} + q^r \sum_{j=0}^{n-1} p^j \binom{j+r-1}{r-1} W_{n-j}^{(k)} \\
&= ap^n q^{r+k} \binom{n+r-1}{r-1} + q^r \sum_{j=0}^{n-1} p^j \binom{j+r-1}{r-1} W_{n-j}^{(k)} \\
&= p^n q^r \binom{n+r-1}{r-1} W_0^{(k)} + q^r \sum_{j=0}^{n-1} p^j \binom{j+r-1}{r-1} W_{n-j}^{(k)} \\
&= q^r \sum_{j=0}^{n} p^j \binom{j+r-1}{r-1} W_{n-j}^{(k)}.
\end{aligned}$$

Hence, from the matrix (18), we obtain

$$a_n^{(r)} = W_n^{(k+r)} = q^r \sum_{j=0}^{n} p^{n-j} \binom{n-j+r-1}{r-1} W_j^{(k)},$$

which gives the formula (17). □

As consequence of Proposition 3.3, we have an expression for the hyper-Horadam numbers in terms of Horadam numbers.

Corollary 3.4. *For any $n \geq 0$ and $k \geq 1$, we have*

$$W_n^{(k)} = \sum_{j=0}^{n} q^k p^{n-j} \binom{n+k-j-1}{k-1} W_j. \tag{19}$$

The following corollary provides the connection between the incomplete Horadam, hyper-Horadam and Horadam's numbers.

Proposition 3.5. *For any $n \geq 0$ and $k \geq 1$, we have*

$$W_{n+2k} = W_{n+2k}(k-1) + W_n^{(k)}. \tag{20}$$

4. Spectral norms of r-circulant matrices

In this section, we evaluate the spectral norms of r-circulant matrices with the incomplete Horadam and the hyper-Horadam numbers, throughout this section we will assume that $p, q, b > 0$ and $a \geq 0$.

A matrix $A_r = [a_{ij}] \in M_{n,n}(\mathbb{C})$ is called r-circulant matrix if it is of the form

$$A_r = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ ra_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ ra_{n-2} & ra_{n-1} & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ra_2 & ra_3 & ra_4 & \cdots & a_0 & a_1 \\ ra_1 & ra_2 & ra_3 & \cdots & ra_{n-1} & a_0 \end{pmatrix}.$$

The matrix A_r is determined by its first row elements $a_0, a_1, \ldots, a_{n-1}$ and by the parameter r, we denote $A_r = circ_n(a_0, a_1, \ldots, a_{n-1})$. for $r = 1$, the matrix A is called a circulant matrix. The circulant matrix with geometric progression $\mathbf{G} = circ_n\left(qp^{n-1}, qp^{n-2}, \ldots, q\right)$ is the matrix of the form

$$\mathbf{G} = \begin{pmatrix} qp^{n-1} & qp^{n-2} & \cdots & qp & q \\ q & qp^{n-1} & \cdots & qp^2 & qp \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ qp^{n-3} & qp^{n-4} & \cdots & qp^{n-1} & qp^{n-2} \\ qp^{n-2} & qp^{n-3} & \cdots & q & qp^{n-1} \end{pmatrix}.$$

For more information about the circulant matrix with geometric progression one can see [7]. Now, we give some results which will be used in this section.

Let $A = \left[a_{ij}\right]$ be an $m \times n$ matrix, the Frobenius (or Euclidean) norm of matrix A is defined by

$$\|A\|_F = \left[\sum_{i=1}^{m} \sum_{j=1}^{n} \left|a_{ij}\right|^2\right]^{\frac{1}{2}},$$

and it's spectral norm is given by [8],

$$\|A\|_2 = \sqrt{\max_{1\le i\le n} |\lambda_i|},$$

where λ_i's are the eigenvalues of matrix $A^H A$ and A^H is conjugate transpose of matrix A.

The following inequalities hold [21],

$$\frac{1}{\sqrt{n}}\|A\|_F \le \|A\|_2 \le \|A\|_F, \tag{21}$$

and

$$\|A\|_2 \le \|A\|_F \le \sqrt{n}\|A\|_2. \tag{22}$$

Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ be $m \times n$-matrices. The Hadamard product of A and B is

$$A \circ B = \left(a_{ij} b_{ij}\right).$$

Lemma 4.1. *[14] Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ be $m \times n$-matrices. Then*

$$\|A \circ B\|_2 \le r_1(A)\, c_1(B),$$

where $r_1(\cdot)$ and $c_1(\cdot)$ are maximum row length norm and maximum column length norm, respectively

$$r_1(A) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} \left|a_{ij}\right|^2} \ \text{and} \ c_1(B) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} \left|b_{ij}\right|^2}.$$

Lemma 4.2. *[9] Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ be $m \times n$-matrices. Then*

$$\|A \circ B\|_2 \le \|A\|_2 \|B\|_2.$$

Lemma 4.3. *[9] et $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ be $m \times n$-matrices. Then*

$$\|A \otimes B\|_2 = \|A\|_2 \|B\|_2.$$

For any positive integers k and h ($h \ge 2k + 2$), let

$$\begin{aligned}
\mathbf{A}_r^{(k,h)} &:= circ_n\left(W_h(k), W_{h+1}(k), \ldots, W_{h+n-1}(k)\right);\\
\mathbf{H}_r^{(k)} &:= circ_n\left(W_0^{(k)}, W_1^{(k)}, \ldots, W_{n-1}^{(k)}\right);\\
\mathbf{F}_r^{(k)} &:= circ_n\left(W_{2k}, W_{2k+1}, \ldots, W_{2k+n-1}\right);
\end{aligned}$$

be a circulant matrices with incomplete Horadam, hyper-Horadam and Horadam's numbers, respectively. We define the matrices $\tilde{\mathbf{A}}_r^{(k,h)}$, $\tilde{\mathbf{H}}_r^{(k)}$ and $\tilde{\mathbf{F}}_r^{(k)}$ by

$$\begin{aligned}
\tilde{\mathbf{A}}_r^{(k,h)} &:= \mathbf{A}_r^{(k,h)} \circ \mathbf{G};\\
\tilde{\mathbf{H}}_r^{(k)} &:= \mathbf{H}_r^{(k)} \circ \mathbf{G};\\
\tilde{\mathbf{F}}_r^{(k)} &:= \mathbf{F}_r^{(k)} \circ \mathbf{G};
\end{aligned}$$

respectively. The matrices $\tilde{\mathbf{A}}_r^{(k,h)}$, $\tilde{\mathbf{H}}_r^{(k)}$ and $\tilde{\mathbf{F}}_r^{(k)}$ correspond to Hadamard product of matrices $\mathbf{A}_r^{(k,h)}$, $\mathbf{H}_r^{(k)}$ and $\mathbf{F}_r^{(k)}$ and circulant matrix with geometric progression $\mathbf{G}$. The first theorem concerns the evaluation of the spectral norm of the circulant matrix with the incomplete Horadam numbers.

Theorem 4.4. *For any $h \geq 2k+2$, let $\tilde{A}_1^{(k,h)}$ be a circulant matrix. Then we have*

$$||\tilde{A}_1^{(k,h)}||_2 = W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1). \tag{23}$$

Proof. Since the circulant matrix $\tilde{\mathbf{A}}_1^{(k,h)}$ is normal, the spectral norm of the matrix $\tilde{\mathbf{A}}_1^{(k,h)}$ is equal to its spectral radius. Furthermore, $\tilde{\mathbf{A}}_1^{(k,h)}$ is irreducible and its entries are nonnegative, its spectral radius is the same as its Perron root. Let u be a vector with all components 1. Then

$$\tilde{\mathbf{A}}_1^{(k,h)}u = \left(\sum_{j=0}^{n-1} qp^{n-1-j}W_{h+j}(k)\right)u.$$

As $\sum_{j=0}^{n-1} qp^{n-1-j}W_{h+j}(k)$ is an eigenvalue of $\tilde{\mathbf{A}}_1^{(k,h)}$ associated with u, which is necessarily the Perron root of $\tilde{\mathbf{A}}_1^{(k,h)}$. Hence from relation (16), we have

$$||\tilde{\mathbf{A}}_1^{(k,h)}||_2 = W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1).$$

□

From the relation (22) and Theorem 4.4 we deduce the upper and lower bounds for the sum of squares of incomplete Horadam numbers.

Corollary 4.5. *For any $h \geq 2k+2$, we have*

$$\begin{aligned}\frac{1}{\sqrt{n}}(W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1)) &\leq \sqrt{\sum_{j=0}^{n-1}\left(qp^{n-j-1}W_{h+j}(k)\right)^2} \\ &\leq W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1).\end{aligned} \tag{24}$$

Theorem 4.6. *For any $n \geq 1$, the spectral norm of the circulant matrix $\tilde{H}_1^{(k)}$ is*

$$||\tilde{H}_1^{(k)}||_2 = W_{n-1}^{(k+1)}. \tag{25}$$

Proof. The result is obtained in the same way to the Theorem 4.4. □

From Theorem 4.6, we deduce the upper and lower bourns of sum of squares of hyper-Horadam numbers

$$\frac{1}{\sqrt{n}}W_{n-1}^{(k+1)} \leq \sqrt{\sum_{j=0}^{n-1} pq^{n-j-1}W_j^{(k)}} \leq \sqrt{n}W_{n-1}^{(k+1)}. \tag{26}$$

Corollary 4.7. *Let $\tilde{F}_1^{(k)}$ be a circulant matrix, then we have the following equality*

$$||\tilde{F}_1^{(k)}||_2 = W_{2k+n+1}(k) - p^{n-1}W_{2k+1} + W_{n-1}^{(k+1)}. \tag{27}$$

Proof. The result is obtained from relations (20), (23) and (25). □

Next, we give upper and lower bounds for the spectral norm of r-circulant matrix with the incomplete Horadam numbers.

Theorem 4.8. *For $h \geq 2k+2$, let $\tilde{A}_r^{(k,h)}$ be a r-circulant matrix.*
(i) For $|r| \geq 1$, we have

$$\frac{1}{\sqrt{n}}(W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1)) \leq ||\tilde{A}_r^{(k,h)}||_2 \leq \sqrt{(n-1)|r|^2+1}\left(W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1)\right).$$

(ii) For $|r| < 1$, we have

$$\frac{|r|}{\sqrt{n}}\left(W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1)\right) \leq \|\tilde{\mathbf{A}}_r^{(k,h)}\|_2 \leq \sqrt{n}\left(W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1)\right).$$

Proof. From the definition of the matrix $\tilde{\mathbf{A}}_r^{(k,h)}$, we have

$$\|\tilde{\mathbf{A}}_r^{(k,h)}\|_F = \sqrt{\sum_{j=0}^{n-1}(n + j(|r|^2 - 1))\left(qp^{n-j-1}W_{h+j}(k)\right)^2}.$$

(i) Since $|r| \geq 1$ and using the inequality (24), we have

$$\|\tilde{\mathbf{A}}_r^{(k,h)}\|_F \geq \sqrt{\sum_{j=0}^{n-1} n\left(qp^{n-j-1}W_{h+j}(k)\right)^2} \geq W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1).$$

From (21), we obtain

$$\frac{1}{\sqrt{n}}(W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1)) \leq \|\tilde{\mathbf{A}}_r^{(k,h)}\|_2.$$

Now, we define the matrices C and D as follows

$$C = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ r & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \cdots & 1 & 1 \\ r & r & \cdots & r & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} qp^{n-1}W_h(k) & qp^{n-2}W_{h+1}(k) & \cdots & qpW_{h+n-2}(k) & qW_{h+n-1}(k) \\ qW_{h+n-1}(k) & qp^{n-1}W_h(k) & \cdots & qp^2W_{h+n-3}(k) & qpW_{h+n-2}(k) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ qp^{n-3}W_{h+2}(k) & qp^{n-4}W_{h+3}(k) & \cdots & qp^{n-1}W_h(k) & qp^{n-2}W_{h+1}(k) \\ qp^{n-2}W_{h+1}(k) & qp^{n-3}W_{h+2}(k) & \cdots & qW_{h+n-1}(k) & qp^{n-1}W_h(k) \end{pmatrix}.$$

such that $\tilde{\mathbf{A}}_r^{(k,h)} = C \circ D$, then we have

$$r_1(C) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|c_{ij}|^2} = \sqrt{\sum_{j=1}^{n}|c_{nj}|^2} = \sqrt{(n-1)|r|^2 + 1},$$

and

$$c_1(D) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|d_{ij}|^2} = \sqrt{\sum_{j=1}^{n}|c_{nj}|^2} = \sqrt{\sum_{j=0}^{n-1}\left(qp^{n-j-1}W_{h+j}(k)\right)^2}.$$

Using Lemma 4.1 and (24), we get

$$\|\tilde{\mathbf{A}}_r^{(k,h)}\|_2 \leq \sqrt{(n-1)|r|^2 + 1}\left(W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1)\right).$$

(ii) Since $|r| \leq 1$, we have

$$\begin{aligned}\|\tilde{\mathbf{A}}_r^{(k,h)}\|_F &= \sqrt{\sum_{j=0}^{n-1}(n + j(|r|^2 - 1))\left(qp^{n-j-1}W_{h+j}(k)\right)^2} \\ &\geq \sqrt{n|r|^2\sum_{j=0}^{n-1}\left(qp^{n-j-1}W_{h+j}(k)\right)^2} \\ &\geq |r|\left(W_{h+n+1}(k+1) - W_{h+1}(k+1)\right).\end{aligned}$$

From Lemma 4.1 and (21), we obtain

$$\|\tilde{\mathbf{A}}_r^{(k,h)}\|_2 \geq \frac{|r|}{\sqrt{n}}\left(W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1)\right).$$

Now, we consider the matrices C and D,

$$C = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ r & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \cdots & 1 & 1 \\ r & r & \cdots & r & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} qp^{n-1}W_h(k) & qp^{n-2}W_{h+1}(k) & \cdots & qpW_{h+n-2}(k) & qW_{h+n-1}(k) \\ qW_{h+n-1}(k) & qp^{n-1}W_h(k) & \cdots & qp^2W_{h+n-3}(k) & qpW_{h+n-2}(k) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ qp^{n-3}W_{h+2}(k) & qp^{n-4}W_{h+3}(k) & \cdots & qp^{n-1}W_h(k) & qp^{n-2}W_{h+1}(k) \\ qp^{n-2}W_{h+1}(k) & qp^{n-3}W_{h+2}(k) & \cdots & qW_{h+n-1}(k) & qp^{n-1}W_h(k) \end{pmatrix}.$$

Then,

$$r_1(C) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|c_{ij}|^2} = \sqrt{\sum_{j=1}^{n}|c_{nj}|^2} = \sqrt{n},$$

and

$$c_1(D) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|d_{ij}|^2} = \sqrt{\sum_{j=1}^{n}|d_{nj}|^2} = \sqrt{\sum_{j=0}^{n-1}(qp^{n-j-1}W_{h+j})^2}.$$

From Lemma 4.1 and (24), we have

$$\|\tilde{\mathbf{A}}_r^{(k,h)}\|_2 \leq \sqrt{n}\left(W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1)\right),$$

which completes the proof. □

Theorem 4.9. *For $\tilde{\boldsymbol{H}}_r^{(k)}$ be a r-circulant matrix. Then*
(i) For $|r| \geq 1$, we have

$$\frac{1}{\sqrt{n}}W_{n-1}^{(k+1)} \leq \|\tilde{\boldsymbol{H}}_r^{(k)}\|_2 \leq \sqrt{(n-1)|r|^2 + 1}\,W_{n-1}^{(k+1)}. \tag{28}$$

(ii) For $|r| < 1$, we have

$$\frac{|r|}{\sqrt{n}} W_{n-1}^{(k+1)} \leq \|\tilde{\boldsymbol{H}}_r^{(k)}\|_2 \leq \sqrt{n} W_{n-1}^{(k+1)}. \tag{29}$$

Proof. The Theorem is obtained by similar way. □

In the following result we give a upper and lower bounds for the spectral norm of a r-circulant matrix with Horadam's numbers in terms of incomplete Horadam and hyper-Horadam numbers.

Theorem 4.10. *For $k \geq 1$, let $\tilde{\boldsymbol{F}}_r^{(k)} = \left(qp^{n-1}W_{2k}, qp^{n-2}W_{2k+1}, \ldots, qW_{2k+n-1}\right)$ be a r-circulant matrix.*
(i) For $|r| \geq 1$, we have

$$\frac{1}{\sqrt{n}}(W_{n-1}^{(k+1)} + W_{2k+n+1}(k) - p^{n-1}W_{2k+1}) \leq \|\tilde{\boldsymbol{F}}_r^{(k)}\|_2 \leq \sqrt{(n-1)|r|^2+1}\left(W_{n-1}^{(k+1)} + W_{2k+n+1}(k) - p^{n-1}W_{2k+1}\right).$$

(ii) For $|r| < 1$, we have

$$\frac{|r|}{\sqrt{n}}\left(W_{n-1}^{(k+1)} + W_{2k+n+1}(k) - p^{n-1}W_{2k+1}\right) \leq \|\tilde{\boldsymbol{F}}_r^{(k)}\|_2 \leq \sqrt{n}\left(W_{n-1}^{(k+1)} + W_{2k+n+1}(k) - p^{n-1}W_{2k+1}\right).$$

Proof. The matrix $\tilde{\mathbf{F}}_r^{(k)}$ is of the form

$$\tilde{\mathbf{F}}_r^{(k)} = \begin{pmatrix} qp^{n-1}W_{2k} & qp^{n-2}W_{2k+1} & \cdots & qpW_{2k+n-2} & qW_{2k+n-1} \\ rqW_{2k+n-1} & qp^{n-1}W_{2k} & \cdots & qp^2W_{2k+n-3} & qpW_{2k+n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ rqp^{n-3}W_{2k+2} & rqp^{n-4}W_{2k+3} & \cdots & qp^{n-1}W_{2k} & qp^{n-2}W_{2k+1} \\ rqp^{n-2}W_{2k+1} & rqp^{n-3}W_{2k+2} & \cdots & rqW_{2k+n-1} & qp^{n-1}W_{2k} \end{pmatrix}.$$

Then, we have

$$\|\tilde{\mathbf{F}}_r^{(k)}\|_F = \sqrt{\sum_{j=0}^{n-1}(n + j(|r|^2-1))\left(qp^{n-1-j}W_{2k+j}\right)^2}.$$

(i) Since $|r| \geq 1$ and by (20), we have

$$\|\tilde{\mathbf{F}}_r^{(k)}\|_F \geq \sqrt{\sum_{j=0}^{n-1} n\left(qp^{n-1-j}W_{2k+j}\right)^2} = \sqrt{\sum_{j=0}^{n-1} n\left(W_j^{(k)} + W_{2k+j}(k-1)\right)^2},$$

from the inequalities (26) and (24),

$$\|\tilde{\mathbf{F}}_r^{(k)}\|_F \geq \sqrt{\sum_{j=0}^{n-1} n\left(W_j^{(k)} + W_{2k+j}(k-1)\right)^2} \geq W_{n-1}^{(k+1)} + W_{2k+n+1}(k+1) - p^{n-1}W_{2k+1},$$

using (21), we obtain

$$\frac{1}{\sqrt{n}}(W_{n-1}^{(k+1)} + W_{2k+n+1}(k+1) - p^{n-1}W_{2k+1}) \leq \|\tilde{\mathbf{F}}_r^{(k)}\|_2.$$

On the other hand, let the matrices C and D be defined by

$$C = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ r & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \cdots & 1 & 1 \\ r & r & \cdots & r & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} qp^{n-1}W_{2k} & qp^{n-2}W_{2k+1} & \cdots & qpW_{2k+n-2} & qW_{2k+n-1} \\ qW_{2k+n-1} & qp^{n-1}W_{2k} & \cdots & qp^2W_{2k+n-3} & qpW_{2k+n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ qp^{n-3}W_{2k+2} & qp^{n-4}W_{2k+3} & \cdots & qp^{n-1}W_{2k} & qp^{n-2}W_{2k+1} \\ qp^{n-2}W_{2k+1} & qp^{n-3}W_{2k+2} & \cdots & qW_{2k+n-1} & qp^{n-1}W_{2k} \end{pmatrix}.$$

such that $\tilde{\mathbf{F}}_r^{(k)} = C \circ D$. Thus, we obtain

$$r_1(C) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} |c_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |c_{nj}|^2} = \sqrt{(n-1)|r|^2+1},$$

and

$$c_1(D) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} |d_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |c_{nj}|^2} = \sqrt{\sum_{j=0}^{n-1} (qp^{n-1-j}W_{2k+j})^2},$$

using the inequalities (26) and (24), we have

$$\sqrt{\sum_{j=0}^{n-1} (qp^{n-1-j}W_{2k+j})^2} = \sqrt{\sum_{j=0}^{n-1} \left(W_j^{(k)} + W_{2k+j}(k-1)\right)^2} \le W_{n-1}^{(k+1)} + W_{2k+n+1}(k) - p^{n-1}W_{2k+1}(k).$$

and from Lemma 4.1, we get

$$\|\tilde{\mathbf{F}}_r^{(k)}\|_2 \le \sqrt{(n-1)|r|^2+1}\left(W_{n-1}^{(k+1)} + W_{2k+n+1}(k) - p^{n-1}W_{2k+1}(k)\right).$$

(ii) Since $|r| \le 1$, we have

$$\begin{aligned} \|\tilde{\mathbf{F}}_r^{(k)}\|_F &= \sqrt{\sum_{j=0}^{n-1} (n + j(|r|^2-1))\left(qp^{n-1-j}W_{2k+j}\right)^2} \\ &\ge \sqrt{n|r|^2 \sum_{j=0}^{n-1} \left(qp^{n-1-j}W_{2k+j}\right)^2} \\ &\ge |r|\left(W_{n-1}^{(k+1)} + W_{2k+n+1}(k) - p^{n-1}W_{2k+1}(k)\right). \end{aligned}$$

From (21), we obtain

$$\|\tilde{\mathbf{F}}_r^{(k)}\|_2 \ge \frac{|r|}{\sqrt{n}}\left(W_{n-1}^{(k+1)} + W_{2k+n+1}(k) - p^{n-1}W_{2k+1}(k)\right).$$

On the other hand, we have

$$r_1(C) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} |c_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |c_{nj}|^2} = \sqrt{n},$$

and

$$c_1(D) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} |d_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |c_{nj}|^2} = \sqrt{\sum_{j=0}^{n-1} (qp^{n-1-j} W_{2k+j})^2}.$$

Using the inequality $\|\tilde{\mathbf{F}}_r^{(k)}\|_2 \le r_1(C)c_1(D)$, we obtain

$$\|\tilde{\mathbf{F}}_r^{(k)}\|_2 \le \sqrt{n}\left(W_{n-1}^{(k+1)} + W_{2k+n+1}(k) - p^{n-1}W_{2k+1}(k)\right).$$

Thus, the proof is completed. □

Corollary 4.11. *For $h \ge 2k+2$, the spectral norm of the Hadamard product of $\tilde{\boldsymbol{A}}_r^{(k,h)}$ and $\tilde{\boldsymbol{H}}_r^{(k)}$ is given by*

(i) For $|r| \ge 1$, we have

$$\|\tilde{\boldsymbol{A}}_r^{(k,h)} \circ \tilde{\boldsymbol{H}}_r^{(k)}\|_2 \le ((n-1)|r|^2+1)W_{n-1}^{(k+1)}\left(W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1)\right).$$

(ii) For $|r| < 1$, we have

$$\|\tilde{\boldsymbol{A}}_r^{(k,h)} \circ \tilde{\boldsymbol{H}}_r^{(k)}\|_2 \le \sqrt{n(n-1)}W_{n-1}^{(k+1)}\left(W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1)\right).$$

Corollary 4.12. *For $h \ge 2k+2$, the spectral norm of the Kronecker product of $\tilde{\boldsymbol{A}}_r^{(k,h)}$ and $\tilde{\boldsymbol{H}}_r^{(k)}$ is given by*

(i) For $|r| \ge 1$, we have

$$\frac{W_{n-1}^{(k+1)}}{n}(W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1)) \le \|\tilde{\boldsymbol{A}}_r^{(k,h)} \otimes \tilde{\boldsymbol{H}}_r^{(k)}\|_2 \le ((n-1)|r|^2+1)W_{n-1}^{(k+1)} \cdot \left(W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1)\right).$$

(ii) For $|r| < 1$, we have

$$\frac{|r|^2 W_{n-1}^{(k+1)}}{n}\left(W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1)\right) \le \|\tilde{\boldsymbol{A}}_r^{(k,h)} \otimes \tilde{\boldsymbol{H}}_r^{(k)}\|_2 \le \sqrt{n(n-1)}W_{n-1}^{(k+1)} \cdot \left(W_{h+n+1}(k+1) - p^{n-1}W_{h+1}(k+1)\right).$$

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