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Shells, Sequences and Intersections

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Abstract. In this paper we consider two facts concerning shells. First, we deal with ”nested” (decreasing or increasing) sequences of shells. We prove that the intersection, as well as the closure of the union of these sequences, is a shell. Secondly, we consider some questions raised in a paper by Stiles on shells, published half century ago. He left open some questions, also connected with ”spheres” (boundaries of balls), and with a finite intersection property. Here we give a new result on these problems.

1. Introduction

Let X be a Banach space over the real field R.

Given a nonempty set A, we denote by $cl(A)$ its closure and (if A is bounded) by $\delta(A)$ its diameter.

We write: $B(c,\rho) = \{x \in X : \|x - c\| \leq \rho\}$ (closed ball with center c and radius ρ).

A *sphere* (of radius $r \geq 0$ and center $c \in X$) is a set of this type:

$S(c,r) = \{x \in X : \|x - c\| = r\} \;\; c \in X; \;\; r \geq 0.$

We call *shell* a set of the form

$\{x \in X : \; r_c^- \leq \|x - c\| \leq r_c^+\}$ for some $c \in X$ and $\;0 \leq r_c^- \leq r_c^+;$

c will be the *center*, which is clearly unique. Shells are closed, bounded but they are convex only if $r_c^- = 0$ (c is the center); in this case the shell is a ball.
These sets were studied in [6]. Note that in two-dimensional spaces, a shell is usually called an annulus (or a ring). The term *shell*, in the literature, is also used to denote other sets (for example, the spheres).

In Section 2 we consider ”nested” (increasing or decreasing) sequences of shells, and we show that the ”limit” preserves the property.

In Section 3 we extend a result proved in [6] concerning shells, and an intersection property considered in that paper.

Finally, Section 4 contains some comments and remarks.

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2. Monotone sequences of shells

In this section we consider decreasing or increasing sequences of shells. Here increasing and decreasing are understood in a "weak" sense (not strictly). It is well known that decreasing sequences of balls have a nonempty intersection which is a ball. Also, for an increasing, bounded sequence of balls the closure of their union is a ball (see [5, Proposition 7]). Here we shall extend these results to the wider class of shells, indicating in details how monotone sequences of shells behave.

We prove first an auxiliary result.

Lemma 2.1. *Let S, S' two shells, with centers c, c', such that $S \subseteq S'$. Then we have:*
(i) $r_c^+ + \|c - c'\| \leq r_{c'}^+$;
(ii) $r_c^+ - r_c^- \leq r_{c'}^+ - r_{c'}^-$.

Proof. First assertion: note that S' will contain all points at a distance r_c^+ from c: this proves (i).
Now we prove (ii). All points at distance r_c^- from c belong to $S \subseteq S'$. Thus $\|c - c'\| + r_c^- \geq r_{c'}^-$. By using this inequality and (i) we obtain:
$r_{c'}^- - r_c^- \leq \|c - c'\| \leq r_{c'}^+ - r_c^+$.
This implies (ii). □

Remark 2.2. *In general $S \subseteq S'$ (c, c' being their centers) does not imply $r_c^- \geq r_{c'}^-$: S' can be a shell containing $B(c, r_c^+)$; in that case:*
(1) $r_{c'}^+ - r_{c'}^- \geq 2r_c^+$.

2.1. Increasing sequences of shells

We start with a lemma.

Lemma 2.3. *Consider an increasing sequence of shells:*
$S_n = \{x \in X : r_{c_n}^- \leq \|x - c_n\| \leq r_{c_n}^+\}$,
with $\bigcup_{n=1}^{\infty} S_n$ bounded. Then the sequences $(c_n)_{n\in N}$, $(r_{c_n}^+)_{n\in N}$ and $(r_{c_n}^-)_{n\in N}$ converge.

Proof. Let $(S_n)_{n\in N}$ as indicated; c_n is the center of S_n, c_{n+k} is the center of S_{n+k} ($k \in N$). By using (i) of the previous lemma, we have:
(2) $r_{c_{n+k}}^+ \geq r_{c_n}^+ + \|c_n - c_{n+k}\|$.
Set $S = cl(\bigcup_{n=1}^{\infty} S_n)$. Recall that for an increasing sequence $(S_n)_{n\in n}$, we have (see [5, Proposition 4]):
(3) $\delta(S) = \lim_{n\to\infty} \delta(S_n)$.
By using (3), we have $\lim_{n\to\infty} r_{c_n}^+ = \lim_{n\to\infty} \frac{\delta(S_n)}{2} = \frac{\delta(S)}{2}$.
Since $(r_{c_n}^+)_{n\in N}$ has the (finite) limit $r^+ = \frac{\delta(S)}{2}$, it is a Cauchy sequence. Thus, by (2) also $(c_n)_{n\in N}$ is Cauchy; so it has a limit, say c.
The sequence $(r_{c_n}^+ - r_{c_n}^-)_{n\in N}$ is increasing (see (ii) of Lemma 2.1), so it has a (finite) limit. Since $(r_{c_n}^+)_{n\in N}$ has a limit, this means that also $(r_{c_n}^-)_{n\in N}$ has a limit, say r^-. □

Remark 2.4. *According to Remark 2.2, we cannot say that, under the assumption done in the previous lemma (increasing sequence), the sequence $(r_{c_n}^-)_{n\in N}$ is decreasing. But $r_{c_n}^- < r_{c_{n+1}}^-$ would then imply, by (1), $r_{c_{n+1}}^+ \geq 2r_{c_n}^+ + r_{c_{n+1}}^- \geq 2r_{c_n}^+$; so the boundedness of S implies that this can happen only for at most finitely many n. Thus the sequence $(r_{c_n}^-)_{n\in N}$ is definitively monotone, which again implies the existence of the limit for such sequence.*

Theorem 2.5. *If* $(S_n)_{n\in N}$ *is an increasing sequence of shells and* $\bigcup_{n=1}^{\infty} S_n$ *is bounded, then* $S = cl(\bigcup_{n=1}^{\infty} S_n)$ *is a shell.*

Proof. Suppose that, for each $n \in N$, $S_n = \{x \in X : r^-_{c_n} \le \|x - c_n\| \le r^+_{c_n}\}$.
According to the previous lemma, there exist c, r^-, r^+ such that

$\lim_{n\to\infty} c_n = c;\ \lim_{n\to\infty} r^+_n = r^+;\ \lim_{n\to\infty} r^-_n = r^-.$

We prove that S is the shell:

$S_0 = \{x \in X : r^- \le \|x - c\| \le r^+\}$.

First we prove that $S \subseteq S_0$.

Take $s \in S$; for $i \in N$ there exists n_i such that the following are true:
$\|s - s_{n_i}\| < 1/i$ for some $s_{n_i} \in S_{n_i}$; $\|c - c_{n_i}\| < 1/i$; $|r^- - r^-_{c_{n_i}}| < 1/i$; $|r^+ - r^+_{c_{n_i}}| < 1/i$.

We have: $r^-_{c_{n_i}} \le \|s_{n_i} - c_{n_i}\| \le r^+_{c_{n_i}}$. Therefore $\|s - c\| \le \|s - s_{n_i}\| + \|s_{n_i} - c_{n_i}\| + \|c_{n_i} - c\| < 1/i + r^+_{c_{n_i}} + 1/i < r^+ + 3/i$.
Since $i \in N$ is arbitrary, this shows that $\|s - c\| \le r^+$.

Similarly: let $r^- > 0$: we assume that $3/i < r^-$. Then $\|s-c\| \ge \|s_{n_i}-c_{n_i}\|-\|s-s_{n_i}\|-\|c_{n_i}-c\| > r^-_{c_{n_i}}-2/i > r^- -3/i$.
Since $i \in N$ is arbitrary, this shows that $\|s - c\| \ge r^-$. If instead $r^- = 0$, then the last inequality is trivially true.
Thus $S \subseteq S_0$.

Now we prove the converse.
Let $r^+ > r^-$. Take $i \in N$ such that $2/i < r^+ - r^-$. Consider a point x such that $r^- + 1/i \le \|x - c\| \le r^+ - 1/i$.

We can find n_i such that $\|c_{n_i} - c\| < 1/2i$; $|r^- - r^-_{c_{n_i}}| < 1/2i$; $|r^+ - r^+_{c_{n_i}}| < 1/2i$.

We have $\|x - c_{n_i}\| \ge \|x - c\| - \|c - c_{n_i}\| > r^- + 1/i - 1/2i > r^-_{c_{n_i}}$.

Similarly $\|x - c_{n_i}\| \le \|x - c\| + \|c - c_{n_i}\| < r^+ - 1/i + 1/2i < r^+_{c_{n_i}}$.

The last two things together say that $x \in S_{n_i} \subseteq \bigcup_{n=1}^{\infty} S_n$.

Since this is true for every $i \in N$, this proves that $\{x \in X : r^- < \|x - c\| < r^+\} \subset \bigcup_{n=1}^{\infty} S_n$.
By passing to the closures, we have: $S_0 \subseteq S$.

If instead $r^+ = r^-$, then S_0 reduces to a sphere; moreover, by (ii) of Lemma 2.1, $r^+_{c_n} = r^-_{c_n}$ for all $n \in N$: therefore all S_n are spheres and $S_n \subseteq S_{n+1}$ for all n implies that these spheres coincide with S_0.

This concludes the proof that $S_0 \subseteq S$. □

2.2. **Decreasing sequences of shells**

Theorem 2.6. *If* $(S_n)_{n\in N}$ *is a decreasing sequence of shells, then* $S = \bigcap_{n=1}^{\infty} S_n$ *is a (nonempty) shell.*

Proof. Let $(S_n)_{n\in N}$ be as indicated, with $S_n = \{x \in X : r^-_{c_n} \le \|x - c_n\| \le r^+_{c_n}\}$. According to (i) of Lemma 2.1 we have:

(4) $r^+_{c_{n+k}} \le r^+_{c_n} - \|c_n - c_{n+k}\|$.

Since $(r^+_{c_n})_{n\in N}$ is decreasing, it has a limit, say r^+, and so it is a Cauchy sequence. Thus, by (4) also $(c_n)_{n\in N}$ is Cauchy, so it has a limit, say c.
The sequence $(r^+_{c_n} - r^-_{c_n})_{n\in N}$ is decreasing (see Lemma 2.1 (ii)), and then it has a limit; since $(r^+_{c_n})_{n\in N}$ has a limit, this means that also $(r^-_{c_n})_{n\in N}$ has a limit, say r^-.

Put $S = \bigcap_{n=1}^{\infty} S_n$ and $S_0 = \{x \in X : r^- \le \|x - c\| \le r^+\}$. We will show that $S = S_0$.

$S \subseteq S_0$. Take $i \in N$; there exists n_i such that $\|c - c_{n_i}\| < 1/i$; $|r^- - r^-_{c_{n_i}}| < 1/i$; $|r^+ - r^+_{c_{n_i}}| < 1/i$.

Suppose that $x \in \bigcap_{n=1}^{\infty} S_n \subset S_{n_i}$; then $r^-_{c_{n_i}} \le \|x - c_{n_i}\| \le r^+_{c_{n_i}}$; thus

$\|x - c\| \le \|x - c_{n_i}\| + \|c_{n_i} - c\| < r^+_{c_{n_i}} + 1/i < r^+ + 2/i$.

Also: $\|x - c\| \ge \|x - c_{n_i}\| - \|c_{n_i} - c\| > r^-_{c_{n_i}} - 1/i > r^- - 2/i$.

Since $i \in N$ is arbitrary, the two inequalities show that $x \in S_0$; thus $\bigcap_{n=1}^{\infty} S_n \subseteq S_0$.

Now we prove the reverse inclusion ($S_0 \subseteq S$).

Let $r^+ > r^- \ge 0$.

Suppose that $r^- < \|x - c\| < r^+$. So also for some $i \in N$ we have: $r^- + 1/i < \|x - c\| < r^+ - 1/i$.

There exists n_i such that for all $n > n_i$ we have:

$\|c - c_n\| < 1/2i$; $|r^- - r^-_{c_n}| < 1/2i$; $|r^+ - r^+_{c_n}| < 1/2i$.

Thus, for all $n > n_i$ we have:

$\|x - c_n\| \le \|x - c\| + \|c - c_n\| < r^+ - 1/i + 1/2i < r^+_{c_n}$,

and

$\|x - c_n\| \ge \|x - c\| - \|c - c_n\| > r^- + 1/i - 1/2i > r^-_{c_n}$.

This shows that all points interior to S_0 are in S_n for all $n > n_i$, so for all S_n (which are closed). Passing to the closure, we obtain $S_0 \subseteq \bigcap_{n=1}^{\infty} S_n = S$.

Now let $r^+ = r^-$: call this number r. In this case S_0 is the sphere $\{x \in X : \|x - c\| = r\}$. Assume, by contradiction, that there exists $x \in S_0 \setminus \bigcap_{n=1}^{\infty} S_n$. Let $\bar{n} \in N$ be such that $x \notin S_{\bar{n}}$ and let $\varepsilon \in \left(0, \frac{1}{2}\,\mathrm{distance}(x, S_{\bar{n}})\right)$. The same is true for all $n > \bar{n}$. Therefore, for these n, either $\|x - c_n\| \ge r^+_{c_n} + 2\varepsilon$, or $\|x - c_n\| \le r^-_{c_n} - 2\varepsilon$.

Note that if $r = 0$, then $r^-_{c_n} = 0$ for every $n \in N$ and the last inequality is not possible.

Now choose n large enough so that $\|c - c_n\| < \varepsilon$; $|r^+ - r^+_{c_n}| < \varepsilon$, and (if $r^- > 0$) $|r^- - r^-_{c_n}| < \varepsilon$. Then we obtain $\|x - c_n\| \le \|x - c\| + \|c - c_n\| < r + \varepsilon < r^+_{c_n} + 2\varepsilon$.

Also (if $r^- > 0$) : $\|x - c_n\| \ge \|x - c\| - \|c - c_n\| > r - \varepsilon > r^-_{c_n} - 2\varepsilon$.

In any case we get a contradiction. Thus $S_0 \subseteq S$. This concludes the proof. □

Remark 2.7. *According to Remark 2.2, we cannot say that, for a decreasing sequence of shells, the sequence* $(r^-_{c_n})_{n\in N}$ *is increasing. But according to (1),* $r^-_{c_n} > r^-_{c_{n+1}}$ *would then imply* $2r^+_{c_{n+1}} \le r^+_{c_n} - r^-_{c_n} \le r^+_{c_n}$*. If this happens for infinitely many n, then* $\lim_{n\to\infty} r^+_{c_n} = (1/2)\lim_{n\to\infty} \delta(S_n) = 0$*. In this case, according to Cantor's intersection theorem, the intersection of the sequence is a singleton. Otherwise, if this happens for at most finitely many n, then the sequence* $(r^-_{c_n})_{n\in N}$ *is definitively monotone, so convergent. Our proof of Theorem 2.6 (which also describes the form that the intersection shell has) does not use this fact.*

3. Shells and the finite intersection property

The following definition was given in [6].

We say that a collection of sets has the *finite intersection property*, (fip) for short, if the intersection of the sets in any finite subcollection is not empty.

We say that a class of sets has (FIP) if any collection of sets in that class with (fip) has nonempty intersection. In general a space is said to have FIP if the class of balls has (FIP); recall that this last property has largely been studied (see for example [4]).

The following result was proved in [6]. It says that only in finite dimensional spaces shells have (FIP).

THEOREM. Let X be a normed space; then X is infinite dimensional if and only if there exists a bounded collection of shells with (fip), but whose intersection is empty.

In the same paper, the author wrote: *we do not know if the THEOREM is true when "shells" are replaced by "spheres", or by "spheres of radius one"*. Then he gave a partial positive solution, when X is an inner product space.

As far as we know, no paper concerning this problem appeared later (and [6] received no citation).

We consider the following properties for a space X, $\mathcal{A}$ denoting a family of indexes:

(a) X is infinite dimensional;

(b) there exists a bounded collection $(S_a)_{a\in\mathcal{A}}$ of shells with the (fip), such that $\bigcap_{a\in\mathcal{A}} S_a = \emptyset$;

(c) there exists a bounded collection $(C_a)_{a\in\mathcal{A}}$ of spheres with the (fip), such that $\bigcap_{a\in\mathcal{A}} C_a = \emptyset$;

(d) there exists a bounded collection $(C_a)_{a\in\mathcal{A}}$ of spheres of radius 1, with the (fip), such that $\bigcap_{a\in\mathcal{A}} C_a = \emptyset$.

It is clear that (d) $\Rightarrow$ (c) $\Rightarrow$ (b), while the above THEOREM shows that (a) $\Leftrightarrow$ (b).

More precisely, in [6] it was only proved that (a) $\Rightarrow$ (b); the reverse implication is a direct consequence of the fact that closed and bounded sets in a finite dimensional space are compact.

For simplicity, from now on, we only deal with Banach spaces.

Still in [6], as said, it was shown that in Hilbert spaces (a) $\Rightarrow$ (d), so all the four properties are equivalent in these spaces. Here we shall extend this result to a large class of Banach spaces.

We shall say that a set E in a Banach space is *equilateral* if there exists some $\lambda \in R$ such that $\|x - y\| = \lambda$ for all pairs of points $x, y \in E$.

In recent years there has been an increasing interest on the existence of these sets. The first example of a space where no infinite equilateral set exists was given in [7]. In "many" (infinite dimensional) Banach spaces there exist infinite equilateral sets: for example, it is simple to see that this happens in several classical Banach spaces; these sets exist whenever X is uniformly smooth (see [3]) as well as if X has a "large" density character (see [7]).

Theorem 3.1. *Let X be an infinite dimensional Banach space containing an infinite equilateral set. Then the four conditions (a), (b), (c), and (d) are equivalent.*

Proof. We want to prove that (a) $\Rightarrow$ (d). Let $A = \{x_1, x_2, ..., x_n, ...\}$ be an equilateral set. Eventually after a translation and a rescaling, we can suppose that $x_1 = \theta$ (the origin of X) and that $\|x_i - x_j\| = 1$ for $i, j \in N$; $i \neq j$.

According to Zorn's Lemma, we can embed A in a maximal equilateral set A_m. Then $\{S(x, 1) : x \in A_m\}$ has (fip): in fact any finite subset A_f of A_m is equilateral; so all points in $A \setminus A_f$ belong to the intersection of the spheres $S(x, 1)$, centered at the points in $x \in A_f$. But the intersection of all the spheres is empty: in fact, the existence of a point c in the intersection, would imply that $A_m \cup \{c\}$ is an equilateral set strictly containing A_m. $\square$

Remark 3.2. *We are not saying that the validity of Theorem 3.1 relies on the existence of equilateral sets in X. For example, we can see that the result is valid ((d) holds) in the space constructed in [7]. We conjecture that the equivalence of the four conditions holds in any space.*

4. Concluding remarks

We have indicated some properties of shells, a class of sets studied in [6]. Since shells are not convex, we cannot apply to them, for example, the many results dealing with intersections of (bounded, closed) convex sets. Shells are "centred" sets in the language of [2] (closed sets which are "symmetric" with respect to a "center"). Thus [2, Corollary 1] implies the following result, contained in our Theorem 2.6: decreasing sequences of shells have nonempty intersection if the space does not contain a subspace isomorphic to c_0.

It can also been shown, easily, that the convergence of monotone sequences of shells is also a Hausdorff convergence. This fact (known for balls) is not true in general; for example, for monotone sequences of bounded closed convex sets: see [5, Examples 2 and 6].

We include a couple of simple examples, concerning the problems we have discussed. They are given in c_0, the space of real sequences converging to 0 with the max norm.

Example 4.1. *Consider the following (not nested) sequence of balls, with the same radius 1:* $(B(x_n, 1))_{n\in N}$*, where* $x_n = (2, ..., 2, 0, 0, 0......)$ *(the first* n *components are 2, the remaining are 0). This collection of balls has* (fip)*, but their intersection is empty.*

Example 4.2. *. Consider the following sets (they are centred, bounded, closed and convex):* $C_n = \{x \in X : x_i = 1 \text{ for } i = 1, \dots, n; \ -1 \leq x_i \leq 1 \text{ for } i > n\}$*; the center of* C_n *is* $e_1 + ... + e_n$*. They form a decreasing sequence of sets with* (fip)*, whose intersection is empty.*

Dealing with sets lying on spheres is in general not easy. For example, in the paper [1], facts of the following type are studied (and not completely described): look for spaces where given any finite set $\{x_1, ..., x_n\}$, for some real number λ the spheres $S(x_i, \lambda)$ $(i = 1, ..., n)$ intersect.

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