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Solutions for a Singular Elliptic Problem Involving the $p(x)$-Laplacian

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Abstract. Here, a singular elliptic problem involving $p(x)-$Laplacian operator in a bounded domain in $\mathbb{R}^N$ is considered. Due to this, the existence of critical points for the energy functional which is unbounded below and satisfies the Palais-Smale condition are proved.

1. Introduction

A large number of researcher study the elliptic equations and variational problems with variable exponent, because of its importance in the theory of partial differential equations. Some of these problems come from different areas of applied mathematics and physics such as Micro Electro-Mechanical systems, modeling of Electrorheological Fluids, Surface Diffusion on Solids or Image Processing and Restoration.

On of the famous elliptic equations is Laplacian. This operator can be categorized as follows:

- Laplacian $\Delta := \sum_j \partial_j^2$ is linear and homogeneous.
- $p-$Laplacian $\Delta_p u(x) := div(|Du|^{p-2}Du)$ is nonlinear but homogeneous.
- $p(x)-$Laplacian $\Delta_{p(x)}u := div(|\nabla|^{p(x)-2}\nabla u)$ is nonlinear and nonhomogeneous.

Thus it seems the problems involving $p(x)-$Laplacian are usually much harder than those involving Laplacian or $p-$Laplacian from this point of view. Moreover, the singular boundary value problems involving the $p-$Laplacian operator have been studied by many researchers [2, 5–9, 13, 14]. Finally, the singularity elliptic problem involving $p(x)-$Laplacian operator is studied (see [10], [20]).

In this paper we consider the following problem

$$(P_\lambda) \quad \begin{cases} -\Delta_{p(x)}u + \frac{|u|^{s-2}u}{|x|^s} = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

- $\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u)$, denotes $p(x)-$Laplacian operator,
- $p(x) \in C(\bar{\Omega})$,

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- $1 < s < p(x) < \infty$,
- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary,
- $\lambda > 0$ is a real parameter and
- $q(x) \in C(\bar{\Omega})$ with $1 < q(x) < p^*(x)$ where

$$P^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & p(x) < N, \\ \infty & p(x) \geq N. \end{cases}$$

- $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$|f(x,t)| \leq a_1 + a_2|t|^{q(x)-2}, \text{ for all } (x,t) \in \Omega \times \mathbb{R} \tag{1}$$

  where a_1, a_2 are two positive constants.

In order to be familiar with the notations and literature associated to the problem, the following preliminaries are presented.

Definition 1.1. *[15] The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, if*

- *$x \to f(x,t)$ is measurable for every $t \in \mathbb{R}$.*
- *$t \to f(x,t)$ is continuous for almost every where $x \in \Omega$.*

Definition 1.2. *[22] Let X be a real Banach space. The operator $T : X \to X^*$ is said strictly monotone if $\langle T(u) - T(v)\ ,\ u - v\rangle \geq 0$ for each u, v in X.*

Definition 1.3. *[16] Let $1 < s < N$, we recall the classical Hardy's inequality, which says that*

$$\int_\Omega \frac{|u(x)|^s}{|x|^s} dx \leq \frac{1}{H} \int_\Omega |\nabla u(x)|^s dx, \qquad \text{for all } u \in X, \tag{2}$$

where $H := (\frac{n-s}{s})^s$.

Definition 1.4. *[21] Let X be a reflexive real Banach space. The operator $T : X \to X^*$ is said the (S_+) condition if the assumptions $\limsup_{n\to+\infty} \langle T(u_n) - T(u_0)\ , u_n - u_0\rangle \leq 0$ and $u_n \rightharpoonup u_0$ in X imply $u_n \to u_0$ in X.*

Definition 1.5. *[1] Let X be a Banach space and $\Phi : X \to \mathbb{R}$ a C^1-functional, we say that Φ satisfies the Palais-Smale condition, denoted by (PS), if any sequence u_n in X such that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$ admits a convergent subsequence.*

Let Ω be a bounded subset of $\mathbb{R}^N$, $(N \geq 2)$ and $p(x) \in C(\bar{\Omega})$. The space $L^{p(x)}(\Omega)$ is defined, see as e.g. [18] and [19], as

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R}; u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

and it is endowed with following norm

$$\|u\|_{L^{p(x)}} := inf \left\{ \lambda > 0; \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Moreover, the space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}; |\nabla u| \in L^{p(x)} \right\},$$

and norm in $W^{1,p(x)}$ is

$$\|u\|_{W^{1,p(x)}} := \|u\|_{L^{p(x)}} + \||\nabla u|\|_{L^{p(x)}}.$$

Also

$$W_0^{1,p(x)}(\Omega) := \left\{u \in W^{1,p(x)};\ u|_{\partial\Omega} = 0\right\}.$$

Let $X := W_0^{1,p(x)}(\Omega)$ endowed with norm

$$\|u\| = \||\nabla u|\|_{L^{p(x)}},$$

the compact embedding $X \hookrightarrow L^{q(x)}(\Omega)$, shows the existence of a $C_q > 0$ such that

$$\|u\|_{L^{q(x)}} \leq C_q\|u\|, \tag{3}$$

where $1 < q(x) < p^*(x)$, for all $x \in \Omega$, (see [11, Proposition 2.5]). Set

$$p^- := \inf_{x\in\Omega} p(x),\ p^+ := \sup_{x\in\Omega} p(x)$$

Assume that $\Phi : X \to \mathbb{R}$ is a functional defined by

$$\Phi(u) = \int_\Omega (\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{|u|^s}{s|x|^s})dx \tag{4}$$

where $1 < s < p^- \leq p(x) \leq p^+ < \infty$. By [17] and [11, Theorem 3.1],

- $\Phi \in C^1$.
- It is continuously Gâteaux differentiable functional.
- For all $u, v \in X$,

$$\Phi'(u)(v) = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u\nabla v + \frac{|u|^{s-2}uv}{|x|^s})dx. \tag{5}$$

- The operator $\Phi' : X \to X^*$ defined by (5) is strictly monotone.
- The operator $\Phi' : X \to X^*$ is homeomorphism and satisfies the condition (S_+).

Moreover, by [12, Theorem 1.3] we have:

Proposition 1.6. *Let $u \in W_0^{1,p(x)}$ and $\rho_p(u) := \int_\Omega |u(x)|^{p(x)}dx$. Then*

(*i*) $\|u\| < 1(= 1 :> 1) \Longleftrightarrow \rho_p(|\nabla u|) < 1(= 1 :> 1)$;
(*ii*) $\|u\| > 1$, *then* $\frac{1}{p^+}\|u\|^{p^-} \leq \Phi(u) \leq \frac{1}{p^-}\|u\|^{p^+} + \int_\Omega \frac{|u|^s}{s|x|^s}dx$;
(*iii*) $\|u\| < 1$, *then* $\frac{1}{p^+}\|u\|^{p^+} \leq \Phi(u) \leq \frac{1}{p^-}\|u\|^{p^-} + \int_\Omega \frac{|u|^s}{s|x|^s}dx$.

Now, let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function.

- For all $(x, \xi) \in X$, define

$$F(x, \xi) := \int_\Omega^\xi f(x, t)dt. \tag{6}$$

- For $u \in X$, define $\Psi : X \to \mathbb{R}$ by

$$\Psi(u) := \int_\Omega F(x, u(x))dx, \tag{7}$$

- Ψ is continuously Gâteaux differentiable functional.
- $\Psi \in C^1$ and has compact derivative such that

$$\Psi'(u)(v) := \int_\Omega f(x, u(x))v(x)dx, \tag{8}$$

for u, v in X (see [17]).

- Define $I := \Phi - \lambda\Psi$. Let $I'(u) = 0$. So

$$\int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla v + \frac{|u|^{s-2}uv}{|x|^s})dx = \lambda \int_\Omega f(x, u(x))v(x)dx, \tag{9}$$

for $u, v \in X$. Thus the critical points of I are the weak solutions of problem P_λ.

2. Two weak solutions

First we recall the following Bonanno's theorem [4].

Theorem 2.1. *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ and assume that, for each*

$$\lambda \in]0, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}[$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies (PS) condition and it is unbounded from below. Then, for each

$$\lambda \in]0, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}[$$

the functional I_λ admits two distinct critical points.

We present the existence of two weak solutions by applying Theorem 2.1 in case $r = 1$.

Theorem 2.2. *Let f satisfies (1), F be in (6), and there exist $\theta > p^+$ and $r > 0$ such that*

$$0 < \theta F(x, t) \le t f(x, t). \tag{10}$$

Then, for $\lambda \in]0, \lambda^*[$, the problem (P_λ) admits two weak solutions, where

$$\lambda^* := \frac{1}{a_1 C_1 (p^+)^{\frac{1}{p^-}} + \frac{a_2}{q^-}[C_q]^q (p^+)^{\frac{q^+}{p^-}}}. \tag{11}$$

Proof. Let Φ and Ψ are defined by (4) and (7), respectively. We prove the following steps:
Step 1. $I := \Phi - \lambda\Psi$ satisfies (PS) condition.
Assume $\{u_n\}$ is a sequence in X such that

$$d := \sup_{n \to +\infty} I(u_n) < \infty, \ \ \|I'(u_n)\|_{X^*} \to 0,$$

thus

$$\begin{aligned} d &\geq I(u_n) \\ &\geq \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \frac{1}{s}\int_\Omega \frac{|u_n|^s}{|x|^s}dx - \lambda\int_\Omega F(x,u_n)dx \\ &\geq \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \frac{1}{s}\int_\Omega \frac{|u_n|^s}{|x|^s}dx - \frac{\lambda}{\theta}\int_\Omega f(x,u_n)u_n dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\int_\Omega |\nabla u_n|^{p(x)}dx + \frac{1}{\theta}\|I'(u_n)\|\,\|u_n\|, \end{aligned}$$

so $\|u_n\|$ is bounded. Therefore, if $u_n \rightharpoonup u$ so $\Psi'(u_n) \to \Psi'(u)$, since $I'(u_n) = \Phi'(u_n) - \lambda\Psi'(u_n) = 0$ then $\Phi'(u_n) \to \lambda\Psi'(u_n)$, thus $u_n \to u$ (because Φ' is homeomorphism). So I satisfies the condition (PS).

Step 2. I is unbounded from below.
First we show, there exists $M \in \mathbb{R}^+$ such that for $x \in \Omega$ and $|t| > M$

$$F(x,\xi) \geq K|\xi|^\theta. \tag{12}$$

(10) implies

$$0 < \theta F(x,\xi t) \leq \xi t f(x,\xi t), \text{ for all } \xi > 0.$$

Let $m(x) := \min_{|\xi|=M} F(x,\xi)$ and $g_t(z) := F(x,zt)$ for all $z > 0$ thus

$$0 < \theta g_t(z) = \theta F(x,zt) \leq zt f(x,zt) = z g_t'(z)$$

for all $z > \frac{M}{|t|}$, so

$$\int_{\frac{M}{|t|}}^1 \frac{g_t'(z)}{g_t(z)}dz \geq \int_{\frac{M}{|t|}}^1 \frac{\theta}{z}dz,$$

then

$$Ln\left(\frac{g_t(1)}{g_t(\frac{M}{|t|})}\right) \geq Ln\left(\frac{|t|^\theta}{M^\theta}\right)$$

therefore

$$F(x,t) = g_t(1) > F\left(x,\frac{M}{|t|}t\right)\frac{|t|^\theta}{M^\theta} \geq m(x)\frac{|t|^\theta}{M^\theta} \geq K|t|^\theta$$

so (12) is established.

Fixed $v \in X - \{0\}$, for each $t > 1$ one has

$$\begin{aligned} I(tv) &= \int_\Omega \frac{1}{p(x)}|t\nabla v|^{p(x)}dx + \frac{1}{s}\int_\Omega \frac{|tv|^s}{|x|^s}dx - \lambda\int_\Omega F(x,tv)dx \\ &\leq t^{p^+}\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}dx + \frac{t^s}{sH}\int_\Omega \frac{|\nabla v|^s}{|x|^s}dx - \lambda K t^\theta\int_\Omega |v|^\theta dx - C_1 \\ &\leq t^{p^+}\left(\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}dx + \frac{1}{sH}\int_\Omega \frac{|\nabla v|^s}{|x|^s}dx\right) - \lambda K t^\theta\int_\Omega |v|^\theta dx - C_1, \end{aligned}$$

where H as in Definition (1.3). Since $p^+ < \theta$ if $t \to +\infty$ then $I \to -\infty$.

Fix $\lambda \in]0,\lambda^*[$ where λ^* is defined as (11) and will be given later. Proposition 1.6 for each $u \in \Phi^{-1}(]-\infty,1[)$ implies

$$\|u\| \leq [p^+\Phi(u)]^{\frac{1}{p}} \leq [p^+]^{\frac{1}{p}} = (p^+)^{\frac{1}{p^-}}. \tag{13}$$

by (3) and Proposition 1.6

$$\int_{\Omega} |u(x)|^{q(x)}dx = \rho_q(u) \leq [||u||_{L^{q(x)}(\Omega)}]^q \leq [C_q||u||]^q \tag{14}$$

for $u \in X$. By the compact embedding $X \hookrightarrow L^1(\Omega),\ X \hookrightarrow L^q(\Omega)$ there exist $C_1, C_q > 0$ and by (1), (10), (13) and (14)

$$\begin{aligned}
\Psi(u) = & \int_{\Omega} F(x,u)dx \\
\leq & \ a_1 \int_{\Omega} |u(x)|dx + \frac{a_2}{q^-} \int_{\Omega} |u(x)|^{q(x)}dx \\
\leq & \ a_1 C_1 ||u|| + \frac{a_2}{q^-}[C_q||u||]^q \\
\leq & \ a_1 C_1 [p^+]^{\frac{1}{p^-}} + \frac{a_2}{q^-}[C_q]^q (p^+)^{\frac{q^+}{p^-}} \\
= & \ \frac{1}{\lambda^*} \\
< & \ \frac{1}{\lambda},
\end{aligned}$$

therefore $\lambda < \frac{1}{\sup\limits_{u\in\Phi^{-1}(]-\infty,1[)} \Psi(u)}$. Let $I := I_\lambda$ in Theorem 2.1, thus by Theorem 2.1 problem (P_λ) admits two weak solutions. □

Here we present two examples to show the validity of Condition (10) of Theorem 2.2. In other words, in each example we present a function $f(x,t)$ to check the validity of Condition (10). Consequently, by Theorem 2.2 we can guarantee the existence of two weak solutions for each problem presented in the examples.

Example 2.3. *Consider the following problem*

$$\begin{cases} -\Delta_{p(x)}u + \frac{|u|^{s-2}u}{|x|^s} = \lambda q(x) sinht, & in\ \Omega, \\ u = 0, & on\ \partial\Omega. \end{cases}$$

For $x \in \mathbb{R},\ 1 < p^+ < \theta < |t| < q(x) < \infty$, we have

$$F(x,t) = q(x)[cosht - 1].$$

We prove $\theta F(x,t) < tf(x,t)$, or equivalently $\theta cosht - tsinht - \theta \leq 0$, for $x \in \mathbb{R}$ and $1 < \theta < |t|$. For this, we discuss two following cases:
(i) if $t > \theta > 1$ then

$$\begin{aligned} \theta cosht - tsinht - \theta & < \theta cosht - tsinht - \theta \\ & < \theta e^{-t} - \theta < 0. \end{aligned}$$

(ii) if $t < -\theta < -1$ hence

$$\begin{aligned} \theta cosht - tsinht - \theta & < \theta cosht + tsinht - \theta \\ & < \theta e^{t} - \theta < 0. \end{aligned}$$

Therefore, the function f satisfies Condition (10), so by Theorem 2.2 this problem has two weak solutions.

Remark 2.4. *In Example 2.3, it is easily seen that the function $q(x)$ can be replaced by all positive functions* $\cosh x, e^x, x^2$.

The following example (given in [3]) presents a function $f(x,t)$ to satisfy Condition (10).

Example 2.5. *Let a and b be two positive constants and*

$$f(x,t) = \begin{cases} a + bq(x)t^{q(x)-1} & x \in \Omega,\ t \geq 0 \\ \\ a - bq(x)(-t)^{q(x)-1} & x \in \Omega,\ t < 0 \end{cases}$$

for $(x,t) \in \Omega \times \mathbb{R}$ *and* $1 < p^+ < q(x) < \infty$. *Thus f satisfies Condition (10) of Theorem 2.2.*

3. Three weak solutions

First we recall the following Bonanno's theorem [4].

Theorem 3.1. *Let X be a reflexive real Banach space,* $\Phi : X \to \mathbb{R}$ *be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on* X^*, $\Psi : X \to \mathbb{R}$ *be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_{x \in \Omega} \Phi(x) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ *and* $\overline{x} \in X$ *with* $\Phi(\overline{x}) < r$ *such that*

(i) $\frac{\sup\limits_{\Phi(x)<r} \Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$.

(ii) for each $\lambda \in \Lambda :=]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup\limits_{\Phi(x)<r} \Psi(x)}[$ *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then for each $\lambda \in \Lambda$, *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points in X.*

In order to percent the existence of at least three weak solutions set

$$\delta(x) = sup\left\{\delta > 0 : B(x,\delta) \subseteq \Omega\right\},$$

$$D = \sup_{x \in \Omega} \delta(x),$$

for $x \in \Omega$, it is easy to see that there exists $x_0 \in \Omega$ such that

$$B(x_0, D) \subseteq \Omega.$$

Also, for $a > 0$ and $q(x) \in C(\overline{\Omega})$ with

$$1 < q^- := \inf_{x \in \Omega} q(x) < q(x) < q^+ := \sup_{x \in \Omega} q(x) < 0,$$

we have

$$[a]^{q(x)} := max\left\{a^{q^-}, a^{q^+}\right\},$$

$$[a]_{q(x)} := min\left\{a^{q^-}, a^{q^+}\right\},$$

where $x \in \Omega$. Let $r > 0$, set

$$\overline{\omega} := \frac{1}{r}\left\{a_1 C_1 (p^+)^{\frac{1}{p^-}} [r]^{\frac{1}{p}} + \frac{a_2}{q^-}[C_q]^q (p^+)^{\frac{q^+}{p^-}} [[r]^{\frac{1}{p}}]^q\right\} \tag{15}$$

where a_1, a_2 are positive numbers and C_1, C_q are ordinary embedding constants $X \hookrightarrow L^1(\Omega)$ and $X \hookrightarrow L^{q(x)}(\Omega)$, respectively.

Theorem 3.2. *Assume that f, F satisfy* (6) *and*

$$F(x,t) \geq 0, \tag{16}$$

for $(x,t) \in \Omega \times \mathbb{R}^+$. Suppose there exists $C \in [0,\infty)$ such that

$$F(x,t) \leq C(1+|t|^{q(x)}), \tag{17}$$

for $(x,t) \in \Omega \times \mathbb{R}$, $q(x) \in C(\overline{\Omega})$ and $1 < q^- < q(x) < q^+ < p^-$. Moreover, there exist $\delta, r > 0$ with $r < \frac{1}{p^+}[\frac{2\delta}{D}]_p m(D^N - (\frac{D}{2})^N)$ such that

$$\overline{\omega} < \frac{(H+1)[\frac{2\delta}{D}]^p(2^N-1)}{sH \inf\limits_{x\in\Omega} F(x,\delta)}.$$

Then, for every $\lambda \in \Lambda$, the problem (P_λ) has at least three weak solutions, where

$$\Lambda :=]\frac{(H+1)[\frac{2\delta}{D}]^p(2^N-1)}{sH \inf\limits_{x\in\Omega} F(x,\delta)}, \frac{1}{\overline{\omega}}[$$

and $\overline{\omega}$ is in (15)*, $m := \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})}$ is measure of unit of $\mathbb{R}^{\mathbb{N}}$ and Γ is the Gamma function.*

Proof. Let X, Φ, Ψ and I be the same as the last section. We investigate the conditions (i), (ii) of Theorem 3.1. Let $\overline{u} \in X$ such that

$$\overline{u}(x) = \begin{cases} 0 & x \in \Omega \backslash B(x_0, D), \\ \delta_\lambda & x \in B(x_0, \frac{D}{2}), \\ \frac{2\delta_\lambda}{D}(D - |x - x_0|) & x \in B(x_0, D) \backslash B(x_0, \frac{D}{2}), \end{cases}$$

where $|.|$ is Euclidean norm on $\mathbb{R}^{\mathbb{N}}$. By Proposition 1.6 and hypothesis of theorem

$$\begin{aligned} r &< \tfrac{1}{p^+}[\tfrac{2\delta}{D}]_p m(D^N - (\tfrac{D}{2})^N) \\ &\leq \int_\Omega (\tfrac{1}{p(x)}|\nabla \overline{u}|^{p(x)} + \tfrac{|\overline{u}|^s}{s|x|^s})dx = \Phi(\overline{u}) \\ &\leq \int_\Omega \tfrac{1}{s}|\nabla \overline{u}|^{p(x)}dx + \tfrac{1}{sH}\int_\Omega |\nabla \overline{u}|^s dx \\ &\leq \tfrac{1}{s}[\tfrac{2\delta}{D}]^p m(D^N - (\tfrac{D}{2})^N) + \tfrac{1}{sH}[\tfrac{2\delta}{D}]^s m(D^N - (\tfrac{D}{2})^N) \\ &\leq \tfrac{H+1}{sH}[\tfrac{2\delta}{D}]^p m(D^N - (\tfrac{D}{2})^N). \end{aligned}$$

Therefore

$$\frac{\Psi(\overline{u})}{\Phi(\overline{u})} \geq \frac{sH \inf\limits_{x\in\Omega} F(x,\delta)}{(H+1)[\frac{2\delta}{D}]^p(2^N-1)}, \tag{18}$$

because

$$\Psi(\overline{u}) \geq \int_{B(x_0,D)} F(x, \overline{u}(x))dx \geq \inf_{x\in\Omega} F(x,\delta) m(\frac{D}{2})^N$$

thus, for every $u \in \Phi^{-1}(-\infty, r]$ by Proposition 1.6

$$\|u\| \leq [p^+\Phi(u)]^{\frac{1}{p}} \leq (p^+)^{\frac{1}{p^-}}[r]^{\frac{1}{p}}. \tag{19}$$

Thanks to (19), (1), the compact embedding $X \hookrightarrow L^1(\Omega)$ and $X \hookrightarrow L^{q(x)}(\Omega)$, we have

$$\begin{aligned}\Psi(u) &= \int_\Omega F(x,u(x))dx \\ &\le a_1 \int_\Omega |u(x)|dx + \frac{a_2}{q^-}\int_\Omega |u(x)|^{q(x)}dx \\ &\le a_1C_1\|u\| + \frac{a_2}{q^-}[C_q]^q(p^+)^{\frac{q^+}{p^-}}[[r]^{\frac{1}{p}}] \\ &\le a_1C_1(p^+)^{\frac{1}{p^-}}[r]^{\frac{1}{p}} + \frac{a_2}{q^-}[C_q]^q(p^+)^{\frac{q^+}{p^-}}[[r]^{\frac{1}{p}}]\end{aligned}$$

for $u \in \Phi^{-1}(-\infty, r]$, therefore

$$\frac{1}{r}\sup_{u\in\Phi^{-1}(-\infty,r]}\Psi(u) \le \frac{1}{r}\left\{a_1C_1(p^+)^{\frac{1}{p^-}}[r]^{\frac{1}{p}} + \frac{a_2}{q^-}[C_q]^q(p^+)^{\frac{q^+}{p^-}}[[r]^{\frac{1}{p}}]\right\} < \frac{\Psi(\overline{u})}{\Phi(\overline{u})}$$

this implies part (i) of Theorem 3.1 is satisfied.
Here, we show that for each $\lambda > 0$, $I := \Phi - \lambda\Psi$ is coercive. Let $u \in X$ with $\|u\| \ge \left\{1, \frac{1}{C_q}\right\}$, by (3) and (17), we have

$$\begin{aligned}\Psi(u) = \int_\Omega F(x,t)dx &\le \int_\Omega (C(1+|t|^{q(x)}))dx \\ &\le C(|\Omega| + [C_q\|u\|]^{q^+}),\end{aligned}$$

therefore

$$I(u) = \Phi(u) - \lambda\Psi(u) \ge \frac{1}{p^+}\|u\|^{p^-} - \lambda C(|\Omega| + C_q^{q^+}\|u\|^{q^+}),$$

hence, according $q^+ < p^-$ implies I is coercive. So by Theorem 3.1 the problem P_λ has at least three weak solutions. □

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