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Further Results on Non-Self-Centrality Measures of Graphs

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Abstract. For indicating the non-self-centrality extent of graphs, two new eccentricity-based measures namely third Zagreb eccentricity index $E_3(G)$ and non-self-centrality number $N(G)$ of a connected graph G have recently been introduced as $E_3(G) = \sum_{uv\in E(G)} |\varepsilon_G(u) - \varepsilon_G(v)|$ and $N(G) = \sum_{\{u,v\}\subseteq V(G)} |\varepsilon_G(u) - \varepsilon_G(v)|$, where $\varepsilon_G(u)$ denotes the eccentricity of a vertex u in G. In this paper, we find relation between the third Zagreb eccentricity index of graphs with some eccentricity-based invariants such as second Zagreb eccentricity index and second eccentric connectivity index. We also give sharp upper and lower bounds on the non-self-centrality number of graphs in terms of some structural parameters and relate it to two well-known eccentricity-based invariants namely total eccentricity and first Zagreb eccentricity index. Furthermore, we present exact expressions or sharp upper bounds on the third Zagreb eccentricity index and non-self-centrality number of several graph operations such as join, disjunction, symmetric difference, lexicographic product, strong product, and generalized hierarchical product. The formulae for Cartesian product and rooted product as two important special cases of generalized hierarchical product and the formulae for corona product as a special case of rooted product are also given.

1. Introduction

All graphs considered in this paper are finite, simple and connected. Let G be a graph on n vertices and m edges. We denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. The degree $\deg_G(u)$ of a vertex $u \in V(G)$ is the number of edges incident to u. The distance $d_G(u, v)$ between the vertices $u, v \in V(G)$ is defined as the length of a shortest path in G connecting u and v. The eccentricity $\varepsilon_G(u)$ of a vertex $u \in V(G)$ is the largest distance between u and any other vertex v of G, i.e., $\varepsilon_G(u) = \max_{v\in V(G)} d_G(u, v)$. The maximum (minimum, resp.) eccentricity over all vertices of G is called the diameter (radius, resp.) of G and denoted by $d(G)$ ($r(G)$, resp.). Some novel applications of eccentricity in networks were given in [23]. A vertex $u \in V(G)$ is called a universal vertex if u is adjacent to every other vertex of G, i.e., $deg_G(u) = n - 1$. We denote the number of universal vertices of G by $n_{n-1}(G)$. Obviously, the eccentricity of a universal vertex in a non-trivial graph is equal to 1.

A topological index (also known as graph invariant or molecular descriptor) is a numeric quantity that is mathematically derived in a direct and unambiguous manner from the structural graph of a molecule. It is used in theoretical chemistry for the design of chemical compounds with given physico-chemical properties or given pharmacologic and biological activities [20]. From the graph theoretical point of view,

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a topological index can be viewed as a graph invariant under automorphisms of graphs. Many topological indices have been proposed and employed to date with various degrees of success in QSAR/QSPR studies [15, 28]. Some of them, based on vertex eccentricity have attracted much attention in chemistry.

The eccentric connectivity index was introduced by Sharma *et al.* [26] in 1997. The eccentric connectivity index $\xi^c(G)$ of a graph G is defined as

$$\xi^c(G) = \sum_{u \in V(G)} deg_G(u)\varepsilon_G(u) = \sum_{uv \in E(G)} \Big(\varepsilon_G(u) + \varepsilon_G(v)\Big).$$

The sum of eccentricities of all vertices of a given graph G is called the total eccentricity of G and denoted by $\tau(G)$.

The first and second Zagreb eccentricity indices of a graph G were introduced by Vukičević and Graovac [29] in 2010 as follows:

$$E_1(G) = \sum_{u \in V(G)} \varepsilon_G(u)^2, \ \ E_2(G) = \sum_{uv \in E(G)} \varepsilon_G(u)\varepsilon_G(v).$$

Note that two well-known topological indices namely first and second Zagreb indices [21, 22] of a graph are similarly defined but based on the degrees of vertices. We refer the reader to [4, 13, 19] for some recent surveys on Zagreb indices and to [5, 10, 11, 14, 16, 24, 30–32, 35, 36] for some results on mathematical properties and applications of eccentricity-based topological indices.

A graph G is called a self-centered graph if all of its vertices have a same eccentricity. Otherwise, it is called non-self-centered. However, in many applications and problems of graph theory, it is of great importance to measure the non-self-centrality extent of graphs. Recently, Xu *et al.* [33] proposed two novel eccentricity-based invariants named as third Zagreb eccentricity index and non-self-centrality number (NSC number for short) for that purpose. The third Zagreb eccentricity index and non-self-centrality number of a graph G are denoted by $E_3(G)$ and $N(G)$, respectively and defined as

$$E_3(G) = \sum_{uv \in E(G)} |\varepsilon_G(u) - \varepsilon_G(v)|, \ \ N(G) = \sum_{\{u,v\} \subseteq V(G)} |\varepsilon_G(u) - \varepsilon_G(v)|,$$

where the second summation is taken over all unordered pairs of vertices of G. The NSC number of G can also be expressed as

$$N(G) = \frac{1}{2} \sum_{u,v \in V(G)} |\varepsilon_G(u) - \varepsilon_G(v)|,$$

where the summation is taken over all ordered pairs of vertices of G. It is easy to see that $E_3(G) = N(G) = 0$ if and only if G is a self-centered graph. Note that these two graph non-self-centrality measures are defined analogously to two well-known graph irregularity measures namely the irregularity [3] (also called third Zagreb index [18]) and the total irregularity [1] by replacing the vertex degrees with the vertex eccentricities. Since the third Zagreb eccentricity index and NSC number are newly-introduced graph invariants, only a few mathematical results on these invariants have been obtained. Xu *et al.* [33] determined some lower and upper bounds on NSC number and characterized the corresponding graphs at which the lower and upper bounds are attained. In particular, they proved that for any tree T, $N(T)$ uniquely attains the maximum value at the $n-$vertex path P_n and the minimum value at the $n-$vertex star S_n, while for any tree T of order $n \geq 3$, $E_3(T)$ has only two values: $n-2$ for trees with odd diameter and $n-1$ for trees with even diameter. From this fact, $N(G)$ is better than $E_3(G)$ for indicating the non-self-centrality of a graph. In [34], the relations between NSC number and total irregularity of graphs were established. The purpose of this paper is to further study mathematical properties of these new graph invariants.

This paper is organized as follows. In Section 2, we find relation between the third Zagreb eccentricity index and some eccentricity-based invariants such as second Zagreb eccentricity index and second eccentric connectivity index. We also give sharp upper and lower bounds on the NSC number of graphs in terms of some structural parameters and relate it to two well-known eccentricity-based invariants namely total eccentricity and first Zagreb eccentricity index. In Section 3, we study the third Zagreb eccentricity index and

NSC number under several graph operations such as join, disjunction, symmetric difference, lexicographic product, strong product, and generalized hierarchical product and give exact expressions or sharp upper bounds on these eccentricity-based invariants of the above-mentioned graph operations. In Section 4, we apply our results to compute the third Zagreb eccentricity index and NSC number of Cartesian product and rooted product which are two important special cases of generalized hierarchical product, and of corona product which is a special case of rooted product.

2. Relation with some eccentricity-based invariants

In this section, we relate the third Zagreb eccentricity index and non-self-centrality number of graphs to some other eccentricity-based invariants such as total eccentricity, first and second Zagreb eccentricity indices, and second eccentric connectivity index.

We define the second eccentric connectivity index of a graph G as follows:

$$\xi^{(2)}(G) = \sum_{u\in V(G)} deg_G(u)\varepsilon_G(u)^2 = \sum_{uv\in E(G)} \Big(\varepsilon_G(u)^2 + \varepsilon_G(v)^2\Big).$$

In the following theorem, we show that the second eccentric connectivity index is a linear combination of the third Zagreb eccentricity index and the second Zagreb eccentricity index.

Theorem 2.1. *For any graph G,*

$$\xi^{(2)}(G) = E_3(G) + 2E_2(G). \tag{1}$$

Proof. It is clear that for each $uv \in E(G)$, $|\varepsilon_G(u) - \varepsilon_G(v)| = 0$ or $|\varepsilon_G(u) - \varepsilon_G(v)| = 1$. So for each $uv \in E(G)$, $|\varepsilon_G(u) - \varepsilon_G(v)|^2 = |\varepsilon_G(u) - \varepsilon_G(v)|$. Hence

$$\begin{aligned}\xi^{(2)}(G) &= \sum_{uv\in E(G)} \Big(\varepsilon_G(u)^2 + \varepsilon_G(v)^2\Big) = \sum_{uv\in E(G)} \Big((\varepsilon_G(u) - \varepsilon_G(v))^2 + 2\varepsilon_G(u)\varepsilon_G(v)\Big)\\ &= \sum_{uv\in E(G)} \Big(|\varepsilon_G(u) - \varepsilon_G(v)|^2 + 2\varepsilon_G(u)\varepsilon_G(v)\Big) = \sum_{uv\in E(G)} \Big(|\varepsilon_G(u) - \varepsilon_G(v)| + 2\varepsilon_G(u)\varepsilon_G(v)\Big)\\ &= E_3(G) + 2E_2(G),\end{aligned}$$

and Eq. (1) holds. □

In the following theorem, we give a sharp upper bound on the NSC number of a graph G in terms of the order, total eccentricity, and first Zagreb eccentricity index of G.

Theorem 2.2. *Let G be a graph of order n. Then*

$$N(G) \leq \sqrt{\tbinom{n}{2}\Big(nE_1(G) - \tau(G)^2\Big)}, \tag{2}$$

with equality if and only if G is a self-centered graph.

Proof. By Cauchy–Schwarz inequality, we have

$$\begin{aligned}N(G)^2 = \Big(\sum_{\{u,v\}\subseteq V(G)} |\varepsilon_G(u) - \varepsilon_G(v)|\Big)^2 &\leq \binom{n}{2} \sum_{\{u,v\}\subseteq V(G)} |\varepsilon_G(u) - \varepsilon_G(v)|^2\\ &= \frac{1}{2}\binom{n}{2} \sum_{u,v\in V(G)} \Big(\varepsilon_G(u)^2 + \varepsilon_G(v)^2 - 2\varepsilon_G(u)\varepsilon_G(v)\Big)\\ &= \frac{1}{2}\binom{n}{2}\Big(2nE_1(G) - 2\tau(G)^2\Big)\\ &= \binom{n}{2}\Big(nE_1(G) - \tau(G)^2\Big),\end{aligned}$$

from which we straightforwardly arrive at Eq. (2). By Cauchy–Schwarz inequality, the equality holds in (2) if and only if for every $\{u, v\} \subseteq V(G)$, $|\varepsilon_G(u) - \varepsilon_G(v)|$ is constant, which implies that G is a self-centered graph. □

In the following theorem, we give a sharp lower bound on the NSC number of a non-self-centered graph G in terms of the order, radius, diameter, total eccentricity, and first Zagreb eccentricity index of G.

Theorem 2.3. *Let G be a non-self-centered graph of order n. Then*

$$N(G) \geq \frac{nE_1(G) - \tau(G)^2}{d(G) - r(G)}, \tag{3}$$

with equality if and only if $d(G) - r(G) = 1$.

Proof. It is easy to see that for every $\{u, v\} \subseteq V(G)$, $|\varepsilon_G(u) - \varepsilon_G(v)| \leq d(G) - r(G)$. Hence

$$nE_1(G) - \tau(G)^2 = \sum_{\{u,v\} \subseteq V(G)} |\varepsilon_G(u) - \varepsilon_G(v)|^2 \leq \big(d(G) - r(G)\big) \sum_{\{u,v\} \subseteq V(G)} |\varepsilon_G(u) - \varepsilon_G(v)| = \big(d(G) - r(G)\big)N(G),$$

which is easily transformed into Eq. (3). The equality holds in (3) if and only if for every $\{u, v\} \subseteq V(G)$, $|\varepsilon_G(u) - \varepsilon_G(v)| = d(G) - r(G)$ or $|\varepsilon_G(u) - \varepsilon_G(v)| = 0$. If $d(G) - r(G) = 1$, then for every $\{u, v\} \subseteq V(G)$, $0 \leq |\varepsilon_G(u) - \varepsilon_G(v)| \leq d(G) - r(G) = 1$. This implies that, for every $\{u, v\} \subseteq V(G)$, $|\varepsilon_G(u) - \varepsilon_G(v)| = 1 = d(G) - r(G)$ or $|\varepsilon_G(u) - \varepsilon_G(v)| = 0$ and the equality holds in (3). If the equality holds in (3), then for every $\{u, v\} \subseteq V(G)$, $|\varepsilon_G(u) - \varepsilon_G(v)| = d(G) - r(G)$ or $|\varepsilon_G(u) - \varepsilon_G(v)| = 0$. It is clear that for each $uv \in E(G)$, $|\varepsilon_G(u) - \varepsilon_G(v)| = 1$ or $|\varepsilon_G(u) - \varepsilon_G(v)| = 0$. Since G is a non-self-centered graph, so there exists an edge $uv \in E(G)$ such that $|\varepsilon_G(u) - \varepsilon_G(v)| = 1$. This implies that $d(G) - r(G) = 1$. □

3. Graph operations

In this section, we present exact expressions or sharp upper bounds on the third Zagreb eccentricity index and NSC number of several graph operations in terms of the respective indices of the components, the number of their vertices, the number of their universal vertices, and in some cases also the number of their edges. The considered operations are binary. Hence, we will deal with two graphs G_1 and G_2 which are considered to be simple connected graphs. For given component graphs G_i, the number of vertices and edges are denoted by n_i and m_i, respectively, where $i = 1, 2$. When more than two graphs can be combined using a given operation, the values of subscripts will vary accordingly. We refer the reader to monograph [25] for detailed exposition on graph operations and to [2, 6–10, 17, 27] for more information on computing topological invariants of graph operations.

We start with the following lemma which will be used in the rest of the paper.

Lemma 3.1. *Let G be a graph of order n with diameter at most two. Then*

$$E_3(G) = N(G) = n_{n-1}(G)\big(n - n_{n-1}(G)\big).$$

Proof. By proof of Theorem 3.1 in [33], $E_3(G) = N(G)$ if and only if the diameter of G is at most two. On the other hand, by definition of the NSC number, we have

$$\begin{aligned} N(G) = \sum_{\{u,v\} \subseteq V(G)} |\varepsilon_G(u) - \varepsilon_G(v)| &= \sum_{\substack{\{u,v\} \subseteq V(G): \\ \varepsilon_G(u) = \varepsilon_G(v) = 1}} |1 - 1| + \sum_{\substack{\{u,v\} \subseteq V(G): \\ \varepsilon_G(u) = 1, \varepsilon_G(v) = 2}} |1 - 2| + \sum_{\substack{\{u,v\} \subseteq V(G): \\ \varepsilon_G(u) = \varepsilon_G(v) = 2}} |2 - 2| \\ &= n_{n-1}(G)\big(n - n_{n-1}(G)\big). \end{aligned}$$

This completes the proof of the lemma. □

3.1. Join

The join $G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph union $G_1 \cup G_2$ together with all the edges joining $V(G_1)$ and $V(G_2)$. The definition generalizes to the case of $k \geq 3$ graphs in a straightforward manner. The join of graphs is also known as their sum. The eccentricity of a vertex $u \in V(G_1 + G_2 + ... + G_k)$ is given by

$$\varepsilon_{G_1+G_2+...+G_k}(u) = \begin{cases} 1 & if\ \varepsilon_{G_i}(u) = 1, \\ 2 & if\ \varepsilon_{G_i}(u) \geq 2. \end{cases}$$

Here G_i denotes the component of $G_1, G_2, \ldots,$ or G_k containing vertex u.

Theorem 3.2. *The third Zagreb eccentricity index and NSC number of $G_1 + G_2 + ... + G_k$ are given by*

$$E_3(G_1 + G_2 + ... + G_k) = N(G_1 + G_2 + ... + G_k) = \Big(\sum_{i=1}^{k} n_{n_i-1}(G_i)\Big)\Big(n - \sum_{i=1}^{k} n_{n_i-1}(G_i)\Big), \tag{4}$$

where $n = \sum_{i=1}^{k} n_i$.

Proof. The graph $G_1 + G_2 + ... + G_k$ is an n-vertex graph with diameter at most two and the number of universal vertices of this graph is $n_{n-1}(G_1 + G_2 + ... + G_k) = \sum_{i=1}^{k} n_{n_i-1}(G_i)$. Now by Lemma 3.1 we can get Eq. (4). □

3.2. Disjunction

The disjunction $G_1 \vee G_2$ of graphs G_1 and G_2 is a graph with the vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent, whenever $u_1v_1 \in E(G_1)$ or $u_2v_2 \in E(G_2)$. The eccentricity of a vertex (u_1, u_2) in $G_1 \vee G_2$ is given by

$$\varepsilon_{G_1 \vee G_2}((u_1, u_2)) = \begin{cases} 1 & if\ \varepsilon_{G_1}(u_1) = \varepsilon_{G_2}(u_2) = 1, \\ 2 & if\ \varepsilon_{G_1}(u_1) \geq 2\ \ or\ \ \varepsilon_{G_2}(u_2) \geq 2. \end{cases}$$

Theorem 3.3. *The third Zagreb eccentricity index and NSC number of $G_1 \vee G_2$ are given by*

$$E_3(G_1 \vee G_2) = N(G_1 \vee G_2) = n_{n_1-1}(G_1)n_{n_2-1}(G_2)\Big(n_1n_2 - n_{n_1-1}(G_1)n_{n_2-1}(G_2)\Big). \tag{5}$$

Proof. The graph $G_1 \vee G_2$ is an n_1n_2-vertex graph with diameter at most two and the number of universal vertices of this graph is $n_{n_1n_2-1}(G_1 \vee G_2) = n_{n_1-1}(G_1)n_{n_2-1}(G_2)$. Now by Lemma 3.1 we can get Eq. (5). □

3.3. Symmetric difference

The symmetric difference $G_1 \oplus G_2$ of graphs G_1 and G_2 is a graph with vertex set $V(G_1) \times V(G_2)$ and edge set $E(G_1 \oplus G_2) = \{(u_1, u_2)(v_1, v_2) : u_1v_1 \in E(G_1)$ *or* $u_2v_2 \in E(G_2)$ *but not both*$\}$. The eccentricity of every vertex (u_1, u_2) in $G_1 \oplus G_2$ is given by

$$\varepsilon_{G_1 \oplus G_2}((u_1, u_2)) = 2.$$

So, the symmetric difference $G_1 \oplus G_2$ is self-centered and we easily arrive at:

Theorem 3.4. *The third Zagreb eccentricity index and NSC number of $G_1 \oplus G_2$ are given by*

$$E_3(G_1 \oplus G_2) = N(G_1 \oplus G_2) = 0.$$

3.4. Lexicographic product

The lexicographic product $G_1[G_2]$ of graphs G_1 and G_2 is a graph with vertex set $V(G_1)\times V(G_2)$ and two vertices (u_1,u_2) and (v_1,v_2) are adjacent if and only if $u_1v_1\in E(G_1)$ or $[u_1=v_1\in V(G_1)$ and $u_2v_2\in E(G_2)]$. The lexicographic product of two graphs is also known as their composition. The eccentricity of a vertex (u_1,u_2) in $G_1[G_2]$ is given by

$$\varepsilon_{G_1[G_2]}((u_1,u_2))=\begin{cases}1 & if\ \varepsilon_{G_1}(u_1)=\varepsilon_{G_2}(u_2)=1,\\ 2 & if\ \varepsilon_{G_1}(u_1)=1,\ \varepsilon_{G_2}(u_2)\geq 2,\\ \varepsilon_{G_1}(u_1) & if\ \varepsilon_{G_1}(u_1)\geq 2.\end{cases}$$

Theorem 3.5. *If G_1 contains universal vertices, then*

$$E_3(G_1[G_2])=N(G_1[G_2])=n_{n_1-1}(G_1)n_{n_2-1}(G_2)\Big(n_1n_2-n_{n_1-1}(G_1)n_{n_2-1}(G_2)\Big); \tag{6}$$

whereas otherwise

$$E_3(G_1[G_2])=n_2^2E_3(G_1), \tag{7}$$

$$N(G_1[G_2])=n_2^2N(G_1). \tag{8}$$

Proof. If G_1 contains universal vertices, then $G_1[G_2]$ is an n_1n_2-vertex graph with diameter at most two and the number of universal vertices of this graph is $n_{n_1n_2-1}(G_1[G_2])=n_{n_1-1}(G_1)n_{n_2-1}(G_2)$. Now by Lemma 3.1 we can get Eq. (6). Now, let G_1 contain no universal vertices. By definition of the third Zagreb eccentricity index we obtain

$$\begin{aligned}E_3(G_1[G_2])=&\sum_{(u_1,u_2)(v_1,v_2)\in E(G_1[G_2])}\left|\varepsilon_{G_1[G_2]}((u_1,u_2))-\varepsilon_{G_1[G_2]}((v_1,v_2))\right|\\ =&\sum_{u_1v_1\in E(G_1)}\sum_{u_2,v_2\in V(G_2)}\left|\varepsilon_{G_1}(u_1)-\varepsilon_{G_1}(v_1)\right|+\sum_{u_1\in V(G_1)}\sum_{u_2v_2\in E(G_2)}\left|\varepsilon_{G_1}(u_1)-\varepsilon_{G_1}(u_1)\right|\\ =&{n_2}^2E_3(G_1),\end{aligned}$$

and Eq. (7) holds.

By definition of the non-self-centrality number we obtain

$$\begin{aligned}N(G_1[G_2])=&\frac{1}{2}\sum_{(u_1,u_2),(v_1,v_2)\in V(G_1[G_2])}\left|\varepsilon_{G_1[G_2]}((u_1,u_2))-\varepsilon_{G_1[G_2]}((v_1,v_2))\right|\\ =&\frac{1}{2}\sum_{u_1,v_1\in V(G_1)}\sum_{u_2,v_2\in V(G_2)}\left|\varepsilon_{G_1}(u_1)-\varepsilon_{G_1}(v_1)\right|\\ =&{n_2}^2N(G_1),\end{aligned}$$

and Eq. (8) holds. □

3.5. Strong product

The strong product $G_1\boxtimes G_2$ of graphs G_1 and G_2 is a graph with vertex set $V(G_1)\times V(G_2)$ and two vertices (u_1,u_2) and (v_1,v_2) are adjacent if and only if $[u_1=v_1\in V(G_1)$ and $u_2v_2\in E(G_2)]$ or $[u_2=v_2\in V(G_2)$ and $u_1v_1\in E(G_1)]$ or $[u_1v_1\in E(G_1)$ and $u_2v_2\in E(G_2)]$. The eccentricity of a vertex (u_1,u_2) of $G_1\boxtimes G_2$ was given in [27],

$$\varepsilon_{G_1\boxtimes G_2}((u_1,u_2))=\max\{\varepsilon_{G_1}(u_1),\varepsilon_{G_2}(u_2)\}.$$

Theorem 3.6. *Let $r(G_2)\geq d(G_1)$. The third Zagreb eccentricity index and NSC number of $G_1\boxtimes G_2$ are given by*

$$E_3(G_1\boxtimes G_2)=(n_1+2m_1)E_3(G_2), \tag{9}$$

$$N(G_1\boxtimes G_2)={n_1}^2N(G_2). \tag{10}$$

Proof. Under the condition $r(G_2) \geq d(G_1)$, for every vertex $u_1 \in V(G_1)$, $u_2 \in V(G_2)$, $\varepsilon_{G_2}(u_2) \geq \varepsilon_{G_1}(u_1)$, so $\varepsilon_{G_1 \boxtimes G_2}((u_1, u_2)) = \varepsilon_{G_2}(u_2)$. Now by definition of the third Zagreb eccentricity index we obtain

$$\begin{aligned} E_3(G_1 \boxtimes G_2) &= \sum_{(u_1,u_2)(v_1,v_2)\in E(G_1\boxtimes G_2)} \left|\varepsilon_{G_1\boxtimes G_2}((u_1,u_2)) - \varepsilon_{G_1\boxtimes G_2}((v_1,v_2))\right| \\ &= \sum_{u_1\in V(G_1)} \sum_{u_2v_2\in E(G_2)} \left|\varepsilon_{G_2}(u_2) - \varepsilon_{G_2}(v_2)\right| + \sum_{u_2\in V(G_2)} \sum_{u_1v_1\in E(G_1)} \left|\varepsilon_{G_2}(u_2) - \varepsilon_{G_2}(u_2)\right| \\ &\quad + \sum_{u_1v_1\in E(G_1)} \sum_{u_2v_2\in E(G_2)} \left[\left|\varepsilon_{G_2}(u_2) - \varepsilon_{G_2}(v_2)\right| + \left|\varepsilon_{G_2}(v_2) - \varepsilon_{G_2}(u_2)\right|\right] \\ &=(n_1 + 2m_1)E_3(G_2), \end{aligned}$$

and Eq. (9) holds.

By definition of the NSC number we obtain

$$\begin{aligned} N(G_1 \boxtimes G_2) &=\frac{1}{2} \sum_{(u_1,u_2),(v_1,v_2)\in V(G_1\boxtimes G_2)} \left|\varepsilon_{G_1\boxtimes G_2}((u_1,u_2)) - \varepsilon_{G_1\boxtimes G_2}((v_1,v_2))\right| \\ &=\frac{1}{2} \sum_{u_1,v_1\in V(G_1)} \sum_{u_2,v_2\in V(G_2)} \left|\varepsilon_{G_2}(u_2) - \varepsilon_{G_2}(v_2)\right| \\ &={n_1}^2 N(G_2), \end{aligned}$$

and Eq. (10) holds. □

3.6. Generalized hierarchical product

Let $\phi \neq U \subseteq V(G_1)$. The generalized hierarchical product $G_1(U) \sqcap G_2$ of graphs G_1 and G_2 is a graph with vertex set $V(G_1) \times V(G_2)$ and vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $[u_1 = v_1 \in U$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \in V(G_2)$ and $u_1v_1 \in E(G_1)]$.

For $\phi \neq U \subseteq V(G)$, a path between vertices $u, v \in V(G)$ through U is a uv-path in G containing some vertex $z \in U$ (vertex z could be the vertex u or vertex v). The distance between u and v through U, denoted by $d_{G(U)}(u, v)$, is the length of a shortest path between u and v through U. Note that if one of the vertices u and v belongs to U, then $d_{G(U)}(u, v) = d_G(u, v)$. For $u \in V(G)$, we define

$$\varepsilon_{G(U)}(u) = \max_{v\in V(G)} d_{G(U)}(u, v).$$

For notational convenience, we introduce the invariants $E_3(G(U))$ and $N(G(U))$ as follows:

$$\begin{aligned} E_3(G(U)) &= \sum_{uv\in E(G)} \left|\varepsilon_{G(U)}(u) - \varepsilon_{G(U)}(v)\right|, \\ N(G(U)) &=\frac{1}{2} \sum_{u,v\in V(G)} \left|\varepsilon_{G(U)}(u) - \varepsilon_{G(U)}(v)\right|. \end{aligned}$$

The eccentricity of a vertex (u_1, u_2) in $G_1(U) \sqcap G_2$ was given in [12],

$$\varepsilon_{G_1(U)\sqcap G_2}((u_1, u_2)) = \varepsilon_{G_1(U)}(u_1) + \varepsilon_{G_2}(u_2).$$

Theorem 3.7. *Let $\phi \neq U \subseteq V(G_1)$. The third Zagreb eccentricity index and NSC number of $G_1(U) \sqcap G_2$ are given by*

$$E_3(G_1(U) \sqcap G_2) = n_2 E_3(G_1(U)) + |U|\, E_3(G_2), \tag{11}$$

$$N(G_1(U) \sqcap G_2) \leq {n_2}^2 N(G_1(U)) + {n_1}^2 N(G_2). \tag{12}$$

Equality holds in (12) if and only if for every $u_1, v_1 \in V(G_1)$, $\varepsilon_{G_1(U)}(u_1) = \varepsilon_{G_1(U)}(v_1)$ or G_2 is a self-centered graph.

Proof. By definition of the third Zagreb eccentricity index we obtain

$$\begin{aligned}E_3(G_1(U)\sqcap G_2) =& \sum_{(u_1,u_2)(v_1,v_2)\in E(G_1(U)\sqcap G_2)} \left|\varepsilon_{G_1(U)\sqcap G_2}((u_1,u_2)) - \varepsilon_{G_1(U)\sqcap G_2}((v_1,v_2))\right| \\ =& \sum_{u_1v_1\in E(G_1)}\sum_{u_2\in V(G_2)} \left|\left(\varepsilon_{G_1(U)}(u_1)+\varepsilon_{G_2}(u_2)\right) - \left(\varepsilon_{G_1(U)}(v_1)+\varepsilon_{G_2}(u_2)\right)\right| \\ &+ \sum_{u_1\in U}\sum_{u_2v_2\in E(G_2)} \left|\left(\varepsilon_{G_1(U)}(u_1)+\varepsilon_{G_2}(u_2)\right) - \left(\varepsilon_{G_1(U)}(u_1)+\varepsilon_{G_2}(v_2)\right)\right| \\ =& \sum_{u_1v_1\in E(G_1)}\sum_{u_2\in V(G_2)} \left|\varepsilon_{G_1(U)}(u_1)-\varepsilon_{G_1(U)}(v_1)\right| + \sum_{u_1\in U}\sum_{u_2v_2\in E(G_2)} \left|\varepsilon_{G_2}(u_2)-\varepsilon_{G_2}(v_2)\right| \\ =& n_2E_3(G_1(U)) + |U|\,E_3(G_2),\end{aligned}$$

and Eq. (11) holds.

By definition of the non-self-centrality number we obtain

$$\begin{aligned}N(G_1(U)\sqcap G_2) =& \frac{1}{2}\sum_{(u_1,u_2),(v_1,v_2)\in V(G_1(U)\sqcap G_2)} \left|\varepsilon_{G_1(U)\sqcap G_2}((u_1,u_2)) - \varepsilon_{G_1(U)\sqcap G_2}((v_1,v_2))\right| \\ =& \frac{1}{2}\sum_{u_1,v_1\in V(G_1)}\sum_{u_2,v_2\in V(G_2)} \left|\left(\varepsilon_{G_1(U)}(u_1)+\varepsilon_{G_2}(u_2)\right) - \left(\varepsilon_{G_1(U)}(v_1)+\varepsilon_{G_2}(v_2)\right)\right| \\ =& \frac{1}{2}\sum_{u_1,v_1\in V(G_1)}\sum_{u_2,v_2\in V(G_2)} \left|\left(\varepsilon_{G_1(U)}(u_1)-\varepsilon_{G_1(U)}(v_1)\right) + \left(\varepsilon_{G_2}(u_2)-\varepsilon_{G_2}(v_2)\right)\right|.\end{aligned}$$

Now by triangle inequality we obtain

$$\begin{aligned}N(G_1(U)\sqcap G_2) \leq& \frac{1}{2}\sum_{u_1,v_1\in V(G_1)}\sum_{u_2,v_2\in V(G_2)} \left[\left|\varepsilon_{G_1(U)}(u_1)-\varepsilon_{G_1(U)}(v_1)\right| + \left|\varepsilon_{G_2}(u_2)-\varepsilon_{G_2}(v_2)\right|\right] \\ =& {n_2}^2N(G_1(U)) + {n_1}^2N(G_2),\end{aligned}$$

and Eq. (12) holds. By triangle inequality, the equality holds in (12) if and only if for every $u_1, v_1 \in V(G_1)$, $u_2, v_2 \in V(G_2)$, $\varepsilon_{G_1(U)}(u_1) - \varepsilon_{G_1(U)}(v_1), \varepsilon_{G_2}(u_2) - \varepsilon_{G_2}(v_2) > 0$ or $\varepsilon_{G_1(U)}(u_1) - \varepsilon_{G_1(U)}(v_1), \varepsilon_{G_2}(u_2) - \varepsilon_{G_2}(v_2) < 0$ or $\varepsilon_{G_1(U)}(u_1) - \varepsilon_{G_1(U)}(v_1) = 0$ or $\varepsilon_{G_2}(u_2) - \varepsilon_{G_2}(v_2) = 0$. If the first two cases occur, then for every $u_2, v_2 \in V(G_2)$, $\varepsilon_{G_2}(u_2) - \varepsilon_{G_2}(v_2), \varepsilon_{G_2}(v_2) - \varepsilon_{G_2}(u_2) > 0$ or $\varepsilon_{G_2}(u_2) - \varepsilon_{G_2}(v_2), \varepsilon_{G_2}(v_2) - \varepsilon_{G_2}(u_2) < 0$, which is a contradiction. So the equality holds in (12) if and only if for every $u_1, v_1 \in V(G_1)$, $u_2, v_2 \in V(G_2)$, $\varepsilon_{G_1(U)}(u_1) - \varepsilon_{G_1(U)}(v_1) = 0$ or $\varepsilon_{G_2}(u_2) - \varepsilon_{G_2}(v_2) = 0$, which implies that for every $u_1, v_1 \in V(G_1)$, $\varepsilon_{G_1(U)}(u_1) = \varepsilon_{G_1(U)}(v_1)$ or G_2 is a self-centered graph. □

4. Applications and corollaries

In this section, we apply the results obtained in Subsection 3.6 to compute the third Zagreb eccentricity index and NSC number of three other graph operations namely Cartesian product, rooted product, and corona product. As stated in Section 3, the component G_i of each graph operation is considered to be a simple connected graph with n_i vertices, where $i = 1, 2$.

4.1. Cartesian product

The Cartesian product $G_1 \square G_2$ of graphs G_1 and G_2 is a graph with vertex set $V(G_1) \times V(G_2)$ and vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $[u_1 = v_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \in V(G_2)$ and $u_1v_1 \in E(G_1)]$. The definition generalizes to the case of $k \geq 3$ graphs in a straightforward manner. Note that $G_1 \square G_2 \cong G_1(U) \sqcap G_2$, where $U = V(G_1)$ and by Theorem 3.7 we get the following corollary.

Corollary 4.1. *The third Zagreb eccentricity index and NSC number of $G_1\square G_2$ are given by*

$$E_3(G_1\square G_2) = n_2E_3(G_1) + n_1E_3(G_2), \tag{13}$$

$$N(G_1\square G_2) \leq {n_2}^2N(G_1) + {n_1}^2N(G_2). \tag{14}$$

The equality holds in (14) if and only if G_1 or G_2 is a self-centered graph.

Proof. If $U = V(G_1)$, then for each pair of vertices $u_1, v_1 \in V(G_1)$, $d_{G_1(U)}(u_1, v_1) = d_{G_1}(u_1, v_1)$. So for each vertex $u_1 \in V(G_1)$,

$$\varepsilon_{G_1(U)}(u_1) = \max_{v_1\in V(G_1)} d_{G_1(U)}(u_1, v_1) = \max_{v_1\in V(G_1)} d_{G_1}(u_1, v_1) = \varepsilon_{G_1}(u_1).$$

Hence $E_3(G_1(U)) = E_3(G_1)$ and $N(G_1(U)) = N(G_1)$. Now by Theorem 3.7,

$$E_3(G_1\square G_2) = E_3(G_1(V(G_1)) \sqcap G_2) = n_2E_3(G_1) + n_1E_3(G_2),$$

$$N(G_1\square G_2) = N(G_1(V(G_1)) \sqcap G_2) \leq {n_2}^2N(G_1) + {n_1}^2N(G_2),$$

and Eqs. (13), (14) hold. By Theorem 3.7, the equality holds in (14) if and only if G_1 or G_2 is a self-centered graph. □

It is easy to see that,

$$E_3(P_n) = \begin{cases} n-1 & if\ n\ is\ odd, \\ n-2 & if\ n\ is\ even, \end{cases} \quad N(P_n) = \begin{cases} \frac{n^3-n}{12} & if\ n\ is\ odd, \\ \frac{n^3-4n}{12} & if\ n\ is\ even. \end{cases}$$

Using these results and Corollary 4.1 we get the following corollary.

Corollary 4.2. *The third Zagreb eccentricity index and NSC number of the rectangular lattice $P_n\square P_m$ are given by*

$$E_3(P_n\square P_m) = \begin{cases} m(n-1) + n(m-1) & if\ n, m\ are\ odd, \\ m(n-2) + n(m-2) & if\ n, m\ are\ even, \\ m(n-1) + n(m-2) & if\ n\ is\ odd,\ m\ is\ even, \end{cases} \tag{15}$$

$$N(P_n\square P_m) \leq \begin{cases} \frac{1}{12}[n^2(m^3-m) + m^2(n^3-n)] & if\ n, m\ are\ odd, \\ \frac{1}{12}[n^2(m^3-4m) + m^2(n^3-4n)] & if\ n, m\ are\ even, \\ \frac{1}{12}[n^2(m^3-4m) + m^2(n^3-n)] & if\ n\ is\ odd,\ m\ is\ even. \end{cases} \tag{16}$$

The equality holds in (16) if and only if $n = 2$ or $m = 2$.

Now, we tackle the case when the number of components in Cartesian product is $k \geq 2$.

Corollary 4.3. *The third Zagreb eccentricity index and NSC number of $G_1\square G_2\square ...\square G_k$ are given by*

$$E_3(G_1\square G_2\square ...\square G_k) = n_1n_2...n_k \sum_{i=1}^{k} \frac{E_3(G_i)}{n_i}, \tag{17}$$

$$N(G_1\square G_2\square ...\square G_k) \leq {n_1}^2{n_2}^2...{n_k}^2 \sum_{i=1}^{k} \frac{N(G_i)}{{n_i}^2}. \tag{18}$$

The equality holds in (18) if and only if at most one of the graphs $G_1, G_2, ..., G_k$ is non-self-centered.

Proof. Using Corollary 4.1 we obtain

$$E_3(G_1\square G_2\square ...\square G_k) = E_3\Big((G_1\square G_2\square ...\square G_{k-1})\square G_k\Big) = n_k E_3(G_1\square G_2\square ...\square G_{k-1}) + n_1...n_{k-1}E_3(G_k).$$

Now by induction on k we obtain

$$E_3(G_1\square G_2\square ...\square G_k) = n_1...n_{k-1}n_k \sum_{i=1}^{k-1}\frac{E_3(G_i)}{n_i} + n_1...n_{k-1}E_3(G_k) = n_1n_2...n_k\sum_{i=1}^{k}\frac{E_3(G_i)}{n_i},$$

and Eq. (17) holds. Similarly, by Corollary 4.1,

$$N(G_1\square G_2\square ...\square G_k) = N\Big((G_1\square G_2\square ...\square G_{k-1})\square G_k\Big) \le {n_k}^2 N(G_1\square G_2\square ...\square G_{k-1}) + {n_1}^2{n_2}^2...{n_{k-1}}^2 N(G_k), \tag{19}$$

with equality if and only if $G_1\square G_2\square ...\square G_{k-1}$ or G_k is a self-centered graph. By induction on k we obtain

$$N(G_1\square G_2\square ...\square G_{k-1}) \le {n_1}^2{n_2}^2...{n_{k-1}}^2\sum_{i=1}^{k-1}\frac{N(G_i)}{{n_i}^2}, \tag{20}$$

with equality if and only if at most one of the graphs $G_1, G_2, ..., G_{k-1}$ is non-self-centered.

From Eqs. (19) and (20) we obtain

$$N(G_1\square G_2\square ...\square G_k) \le {n_1}^2{n_2}^2...{n_{k-1}}^2{n_k}^2\sum_{i=1}^{k-1}\frac{N(G_i)}{{n_i}^2} + {n_1}^2{n_2}^2...{n_{k-1}}^2{n_k}^2\frac{N(G_k)}{{n_k}^2} = {n_1}^2{n_2}^2...{n_k}^2\sum_{i=1}^{k}\frac{N(G_i)}{{n_i}^2},$$

and Eq. (18) holds. Equality holds in (18) if and only if the equality in (19) and (20) holds, which implies that at most one of the graphs $G_1, G_2, ..., G_k$ is non-self-centered. □

4.2. Rooted product

The rooted product $G_1\{G_2\}$ of a graph G_1 and a non-trivial rooted graph G_2 is the graph obtained by taking one copy of G_1 and n_1 copies of G_2, and by identifying the root vertex of the ith copy of G_2 with the ith vertex of G_1, for $i = 1, 2, \ldots, n_1$. The rooted product is also known as the cluster product. Let w be a root vertex of G_2. Note that if $U = \{w\} \subset V(G_2)$, then $G_1\{G_2\} \cong G_2(U) \sqcap G_1 \cong G_2(\{w\}) \sqcap G_1$ and by Theorem 3.7 we get the following corollary.

Corollary 4.4. *Let G_2 be a non-trivial rooted graph and let w denote its root vertex. The third Zagreb eccentricity index and NSC number of $G_1\{G_2\}$ are given by*

$$E_3(G_1\{G_2\}) = E_3(G_1) + n_1\sum_{uv\in E(G_2)}\left|d_{G_2}(u,w) - d_{G_2}(v,w)\right|, \tag{21}$$

$$N(G_1\{G_2\}) \le {n_2}^2 N(G_1) + \frac{{n_1}^2}{2}\sum_{u,v\in V(G_2)}\left|d_{G_2}(u,w) - d_{G_2}(v,w)\right|. \tag{22}$$

The equality holds in (22) if and only if G_1 is a self-centered graph.

Proof. For each vertex $u \in V(G_2)$,

$$\varepsilon_{G_2(\{w\})}(u) = \max_{v\in V(G_2)} d_{G_2(\{w\})}(u,v) = \max_{v\in V(G_2)}\Big(d_{G_2}(u,w) + d_{G_2}(w,v)\Big) = d_{G_2}(u,w) + \varepsilon_{G_2}(w). \tag{23}$$

Hence,

$$E_3(G_2(\{w\})) = \sum_{uv\in E(G_2)}\left|\varepsilon_{G_2(\{w\})}(u) - \varepsilon_{G_2(\{w\})}(v)\right| = \sum_{uv\in E(G_2)}\left|\Big(d_{G_2}(u,w) + \varepsilon_{G_2}(w)\Big) - \Big(d_{G_2}(v,w) + \varepsilon_{G_2}(w)\Big)\right|$$
$$= \sum_{uv\in E(G_2)}\left|d_{G_2}(u,w) - d_{G_2}(v,w)\right|.$$

Similarly,

$$N(G_2(\{w\})) = \frac{1}{2} \sum_{u,v\in V(G_2)} \left| d_{G_2}(u,w) - d_{G_2}(v,w) \right|.$$

Now by Theorem 3.7,

$$E_3(G_1\{G_2\}) = E_3(G_2(\{w\}) \sqcap G_1) = E_3(G_1) + n_1 E_3(G_2(\{w\})) = E_3(G_1) + n_1 \sum_{uv\in E(G_2)} \left| d_{G_2}(u,w) - d_{G_2}(v,w) \right|,$$

and Eq. (21) holds. Similarly,

$$\begin{aligned} N(G_1\{G_2\}) =& N(G_2(\{w\}) \sqcap G_1) \leq {n_2}^2 N(G_1) + {n_1}^2 N(G_2(\{w\})) \\ =& {n_2}^2 N(G_1) + \frac{{n_1}^2}{2} \sum_{u,v\in V(G_2)} \left| d_{G_2}(u,w) - d_{G_2}(v,w) \right|, \end{aligned}$$

and Eq. (22) holds. By Theorem 3.7, the equality holds in (22) if and only if for every $u, v \in V(G_2)$, $\varepsilon_{G_2(\{w\})}(u) = \varepsilon_{G_2(\{w\})}(v)$ or G_1 is a self-centered graph. By Eq. (23), the first case occurs if and only if for every $u, v \in V(G_2)$, $d_{G_2}(u,w) = d_{G_2}(v,w)$. This implies that, for every $u \in V(G_2)$, $d_{G_2}(u,w) = d_{G_2}(w,w) = 0$, which is in contradiction to the fact that G_2 is non-trivial. Hence, equality holds in (22) if and only if G_1 is a self-centered graph. □

4.3. Corona product

The corona product $G_1 \circ G_2$ is the graph obtained by taking one copy of G_1 and n_1 copies of G_2, and by joining each vertex of the ith copy of G_2 to the ith vertex of G_1, for $i = 1, 2, \ldots, n_1$. Note that $G_1 \circ G_2 \cong G_1\{K_1 + G_2\}$, where the root vertex of $K_1 + G_2$ is at the single vertex of K_1. Now by Corollary 4.4 we get the following corollary.

Corollary 4.5. *The third Zagreb eccentricity index and NSC number of $G_1 \circ G_2$ are given by*

$$E_3(G_1 \circ G_2) = E_3(G_1) + n_1 n_2, \tag{24}$$

$$N(G_1 \circ G_2) \leq (n_2 + 1)^2 N(G_1) + {n_1}^2 n_2. \tag{25}$$

The equality holds in (25) if and only if G_1 is a self-centered graph.

Proof. Let us denote the single vertex of K_1 by w. By Corollary 4.4,

$$\begin{aligned} E_3(G_1 \circ G_2) =& E_3(G_1\{K_1 + G_2\}) = E_3(G_1) + n_1 \sum_{uv\in E(K_1+G_2)} \left| d_{K_1+G_2}(u,w) - d_{K_1+G_2}(v,w) \right| \\ =& E_3(G_1) + n_1 \Big[\sum_{uv\in E(G_2)} |1-1| + \sum_{u\in V(G_2)} |1-0| \Big] \\ =& E_3(G_1) + n_1 n_2, \end{aligned}$$

and Eq. (24) holds. Similarly,

$$\begin{aligned} N(G_1 \circ G_2) =& N(G_1\{K_1 + G_2\}) \leq (n_2+1)^2 N(G_1) + \frac{n_1^2}{2} \sum_{u,v\in V(K_1+G_2)} \left| d_{K_1+G_2}(u,w) - d_{K_1+G_2}(v,w) \right| \\ =& (n_2+1)^2 N(G_1) + \frac{n_1^2}{2} \Big[\sum_{u,v\in V(G_2)} |1-1| + \sum_{u=w, v\in V(G_2)} |0-1| + \sum_{u\in V(G_2), v=w} |1-0| \Big] \\ =& (n_2+1)^2 N(G_1) + {n_1}^2 n_2, \end{aligned}$$

and Eq. (25) holds. By Corollary 4.4, the equality holds in (25) if and only if G_1 is a self-centered graph. □

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