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On Dimension and Weight of a Local Contact Algebra

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Abstract. As proved in [16], there exists a duality Λ^t between the category **HLC** of locally compact Hausdorff spaces and continuous maps, and the category **DHLC** of complete local contact algebras and appropriate morphisms between them. In this paper, we introduce the notions of weight w_a and of dimension $\dim_a$ of a local contact algebra, and we prove that if X is a locally compact Hausdorff space then $w(X) = w_a(\Lambda^t(X))$, and if, in addition, X is normal, then $\dim(X) = \dim_a(\Lambda^t(X))$.

1. Introduction

According to Stone's famous duality theorem [43], the Boolean algebra $\mathrm{CO}(X)$ of all clopen (= closed and open) subsets of a zero-dimensional compact Hausdorff space X carries the whole information about the space X, i.e. the space X can be reconstructed from $\mathrm{CO}(X)$, up to homeomorphism. It is natural to ask whether the Boolean algebra $\mathrm{RC}(X)$ of all regular closed subsets of a compact Hausdorff space X carries the full information about the space X (see Example 2.5 below for $\mathrm{RC}(X)$). It is well known that the answer is "No". For example, the Boolean algebras of all regular closed subsets of the unit interval $\mathbb{I}$ (with its natural topology) and the absolute $a\mathbb{I}$ of $\mathbb{I}$ (i.e. the Stone dual of $\mathrm{RC}(\mathbb{I})$) are isomorphic but $\mathbb{I}$ and $a\mathbb{I}$ are not homeomorphic because $\mathbb{I}$ is connected and $a\mathbb{I}$ is not (see, e.g., [38] for absolutes). Suppose that **HC** is the category of compact Hausdorff spaces and continuous maps, and that X is a compact Hausdorff space. As shown by H. de Vries [14], all information about the space X is contained in the pair $\langle \mathrm{RC}(X), \rho_X\rangle$, where ρ_X is a binary relation on $\mathrm{RC}(X)$ such that for all $F, G \in \mathrm{RC}(X)$,

$$F\rho_X G \text{ if and only if } F \cap G \neq \emptyset.$$

In order to describe abstractly the pairs $\langle \mathrm{RC}(X), \rho_X\rangle$, he introduced the notion of *compingent Boolean algebra*, and he proved that there exists a duality between the category **HC** and the category **DHC** of complete compingent Boolean algebras and appropriate morphisms between them.

Subsequently, Dimov [16] extended de Vries' duality from the category **HC** to the category **HLC** of locally compact Hausdorff spaces and continuous maps, and, on the base of this result, he also obtained

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an extension of Stone's duality from the category **Stone** of compact zero-dimensional Hausdorff spaces and continuous maps to the category **ZHLC** of zero-dimensional locally compact Hausdorff spaces and continuous maps (see [15, 18]).

The paper [16] has its precursor in results by P. Roeper [40], who showed that all information about a locally compact Hausdorff space X is contained in the triple

$$\langle \mathrm{RC}(X), \rho_X, \mathrm{CR}(X)\rangle,$$

where $\mathrm{CR}(X)$ is the set of all compact regular closed subsets of X. In order to describe abstractly the triples $\langle \mathrm{RC}(X), \rho_X, \mathrm{CR}(X)\rangle$, he introduced the notion of *region-based topology*, and he proved that – up to homeomorphisms, respectively, isomorphisms – there exists a bijection between the class of all locally compact Hausdorff spaces and the class of all complete region-based topologies. The duality theorem proved in [16] says that there exists a duality Λ^t between the category **HLC** and the category **DHLC** of all complete region-based topologies and appropriate morphisms between them. Note that

$$\Lambda^t(X) \stackrel{\mathrm{df}}{=} \langle \mathrm{RC}(X), \rho_X, \mathrm{CR}(X)\rangle,$$

for every locally compact Hausdorff space X. In [19], the dual objects (under the contravariant functor Λ^t) of Euclidean spaces, spheres, tori and Tychonoff cubes are constructed directly (i.e. without the help of the corresponding topological spaces); these algebraical objects completely characterize the mentioned topological spaces.

In [21], the general notion of *Boolean contact algebra* was introduced and, accordingly, "compingent Boolean algebras" were called "normal Boolean contact algebras" (abbreviated as NCAs), and "region-based topologies" were called "local contact Boolean algebras" (abbreviated as LCAs). Typical examples of Boolean contact algebras are the pairs

$$\langle \mathrm{RC}(X), \rho_X\rangle,$$

where X is an arbitrary topological space. We will even use a more general notion, namely, the notion of a *Boolean precontact algebra*, introduced by Düntsch and Vakarelov in [25].

The theory of (local) (pre)contact algebras is a part of *region-based theory of space* which is a kind of point-free geometry and can be considered as an alternative to the well known Euclidean point-based theory of space. Its main idea goes back to Whitehead [47] (see also [46]) and de Laguna [13]. Survey papers describing various aspects and historical remarks on region-based theory of space are [6, 30, 39, 44]. From a Computer Science perspective, (local) (pre)contact algebras are part of *qualitative spatial and temporal reasoning* (which, in turn, is a part of region-based theory of space), an area of artificial intelligence, with applications in geographic information systems, robot navigation, computer aided design, and more. We invite the reader to consult [12], [31] or [48] for details. Let us also mention that region-based theory of space stimulated the appearance of a new area in logic, namely "Spatial Logics" [2], sometimes called "Logics of Space". It could be said that *point-free topology* [32, 37] is also part of the region-based theory of space.

Having a duality Λ^t between the categories **HLC** and **DHLC**, it is natural to look for the algebraic expressions dual to topological properties of locally compact Hausdorff spaces. It is easy to find such an expression for the property of "connectedness" even for arbitrary topological spaces, see [7]. Namely, a Boolean contact algebra $\langle B, C\rangle$ is said to be *connected* if $a \neq 0, 1$ implies that aCa^*; here, a^* is the Boolean complement of a. It was proved in [7] that for a topological space X, the Boolean contact algebra $\langle \mathrm{RC}(X), \rho_X\rangle$ is connected if and only if the space X is connected.

In this paper we introduce the notions of *dimension of a precontact algebra* and *weight of a local contact algebra*, and prove that

1. The weight of a locally compact Hausdorff space X is equal to the weight of the local contact algebra $\Lambda^t(X)$ (Theorem 4.4), and
2. The Čech–Lebesgue dimension of a normal T_1-space X is equal to the dimension of the Boolean contact algebra $\langle \mathrm{RC}(X), \rho_X\rangle$ (Theorem 3.4). In particular, the Čech–Lebesgue dimension of a normal locally compact Hausdorff space X is equal to the dimension of the local contact algebra $\Lambda^t(X)$ (Corollary 3.5).

One cannot define a notion of dimension for Boolean algebras corresponding to the topological notion of dimension via de Vries' or Dimov's dualities because for all positive natural numbers n and m, the Boolean algebras $\mathrm{RC}(\mathbb{R}^n)$ and $\mathrm{RC}(\mathbb{R}^m)$ are isomorphic (see Birkhoff [8, p.177]) but, clearly, for $n \neq m$, $\dim(\mathbb{R}^n) \neq \dim(\mathbb{R}^m)$. Also, one cannot define an adequate (in the same sense) notion of weight for Boolean algebras because, for example, the Boolean algebras $\mathrm{RC}(\mathbb{I})$ and $\mathrm{RC}(a\mathbb{I})$ are isomorphic but $w(\mathbb{I}) = \aleph_0 < 2^{\aleph_0} = w(a\mathbb{I})$ (see [5, Chapter VI, Problem 234(a)]).

The paper is organized as follows. Section 2 contains all preliminary facts and definitions which are used in this paper. In Section 3, we introduce and study the notion of dimension of a precontact algebra. Here we prove Theorem 3.4 and Corollary 3.5, mentioned above. It is shown as well that the dimension of a normal contact algebra is equal to the dimension of its NCA-completion (see [15, 17] for this notion), that the dimension of any NCA of the form $\langle B, \rho_s\rangle$ (where ρ_s is the smallest contact relation on B) is equal to zero (as it should be), and that the dimension of every relative LCA of an LCA $\langle B, \rho, \mathbb{B}\rangle$ is smaller or equal to $\dim_a(\langle B, \rho, \mathbb{B}\rangle)$. Recall that L. Heindorf (cited in [36]) introduced the notion of $\mathcal{A}$-dimension for Boolean algebras, where $\mathcal{A}$ is an arbitrary non-empty class of Boolean algebras. There is, however, no connection between the topological notion of dimension and the notion of $\mathcal{A}$-dimension, so that his investigations are in a different direction from those carried out here. Recall also that M. G. Charalambous [11] introduced and studied a notion of dimension for the (σ-)frames. It corresponds to the so called *localic duality* (see, e.g., [32, Corollary II.1.7]), which is a duality between the category of all spatial frames and all functions between them which preserve finite meets and arbitrary joins and the category of all sober spaces and all continuous maps between them. The dimension of a spatial frame, which is introduced in [11], is equal to the Čech–Lebesgue dimension of its dual sober topological space. Note that the dual object of a sober topological space is its topology regarded as a frame. That is why, the investigations made in [11] and in this paper are completely different.

In Section 4, we introduce and study the notion of weight of a local contact algebra. Here we prove Theorem 4.4, mentioned above. We show as well that the weight of a local contact algebra is equal to the weight of its LCA-completion (see [15, 17] for this notion), find an algebraic analogue of Alexandroff-Urysohn theorem for bases ([27, Theorem 1.1.15]), describe the LCAs whose dual spaces are metrizable, and characterize the LCAs whose dual spaces are zero-dimensional. Furthermore, for a dense Boolean subalgebra A_0 of a Boolean algebra A, we construct an NCA $\langle A, \rho\rangle$ such that $w_a(\langle A, \rho\rangle) = |A_0|$, and if A is complete, then its dual space is homeomorphic to the Stone dual of A_0.

In Section 5, we discuss the relationship between algebraic density and algebraic weight, introduce the notion of a *π-semiregular space*, and show that if X is π-semiregular then $\pi w(X)$ is equal to the density of the Boolean algebra $\mathrm{RC}(X)$. Finally, for every π-semiregular space X with $\pi w(X) \geq \aleph_0$, we prove that there exists a zero-dimensional compact Hausdorff space Y with $w(Y) = \pi w(X)$ such that the Boolean algebras $\mathrm{RC}(X)$ and $\mathrm{RC}(Y)$ are isomorphic.

The results from Sections 4 and 5 are from the arXiv-paper [15].

2. Preliminaries

2.1. Notation and first definitions

Suppose that $\langle P, \leq, 0\rangle$ is a partially ordered set with smallest element 0. If $M \subseteq P$, then $M^+ \stackrel{\text{df}}{=} M \setminus \{0\}$. M is called *dense in P*, if for all $a \in P^+$ there is some $b \in M^+$ such that $b \leq a$.

A *join-semilattice* is a partially ordered set having all finite non-empty joins.

We denote by $\mathbb{N}$ the set of all non-negative integers, by $\mathbb{N}^-$ the set $\mathbb{N} \cup \{-1\}$, by $\mathbb{N}^+$ the set $\mathbb{N} \setminus \{0\}$, by $\mathbb{R}$ the real line (with its natural topology), and by $\mathbb{I}$ the subspace $[0, 1]$ ($\stackrel{\text{df}}{=} \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$) of $\mathbb{R}$.

The power set of a set X is denoted by 2^X; we implicitly suppose that 2^X is a Boolean algebra under the set operations. The cardinality of a set X is denoted, as usual, by $|X|$.

Throughout, $\langle B, \wedge, \vee, {}^*, 0, 1\rangle$ will denote a Boolean algebra unless indicated otherwise; we do not assume that $0 \neq 1$. With some abuse of language, we shall usually identify algebras with their universe, if no confusion can arise.

If B is a Boolean algebra and $b \in B^+$, we let B_b be the relative algebra of B with respect to b [33, Lemma 3.1.].

If $a, b \in B$, then $a \triangle b$ denotes the symmetric difference of a and b, i.e. $a \triangle b \stackrel{\text{df}}{=} (a \wedge b^*) \vee (b \wedge a^*)$. It is well known that $a \triangle b = 0$ if and only if $a = b$.

Throughout, $(X, \mathcal{T})$ will be a topological space. If no confusion can arise, we shall just speak of X. We denote by $\mathrm{CO}(X)$ the set of all clopen (= closed and open) subsets of X; clearly, $\langle \mathrm{CO}(X), \cup, \cap, \setminus, \emptyset, X \rangle$ is a Boolean algebra. A subset F of X is called *regular closed* (resp., *regular open*) if $F = \mathrm{cl}(\mathrm{int}(F))$ (resp., $F = \mathrm{int}(\mathrm{cl}(F))$). We let $\mathrm{RC}(X)$ (resp., $\mathrm{RO}(X)$) be the set of all regular closed (resp., regular open) subsets of X. The space X is called *semiregular* if $\mathrm{RO}(X)$ is an open base for X, or, equivalently, if $\mathrm{RC}(X)$ is a closed base for X.

If $\mathcal{C}$ is a category, we denote by $|\mathcal{C}|$ the class of all objects of the category $\mathcal{C}$, and by $\mathcal{C}(A, B)$ the set of all $\mathcal{C}$-morphisms between the $\mathcal{C}$-objects A and B.

For unexplained notation we invite the reader to consult [33] for Boolean algebras, [1] for category theory, and [27] for topology.

2.2. *Boolean (pre)contact algebras*

In this paper we work mainly with Boolean algebras with supplementary structures on them. In all cases, we will say that the corresponding structured Boolean algebra is *complete* if the underlying Boolean algebra is complete.

Definition 2.1. ([25]) A *Boolean precontact algebra*, or, simply, *precontact algebra* (PCA) (originally, *Boolean proximity algebra* [25]), is a structure $\langle B, C \rangle$, where B is a Boolean algebra, and C a binary relation on B, called a precontact relation, which satisfies the following axioms:

(C1). If aCb then $a \neq 0$ and $b \neq 0$.
(C2). $aC(b \vee c)$ if and only if aCb or aCc; $(a \vee b)Cc$ if and only if aCc or bCc.

Two precontact algebras $\langle B, C \rangle$ and $\langle B_1, C_1 \rangle$ are said to be *PCA-isomorphic* (or, simply, *isomorphic*) if there exists a *PCA-isomorphism* between them, i.e., a Boolean isomorphism $\varphi : B \longrightarrow B_1$ such that, for every $a, b \in B$, aCb iff $\varphi(a) C_1 \varphi(b)$.

The notion of a precontact algebra was defined independently (and in a completely different form) by S. Celani [10]. A duality theorem for precontact algebras was obtained in [20] (see also [22, 23]).

Definition 2.2. A PCA $\langle B, C \rangle$ is called a *Boolean contact algebra* [21] or, briefly, a *contact algebra* (CA), if it satisfies the following additional axioms for all $a, b \in B$:

(C3). If $a \neq 0$ then aCa.
(C4). aCb implies bCa.

The relation C is called a *contact relation*. As usual, if $a \in B$, we set

$$C(a) \stackrel{\text{df}}{=} \{b \in B \mid aCb\}.$$

We shall consider two more properties of contact algebras:

(C5). If $a(-C)b$ then $a(-C)c$ and $b(-C)c^*$ for some $c \in B$.
(C6). If $a \neq 1$ then there exists $b \neq 0$ such that $b(-C)a$.

A contact algebra $\langle B, C \rangle$ is called a *Boolean normal contact algebra* or, briefly, *normal contact algebra* (abbreviated as NCA) [14, 29] if it satisfies (C5) and (C6). The notion of normal contact algebra was introduced by Fedorchuk [29] under the name of *Boolean δ-algebra* as an equivalent expression of the notion of *compingent Boolean algebra* of de Vries (see the definition below). We call such algebras "normal contact algebras" because they form a subclass of the class of contact algebras which naturally arise as canonical algebras in normal Hausdorff spaces (see [21]).

Axiom $(C6)$ is an extensionality axiom since a CA $\langle B, C\rangle$ satisfies $(C6)$ if and only if $(\forall a, b \in B)[C(a) = C(b)$ implies $a = b]$ (see [21, Lemma 2.2]). Keeping this in mind, we call a CA $\langle B, C\rangle$ an *extensional contact algebra* (abbreviated as ECA) if it satisfies $(C6)$. This notion was introduced in [26] under the name of *Boolean contact algebra*, and a representation theorem for ECAs was proved there.

Note that if $0 \neq 1$, then (C1) follows from the axioms (C2), (C4), and $(C6)$.

Definition 2.3. For a PCA $\langle B, C\rangle$, we define a binary relation "$\ll_C$" on B, called *non-tangential inclusion*, by

$$a \ll_C b \text{ if and only if } a(-C)b^*.$$

Here, $-C$ is the set complement of C in $B \times B$. If C is understood, we shall simply write "$\ll$" instead of "$\ll_C$".

The relations C and $\ll$ are inter-definable. For example, normal contact algebras may be equivalently defined – and exactly in this way they were introduced under the name of *compingent Boolean algebras* by de Vries in [14] – as a pair consisting of a Boolean algebra B and a binary relation $\ll$ on B satisfying the following axioms:

($\ll$1). $a \ll b$ implies $a \leq b$.
($\ll$2). $0 \ll 0$.
($\ll$3). $a \leq b \ll c \leq t$ implies $a \ll t$.
($\ll$4). $a \ll c$ and $b \ll c$ implies $a \vee b \ll c$.
($\ll$5). If $a \ll c$ then $a \ll b \ll c$ for some $b \in B$.
($\ll$6). If $a \neq 0$ then there exists $b \neq 0$ such that $b \ll a$.
($\ll$7). $a \ll b$ implies $b^* \ll a^*$.

Indeed, if $\langle B, C\rangle$ is an NCA, then the relation $\ll_C$ satisfies the axioms ($\ll$1) – ($\ll$7). Conversely, having a pair $\langle B, \ll\rangle$, where B is a Boolean algebra and $\ll$ is a binary relation on B which satisfies ($\ll$1) – ($\ll$7), we define a relation $C_\ll$ by $aC_\ll b$ if and only if $a(-\ll)b^*$ (here, $-\ll$ is the set complement of the relation $\ll$ in $B \times B$); then $\langle B, C_\ll\rangle$ is an NCA. Note that the axioms (C5) and (C6) correspond to ($\ll$5) and to ($\ll$6), respectively.

It is easy to see that contact algebras could be equivalently defined as a pair of a Boolean algebra B and a binary relation $\ll$ on B subject to the axioms ($\ll$1) – ($\ll$4) and ($\ll$7); then, clearly, the relation $\ll$ also satisfies the axioms

($\ll$2') $1 \ll 1$;
($\ll$4') ($a \ll c$ and $b \ll c$) implies $(a \vee b) \ll c$.

It is not difficult to see that precontact algebras could be equivalently defined as a pair of a Boolean algebra B and a binary relation $\ll$ on B subject to the axioms ($\ll$2), ($\ll$2'), ($\ll$3), ($\ll$4) and ($\ll$4').

A mapping φ between two contact algebras $\langle B_1, C_1\rangle$ and $\langle B_2, C_2\rangle$ is called a *CA-morphism* ([20]), if $\varphi : B_1 \longrightarrow B_2$ is a Boolean homomorphism, and $\varphi(a)C_2\varphi(b)$ implies aC_1b, for any $a, b \in B_1$. Note that $\varphi : \langle B_1, C_1\rangle \longrightarrow \langle B_2, C_2\rangle$ is a CA-morphism if and only if $a \ll_{C_1} b$ implies $\varphi(a) \ll_{C_2} \varphi(b)$, for any $a, b \in B_1$. (Thus, a CA-morphism is a structure preserving morphism between $\langle B_1, \ll_1\rangle$ and $\langle B_2, \ll_2\rangle$ in the sense of first order logic.) Two CAs $\langle B_1, C_1\rangle$ and $\langle B_2, C_2\rangle$ are *CA-isomorphic* if and only if there exists a bijection $\varphi : B_1 \longrightarrow B_2$ such that φ and φ^{-1} are CA-morphisms.

The following assertion may be worthy of mention:

Proposition 2.4. *If $\langle B_1, C_1\rangle$ and $\langle B_2, C_2\rangle$ are CAs, $\varphi : B_1 \longrightarrow B_2$ is a Boolean homomorphism and φ preserves the contact relation C_1 (i.e., aC_1b implies $\varphi(a)C_2\varphi(b)$, for all $a, b \in B_1$), then φ is an injection.*

Proof. Assume that φ is not injective. Then, there are $a, b \in B_1$ such that $a \neq b$ and $\varphi(a) = \varphi(b)$; hence, $c \stackrel{\text{df}}{=} a \triangle b \neq 0$, and $\varphi(c) = 0$. By (C3), cC_1c, and the fact that φ preserves C_1 implies that $\varphi(c)C_2\varphi(c)$, i.e. $0C_20$. This contradicts (C1). □

The most important "concrete" example of a CA is given by the regular closed sets of an arbitrary topological space.

Example 2.5. Let $(X, \mathcal{T})$ be a topological space. The collection $\mathrm{RC}(X, \mathcal{T})$ becomes a complete Boolean algebra $\langle \mathrm{RC}(X, \mathcal{T}), 0, 1, \wedge, \vee, ^* \rangle$ under the following operations:

$$F \vee G \stackrel{\mathrm{df}}{=} F \cup G, \qquad F \wedge G \stackrel{\mathrm{df}}{=} \mathrm{cl}(\mathrm{int}(F \cap G)), \qquad F^* \stackrel{\mathrm{df}}{=} \mathrm{cl}(X \setminus F), \qquad 0 \stackrel{\mathrm{df}}{=} \emptyset, \qquad 1 \stackrel{\mathrm{df}}{=} X.$$

The infinite operations are given by the formulas

$$\bigvee \{F_\gamma \mid \gamma \in \Gamma\} \stackrel{\mathrm{df}}{=} \mathrm{cl}(\bigcup_{\gamma\in\Gamma} F_\gamma) \ (= \mathrm{cl}(\bigcup_{\gamma\in\Gamma} \mathrm{int}(F_\gamma)) = \mathrm{cl}(\mathrm{int}(\bigcup_{\gamma\in\Gamma} F_\gamma))), \quad \bigwedge \{F_\gamma \mid \gamma \in \Gamma\} \ \stackrel{\mathrm{df}}{=} \mathrm{cl}(\mathrm{int}(\bigcap \{F_\gamma \mid \gamma \in \Gamma\})),$$

Define a relation $\rho_{(X,\mathcal{T})}$ on $\mathrm{RC}(X, \mathcal{T})$ by setting, for each $F, G \in \mathrm{RC}(X, \mathcal{T})$,

$$F \rho_{(X,\mathcal{T})} G \text{ if and only if } F \cap G \neq \emptyset.$$

Clearly, $\rho_{(X,\mathcal{T})}$ is a contact relation, called the *standard contact relation of* $(X, \mathcal{T})$. The complete contact algebra $\langle \mathrm{RC}(X, \mathcal{T}), \rho_{(X,\mathcal{T})} \rangle$ is called a *standard contact algebra*. If no confusion can arise, we shall usually write simply $\mathrm{RC}(X)$ instead of $\mathrm{RC}(X, \mathcal{T})$, and ρ_X instead of $\rho_{(X,\mathcal{T})}$. Note that, for $F, G \in \mathrm{RC}(X)$,

$$F \ll_{\rho_X} G \text{ if and only if } F \subseteq \mathrm{int}_X(G).$$

Thus, if $(X, \mathcal{T})$ is a normal Hausdorff space then the standard contact algebra $\langle \mathrm{RC}(X, \mathcal{T}), \rho_{(X,\mathcal{T})} \rangle$ is a complete NCA.

Instead of looking at regular closed sets, we may, equivalently, consider regular open sets. The collection $\mathrm{RO}(X)$ of regular open sets becomes a complete Boolean algebra by setting

$$U \vee V \stackrel{\mathrm{df}}{=} \mathrm{int}(\mathrm{cl}(U \cup V)), \qquad U \wedge V \stackrel{\mathrm{df}}{=} U \cap V, \qquad U^* \stackrel{\mathrm{df}}{=} \mathrm{int}(X \setminus U), \qquad 0 \stackrel{\mathrm{df}}{=} \emptyset, \qquad 1 \stackrel{\mathrm{df}}{=} X,$$

and

$$\bigwedge_{i\in I} U_i \stackrel{\mathrm{df}}{=} \mathrm{int}(\mathrm{cl}(\bigcap_{i\in I} U_i)) \ (= \mathrm{int}(\bigcap_{i\in I} U_i)), \qquad \bigvee_{i\in I} U_i \stackrel{\mathrm{df}}{=} \mathrm{int}(\mathrm{cl}(\bigcup_{i\in I} U_i)),$$

see [33, Theorem 1.37]. We define a contact relation D_X on $\mathrm{RO}(X)$ as follows:

$$U D_X V \text{ if and only if } \mathrm{cl}(U) \cap \mathrm{cl}(V) \neq \emptyset.$$

Then $\langle \mathrm{RO}(X), D_X \rangle$ is a complete CA.

The contact algebras $\langle \mathrm{RO}(X), D_X \rangle$ and $\langle \mathrm{RC}(X), \rho_X \rangle$ are CA-isomorphic via the mapping $\nu : \mathrm{RO}(X) \longrightarrow \mathrm{RC}(X)$ defined by the formula $\nu(U) \stackrel{\mathrm{df}}{=} \mathrm{cl}(U)$, for every $U \in \mathrm{RO}(X)$.

Example 2.6. Let B be a Boolean algebra. Then there exist a largest and a smallest contact relations on B; the largest one, ρ_l^B, is defined by

$$a \rho_l^B b \iff (a \neq 0 \text{ and } b \neq 0),$$

and the smallest one, ρ_s^B, by

$$a \rho_s^B b \iff a \wedge b \neq 0.$$

When there is no ambiguity, we will simply write ρ_s instead of ρ_s^B, and ρ_l instead of ρ_l^B.

Note that, for $a, b \in B$,

$$a \ll_{\rho_s} b \iff a \leq b;$$

hence $a \ll_{\rho_s} a$, for any $a \in B$. Thus (B, ρ_s) is a normal contact algebra.

2.3. Local contact algebras

Local contact algebras were introduced by Roeper [40] under the somewhat misleading name *region-based topologies*. Since every region-based topology is a contact algebra and also a lattice-theoretical counterpart of Leader's notion of *local proximity* [34], it was suggested in [21] to rename them to *Boolean local contact algebras*.

Definition 2.7. [40] A system $\langle B, \rho, \mathbb{B}\rangle$ is called a *Boolean local contact algebra* or, briefly, *local contact algebra* (abbreviated as LCA or as LC-algebra) if B is a Boolean algebra, ρ is a contact relation on B, and $\mathbb{B}$ is a not necessarily proper ideal of B satisfying the following axioms:

(LC1). If $a \in \mathbb{B}$, $c \in B$ and $a \ll_\rho c$ then $a \ll_\rho b \ll_\rho c$ for some $b \in \mathbb{B}$.
(LC2). If $a\rho b$ then there exists an element c of $\mathbb{B}$ such that $a\rho(c \wedge b)$.
(LC3). If $a \neq 0$ then there exists some $b \in \mathbb{B}^+$ such that $b \ll_\rho a$.

The elements of $\mathbb{B}$ are called *bounded*, and the elements of $B \setminus \mathbb{B}$ are called *unbounded*.

It may be worthy to note that it follows from a result by M. Rubin [41], that the first order theory of LCAs is undecidable.

Two local contact algebras $\langle B, \rho, \mathbb{B}\rangle$ and $\langle B_1, \rho_1, \mathbb{B}_1\rangle)$ are *LCA-isomorphic* if there exists a CA-isomorphism $\varphi : \langle B, \rho\rangle \longrightarrow \langle B_1, \rho_1\rangle$ such that, for any $a \in B$, $\varphi(a) \in \mathbb{B}_1$ if and only if $a \in \mathbb{B}$.

A map $\varphi : \langle B, \rho, \mathbb{B}\rangle \longrightarrow \langle B_1, \rho_1, \mathbb{B}_1\rangle$ is called an *LCA-embedding* if $\varphi : \langle B, \rho\rangle \longrightarrow \langle B_1, \rho_1\rangle$ is a CA–morphism such that for any $a, b \in B$, $a\rho b$ implies $\varphi(a)\rho_1\varphi(b)$, and $\varphi(a) \in \mathbb{B}_1$ if and only if $a \in \mathbb{B}$. Note that the name is justified, since, as it follows from Proposition 2.4, any LCA–embedding is an injection.

If $\langle B, \rho, \mathbb{B}\rangle$ is a local contact algebra and $\mathbb{B} = B$, i.e., $\mathbb{B}$ is an improper ideal, then $\langle B, \rho\rangle$ is a normal contact algebra. Conversely, any normal contact algebra $\langle B, C\rangle$ can be regarded as a local contact algebra of the form $\langle B, C, B\rangle$.

Proposition 2.8. [40, 45] *Let X be a locally compact Hausdorff space. Then the triple $\langle \mathrm{RC}(X), \rho_X, \mathrm{CR}(X)\rangle$, where $\mathrm{CR}(X)$ is the set of all compact regular closed subsets of X, is a complete local contact algebra.*

The complete LCA $\langle \mathrm{RC}(X), \rho_X, \mathrm{CR}(X)\rangle$ is called the *standard local contact algebra* of X.

We will need the following notation: for every function $\psi : \langle B, \rho, \mathbb{B}\rangle \longrightarrow \langle B', \eta, \mathbb{B}'\rangle$ between two LCAs, the function $\psi^\vee : \langle B, \rho, \mathbb{B}\rangle \longrightarrow \langle B', \eta, \mathbb{B}'\rangle$ is defined by

$$\psi^\vee(a) \stackrel{\mathrm{df}}{=} \bigvee\{\psi(b) \mid b \in \mathbb{B}, b \ll_\rho a\},$$

for every $a \in B$.

Definition 2.9. ([16]) Let **DHLC** be the category whose objects are all complete LC-algebras and whose morphisms are all functions $\varphi : \langle B, \rho, \mathbb{B}\rangle \longrightarrow \langle B', \eta, \mathbb{B}'\rangle$ between the objects of **DHLC** satisfying the following conditions:

(DLC1) $\varphi(0) = 0$;
(DLC2) $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$, for all $a, b \in B$;
(DLC3) If $a \in \mathbb{B}, b \in B$ and $a \ll_\rho b$, then $(\varphi(a^*))^* \ll_\eta \varphi(b)$;
(DLC4) For every $b \in \mathbb{B}'$ there exists $a \in \mathbb{B}$ such that $b \leq \varphi(a)$;
(DLC5) $\varphi(a) = \bigvee\{\varphi(b) \mid b \in \mathbb{B}, b \ll_\rho a\}$, for every $a \in B$;

the composition "$\diamond$" of two morphisms $\varphi_1 : \langle B_1, \rho_1, \mathbb{B}_1\rangle \longrightarrow \langle B_2, \rho_2, \mathbb{B}_2\rangle$ and $\varphi_2 : \langle B_2, \rho_2, \mathbb{B}_2\rangle \longrightarrow \langle B_3, \rho_3, \mathbb{B}_3\rangle$ of **DHLC** is defined by the formula $\varphi_2 \diamond \varphi_1 \stackrel{\mathrm{df}}{=} (\varphi_2 \circ \varphi_1)^\vee$.

Note that two complete LCAs are LCA-isomorphic if and only if they are **DHLC**-isomorphic.

Let **HLC** (resp., **HC**) be the category of all locally compact (resp., compact) Hausdorff spaces and all continuous maps between them. The following duality theorem for the category **HLC** was proved in [16].

Theorem 2.10. ([16]) *The categories* **HLC** *and* **DHLC** *are dually equivalent. The contravariant functors which realize this duality are denoted by*

$$\Lambda^t : \mathbf{HLC} \longrightarrow \mathbf{DHLC} \quad and \quad \Lambda^a : \mathbf{DHLC} \longrightarrow \mathbf{HLC}.$$

The contravariant functor Λ^t is defined as follows:

$$\Lambda^t(X) \stackrel{\text{df}}{=} \langle \mathrm{RC}(X), \rho_X, \mathrm{CR}(X)\rangle,$$

for every **HLC**-*object X, and*

$$\Lambda^t(f)(G) \stackrel{\text{df}}{=} \mathrm{cl}(f^{-1}(\mathrm{int}(G))),$$

for every $f \in$ **HLC**(X,Y) *and every $G \in \mathrm{RC}(Y)$.*

In particular, for every complete LCA $\underline{B} \stackrel{\text{df}}{=} \langle B, \rho, \mathbb{B}\rangle$ and every $X \in |\mathbf{HLC}|$, $\underline{B}$ is LCA-isomorphic to $\Lambda^t(\Lambda^a(\underline{B}))$ and X is homeomorphic to $\Lambda^a(\Lambda^t(X))$. (We do not give here the explicit definition of the contravariant functor Λ^a because we will not use it. (It is given in [16].) For our purposes here, it is enough to know that the compositions $\Lambda^a \circ \Lambda^t$ and $\Lambda^t \circ \Lambda^a$ are naturally equivalent to the corresponding identity functors (see, e.g., [1]).)

Also, the restriction of Λ^t to the subcategory **HC** of the category **HLC** coincides with the de Vries duality functor between the category **HC** and the full subcategory **DHC** of the category **DHLC**, having as objects all NCAs.

The next theorem shows how one can construct the dual object $\Lambda^t(F)$ of a regular closed subset F of a locally compact Hausdorff space X using only F and the dual object $\Lambda^t(X)$ of X.

Theorem 2.11. ([17]) *Let X be a locally compact Hausdorff space and $F \in \mathrm{RC}(X)$. Let $B \stackrel{\text{df}}{=} \mathrm{RC}(X)_F$ be the relative algebra of $\mathrm{RC}(X)$ with respect to F,*

$$\mathbb{B}' \stackrel{\text{df}}{=} \{G \wedge F \mid G \in \mathrm{CR}(X)\}$$

and, for every $a, b \in B$, $a\eta b \Leftrightarrow a\rho_X b$ (i.e., $a\eta b \Leftrightarrow a \cap b \neq \emptyset$). Then $\langle B, \eta, \mathbb{B}'\rangle$ is LCA-isomorphic to $\Lambda^t(F)$, where F is regarded as a subspace of X.

We will also need the following definitions and assertions. Note that for $\gamma \in \Gamma$ and $a \in \prod\{A_\gamma \mid \gamma \in \Gamma\}$, a_γ will denote the γ-th coordinate of a.

Definition 2.12. ([17]) Let $\{\langle B_\gamma, \rho_\gamma, \mathbb{B}_\gamma\rangle \mid \gamma \in \Gamma\}$ be a family of LC-algebras and

$$B \stackrel{\text{df}}{=} \prod\{B_\gamma \mid \gamma \in \Gamma\}$$

be the product of the Boolean algebras $\{B_\gamma \mid \gamma \in \Gamma\}$ in the category **Bool** of Boolean algebras and Boolean homomorphisms. Let

$$\mathbb{B} \stackrel{\text{df}}{=} \{b \in \prod\{\mathbb{B}_\gamma \mid \gamma \in \Gamma\} \mid |\{\gamma \in \Gamma \mid b_\gamma \neq 0\}| < \aleph_0\}.$$

For any two points $a, b \in B$, set

$$a\rho b \Longleftrightarrow \text{ there exists } \gamma \in \Gamma \text{ such that } a_\gamma \rho_\gamma b_\gamma.$$

Then the triple $\langle B, \rho, \mathbb{B}\rangle$ is called a *product of the family* $\{\langle B_\gamma, \rho_\gamma, \mathbb{B}_\gamma\rangle \mid \gamma \in \Gamma\}$ *of LC-algebras*; we will denote it by

$$\prod\{\langle B_\gamma, \rho_\gamma, \mathbb{B}_\gamma\rangle \mid \gamma \in \Gamma\}.$$

Theorem 2.13. ([17]) *Let $\mathcal{B} \stackrel{\text{df}}{=} \{\langle B_\gamma, \rho_\gamma, \mathbb{B}_\gamma\rangle \mid \gamma \in \Gamma\}$ be a family of complete LC-algebras, $\langle B, \rho, \mathbb{B}\rangle$ be the product $\prod\{\langle B_\gamma, \rho_\gamma, \mathbb{B}_\gamma\rangle \mid \gamma \in \Gamma\}$ of the family $\mathcal{B}$ and $\pi_\gamma(a) \stackrel{\text{df}}{=} a_\gamma$, for every $a \in B$ and every $\gamma \in \Gamma$. Then the source $\{\pi_\gamma : \langle B, \rho, \mathbb{B}\rangle \longrightarrow \langle B_\gamma, \rho_\gamma, \mathbb{B}_\gamma\rangle \mid \gamma \in \Gamma\}$ is a product of the family $\mathcal{B}$ in the category* **DHLC**.

Definition 2.14. ([15, 17]) Let $\langle B, \rho, \mathbb{B}\rangle$ be an LCA and D be a subset of $\mathbb{B}$. Then we say that D is *dV-dense in* $\langle B, \rho, \mathbb{B}\rangle$ if for each $a, c \in \mathbb{B}$ such that $a \ll_\rho c$, there exists $d \in D$ with $a \leq d \leq c$.

Fact 2.15. ([15, 17]) *If $\langle B, \rho, \mathbb{B}\rangle$ is an LCA and D is a subset of $\mathbb{B}$, then D is dV-dense in $\langle B, \rho, \mathbb{B}\rangle$ if and only if for each $a, c \in \mathbb{B}$ such that $a \ll_\rho c$, there exists $d \in D$ with $a \ll_\rho d \ll_\rho c$.*

Definition 2.16. ([15, 17]) Let $\langle B, \rho, \mathbb{B}\rangle$ be an LCA. A pair $(\varphi, \langle B', \rho', \mathbb{B}'\rangle)$ is called an *LCA-completion* of the LCA $\langle B, \rho, \mathbb{B}\rangle$ if $\langle B', \rho', \mathbb{B}'\rangle$ is a complete LCA, $\varphi : \langle B, \rho, \mathbb{B}\rangle \longrightarrow \langle B', \rho', \mathbb{B}'\rangle$ is an LCA-embedding and $\varphi(\mathbb{B})$ is dV-dense in $\langle B', \rho', \mathbb{B}'\rangle$.

Two LCA-completions $(\varphi, \langle B', \rho', \mathbb{B}'\rangle)$ and $(\psi, \langle B'', \rho'', \mathbb{B}''\rangle)$ of a local contact algebra $\langle B, \rho, \mathbb{B}\rangle$ are said to be *equivalent* if there exists an LCA-isomorphism $\eta : \langle B', \rho', \mathbb{B}'\rangle \longrightarrow \langle B'', \rho'', \mathbb{B}''\rangle$ such that $\psi = \eta \circ \varphi$.

We define analogously the notions of *NCA-completion* and *equivalent NCA-completions*.

Note that condition (LC3) implies that if a set D is dV-dense in an LCA $\langle B, \rho, \mathbb{B}\rangle$, then it is a dense subset of B. Hence, if $(\varphi, \langle B', \rho', \mathbb{B}'\rangle)$ is an LCA-completion of the LCA $\langle B, \rho, \mathbb{B}\rangle$, then (φ, B') is a completion of the Boolean algebra B.

Theorem 2.17. ([15, 17]) *Every local contact algebra $\langle B, \rho, \mathbb{B}\rangle$ has a unique (up to equivalence) LCA-completion $(\varphi, \langle B', \rho', \mathbb{B}'\rangle)$. Every normal contact algebra $\langle B, C\rangle$ has a unique (up to equivalence) NCA-completion.*

2.4. The Čech–Lebesgue dimension and the weight of a topological space

A *cover* of a set X is a family $\mathcal{A}$ of subsets of X for which $\bigcup \mathcal{A} = X$. If $\mathcal{A}, \mathcal{B}$ are covers of X, then $\mathcal{B}$ is a *refinement of* $\mathcal{A}$, if for every $B \in \mathcal{B}$ there is some $A \in \mathcal{A}$ such that $B \subseteq A$. A cover $\mathcal{B} \stackrel{\text{df}}{=} \{B_i \mid i \in I\}$ is a *shrinking of* $\mathcal{A} \stackrel{\text{df}}{=} \{A_i \mid i \in I\}$ if $B_i \subseteq A_i$ for all $i \in I$. If $\mathcal{A} \stackrel{\text{df}}{=} \{A_i \mid i \in I\} \subseteq 2^X$, a family $\mathcal{B} \stackrel{\text{df}}{=} \{B_i \mid i \in I\} \subseteq 2^X$ is called a *swelling of* $\mathcal{A}$, if $A_i \subseteq B_i$ for all $i \in I$, and for all $k \in \mathbb{N}^+$ and $i_1, \ldots, i_k \in I$,

$$A_{i_1} \cap \ldots \cap A_{i_k} = \emptyset \Longleftrightarrow B_{i_1} \cap \ldots \cap B_{i_k} = \emptyset.$$

A cover (refinement, shrinking, swelling) of a topological space X is called *open (regular open, closed, regular closed)* if all of its members are open (regular open, closed, regular closed) subsets of X.

If X is a set and $\mathcal{A} \subseteq 2^X$, the *order of* $\mathcal{A}$ is defined as

$$\operatorname{ord} \mathcal{A} \stackrel{\text{df}}{=} \begin{cases} n, & \text{if } n = \max\{m \in \mathbb{N}^- \mid (\exists A_1, \ldots, A_{m+1} \in \mathcal{A})(\bigcap_{i=1}^{m+1} A_i \neq \emptyset)\}, \\ \infty, & \text{if such } n \text{ does not exist.} \end{cases}$$

It follows that if $\operatorname{ord} \mathcal{A} = n$, then the intersection of every $n + 2$ distinct elements of $\mathcal{A}$ is empty. Also, $\operatorname{ord} \mathcal{A} = -1$ if and only if $\mathcal{A} = \{\emptyset\}$, and $\operatorname{ord} \mathcal{A} = 0$ if and only if $\mathcal{A}$ is a disjoint family of subsets of X which are not all empty.

The *Čech–Lebesgue dimension* of a topological space X, denoted by $\dim(X)$, is defined in layers (see, e.g., [28]). Suppose that $n \in \mathbb{N}^-$.

(*CL*1). If every finite open cover of X has a finite open refinement of order at most n, then $\dim(X) \leq n$.

(*CL*2). If $\dim(X) \leq n$ and $\dim(X) \nleq n - 1$, then $\dim(X) = n$.

(*CL*3). If $n \lneq \dim(X)$ for all $n \in \mathbb{N}^-$, then $\dim(X) = \infty$.

Observe that $\dim(X) = -1$ if and only if $X = \emptyset$.

The above definition was introduced and discussed by E. Čech in [9]. It is related to the following property of covers of the n-cube $\mathbb{I}^n$ discovered by Lebesgue in [35]: for every $\varepsilon > 0$, $\mathbb{I}^n$ can be covered by a finite family of closed sets with diameters less than ε such that all intersections of $n + 2$ members of the family are empty, but $\mathbb{I}^n$ cannot be covered by a finite family of closed sets with diameters less than 1 such that all intersections of $n + 1$ members of the family are empty.

Obviously, if X and Y are two homeomorphic topological spaces, then $\dim(X) = \dim(Y)$.

Let us recall that $\dim(\mathbb{R}^n) = \dim(\mathbb{I}^n) = n$, for every $n \in \mathbb{N}^+$ (see, e.g., [28, Theorem 1.8.2, Corollary 1.8.3] or [27, Theorem 7.3.19, Corollary 7.3.20]).

In what follows, we will often use the following three theorems (see, e.g., [28, Theorems 1.6.10, 1.7.8, 3.1.2]):

Theorem 2.18. *A normal T_1-space X satisfies the inequality $\dim(X) \le n$ if and only if every $(n+2)$-element open cover $\{U_i \mid i = 1,\dots,n+2\}$ of the space X has an open shrinking $\{W_i \mid i = 1,\dots,n+2\}$ of order $\le n$, i.e., such that $\bigcap\{W_i \mid i = 1,\dots,n+2\} = \emptyset$.*

Theorem 2.19. *Every finite open cover $\{U_i \mid i = 1,\dots,k\}$ of a normal T_1-space X has a closed shrinking $\{F_i \mid i = 1,\dots,k\}$.*

Theorem 2.20. *Every finite family $\{F_i \mid i = 1,\dots,k\}$ of closed subsets of a normal T_1-space X has an open swelling $\{U_i \mid i = 1,\dots,k\}$. If, moreover, a family $\{V_i \mid i = 1,\dots,k\}$ of open subsets of X satisfying $F_i \subseteq V_i$, for $i = 1,\dots,k$ is given, then the swelling can be defined in such a way that $\mathrm{cl}(U_i) \subseteq V_i$ for $i = 1,\dots,k$.*

Recall that if $(X,\mathcal{T})$ is a topological space, then a subfamily $\mathcal{B}$ of $\mathcal{T}$ is called a *base for* $(X,\mathcal{T})$ (or, simply, *for* X) if every non-empty open subset of X can be represented as the union of a subfamily of $\mathcal{B}$. It is easy to see that a subfamily $\mathcal{B}$ of $\mathcal{T}$ is a base for $(X,\mathcal{T})$ if and only if for every point $x \in X$ and any neighbourhood V of x there exists $U \in \mathcal{B}$ such that $x \in U \subseteq V$. Obviously, a topological space $(X,\mathcal{T})$ can have many bases. The cardinal number

$$w(X) \stackrel{\text{df}}{=} \min\{|\mathcal{B}| \mid \mathcal{B} \text{ is a base for } X\}$$

is called the *weight of the topological space* $(X,\mathcal{T})$. As in the case of dim, if X and Y are two homeomorphic topological spaces, then $w(X) = w(Y)$. It is easy to see that $w(\mathbb{R}^n) = \aleph_0$, for every $n \in \mathbb{N}^+$. Indeed, for every $n \in \mathbb{N}^+$, the family of all open balls in $\mathbb{R}^n$ having as centres all points of $\mathbb{R}^n$ with rational coordinates and radii equal to $\frac{1}{m}$, for every $m \in \mathbb{N}^+$, is a base for $\mathbb{R}^n$.

The next theorem of Alexandroff and Urysohn [4] (see also [27, Theorem 1.1.15]) will be often used in this paper:

Theorem 2.21. *Let X be a topological space and $\mathcal{B}$ be a base for X. Then there exists a base $\mathcal{B}_0$ for X such that $\mathcal{B}_0 \subseteq \mathcal{B}$ and $|\mathcal{B}_0| = w(X)$.*

3. Dimension of a precontact algebra

The following assertion might be known.

Proposition 3.1. *Let X be a normal T_1-space, and $n \in \mathbb{N}^-$. Then, $\dim(X) \le n$ if and only if for every finite regular open cover $\mathcal{U} \stackrel{\text{df}}{=} \{U_1,\dots,U_{n+2}\}$ of X there exists a regular closed shrinking $\mathcal{F} \stackrel{\text{df}}{=} \{F_1,\dots,F_{n+2}\}$ of $\mathcal{U}$ such that $\bigcap \mathcal{F} = \emptyset$ (i.e., $\mathrm{ord}(\mathcal{F}) \le n$).*

Proof. (⇒) Let $\dim(X) \le n$ and $\mathcal{U} \stackrel{\text{df}}{=} \{U_1,\dots,U_{n+2}\}$ be a regular open cover of X. Then, by Theorem 2.18, $\mathcal{U}$ has an open shrinking $\mathcal{W} \stackrel{\text{df}}{=} \{W_1,\dots,W_{n+2}\}$ such that $\bigcap \mathcal{W} = \emptyset$. Using Theorem 2.19, we find a closed shrinking $\mathcal{F}' \stackrel{\text{df}}{=} \{F_1,\dots,F_{n+2}\}$ of $\mathcal{W}$. Now, Theorem 2.20 gives us an open swelling $\mathcal{V} \stackrel{\text{df}}{=} \{V_1,\dots,V_{n+2}\}$ of $\mathcal{F}'$ such that $\mathrm{cl}(V_i) \subseteq W_i$, for every $i = 1,\dots,n+2$. Set $\mathcal{F} \stackrel{\text{df}}{=} \{\mathrm{cl}(V_1),\dots,\mathrm{cl}(V_{n+2})\}$. Then $\mathcal{F}$ is a regular closed shrinking of $\mathcal{U}$ and $\bigcap \mathcal{F} = \emptyset$.
(⇐) Let $\mathcal{U}' \stackrel{\text{df}}{=} \{U_1,\dots,U_{n+2}\}$ be an open cover of X. Then, by Theorem 2.19, $\mathcal{U}'$ has a closed shrinking $\mathcal{F}' \stackrel{\text{df}}{=} \{F'_1,\dots,F'_{n+2}\}$. Using Theorem 2.20, we obtain an open swelling $\mathcal{V} \stackrel{\text{df}}{=} \{V_1,\dots,V_{n+2}\}$ of $\mathcal{F}'$ such that $\mathrm{cl}(V_i) \subseteq U_i$, for every $i = 1,\dots,n+2$. Then $\mathcal{U} \stackrel{\text{df}}{=} \{\mathrm{int}(\mathrm{cl}(V_1)),\dots,\mathrm{int}(\mathrm{cl}(V_{n+2}))\}$ is a regular open shrinking of $\mathcal{U}'$. By our hypothesis, $\mathcal{U}$ has a regular closed shrinking $\mathcal{F} \stackrel{\text{df}}{=} \{F_1,\dots,F_{n+2}\}$ such that $\bigcap \mathcal{F} = \emptyset$. Then

$\mathcal{F}$ is a closed shrinking of $\mathcal{U}'$. By Theorem 2.20, $\mathcal{F}$ has an open swelling $\mathcal{W} \stackrel{\text{df}}{=} \{W_1, \ldots, W_{n+2}\}$ such that $\text{cl}(W_i) \subseteq U_i$ for every $i = 1, \ldots, n+2$; thus, $\mathcal{W}$ is an open shrinking of $\mathcal{U}'$ and $\bigcap \mathcal{W} = \emptyset$. Thus, by Theorem 2.18, $\dim(X) \leq n$. □

Corollary 3.2. *Let X be a normal T_1-space, and $n \in \mathbb{N}^-$. Then, $\dim(X) \leq n$ if and only if for every finite regular open cover $\mathcal{U} \stackrel{\text{df}}{=} \{U_1, \ldots, U_{n+2}\}$ of X there exists a regular closed shrinking $\mathcal{F} \stackrel{\text{df}}{=} \{F_1, \ldots, F_{n+2}\}$ of $\mathcal{U}$ such that $\bigcap \mathcal{F} = \emptyset$ and $\bigcup_{i=1}^{n+2} \text{int}(F_i) = X$.*

Proof. (⇒) Repeat the proof of the "if" part of Proposition 3.1 rewriting only the last sentence of it as follows: Then $\mathcal{F}$ is a regular closed shrinking of $\mathcal{U}$, $\bigcap \mathcal{F} = \emptyset$ and $\bigcup_{i=1}^{n+2} \text{int}(\text{cl}(V_i)) = X$.
(⇐) This follows from Proposition 3.1. □

Having in mind the proposition above, we introduce the notion of *dimension of a precontact algebra $\langle B, \rho \rangle$*, denoted by $\dim_a(\langle B, \rho \rangle)$.

Definition 3.3. For a precontact algebra $\langle B, \rho \rangle$ and $n \in \mathbb{N}^-$ set

$$\dim_a(\langle B, \rho \rangle) \leq n,$$

if for all $a_1, \ldots, a_{n+2}, b_1, \ldots, b_{n+2} \in B$ such that $\bigvee_{i=1}^{n+2} b_i = 1$ and $b_i \ll a_i$ for all $i = 1, \ldots, n+2$, there exist $c_1, \ldots, c_{n+2}, d_1, \ldots, d_{n+2} \in B$ which satisfy the following conditions:

$(D1)$. $c_i \ll d_i \ll a_i$ for every $i = 1, \ldots, n+2$.
$(D2)$. $\bigvee_{i=1}^{n+2} c_i = 1$ and $\bigwedge_{i=1}^{n+2} d_i = 0$.

Furthermore, set $\dim_a(\langle B, \rho \rangle) \stackrel{\text{df}}{=} -1$ if and only if $|B| = 1$ (i.e., $0 = 1$ in B). Finally, for all $n \in \mathbb{N}$, set

$$\dim_a(\langle B, \rho \rangle) \stackrel{\text{df}}{=} \begin{cases} n, & \text{if } n-1 < \dim_a(\langle B, \rho \rangle) \leq n, \\ \infty, & \text{if } n < \dim_a(\langle B, \rho \rangle) \text{ for all } n \in \mathbb{N}^-. \end{cases}$$

If $\langle B, \rho, \mathbb{B} \rangle$ is an LCA, then we replace $\langle B, \rho \rangle$ in above notation with $\langle B, \rho, \mathbb{B} \rangle$.

Theorem 3.4. *Let $(X, \mathcal{T})$ be a normal T_1-space and $n \in \mathbb{N}^-$. Then, $\dim(X) \leq n$ if and only if $\dim_a(\langle \text{RC}(X), \rho_X \rangle) \leq n$.*

Proof. Set $B \stackrel{\text{df}}{=} \text{RC}(X)$.
(⇒) Let $\dim(X) \leq n$. Let $a_1, \ldots, a_{n+2}$, $b_1, \ldots, b_{n+2} \in B$, $b_i \ll a_i$ for every $i = 1, \ldots, n+2$, and $\bigvee_{i=1}^{n+2} b_i = 1$. Then $b_i \subseteq \text{int}(a_i)$ for every $i = 1, \ldots, n+2$. Since $\bigcup_{i=1}^{n+2} b_i = X$, we obtain that $\mathcal{A} \stackrel{\text{df}}{=} \{\text{int}(a_i) \mid i = 1, \ldots, n+2\}$ is a regular open cover of X. Then, by Corollary 3.2, $\mathcal{A}$ has a regular closed shrinking $\mathcal{D} \stackrel{\text{df}}{=} \{d_1, \ldots, d_{n+2}\}$ such that $\bigcap \mathcal{D} = \emptyset$ and $\bigcup_{i=1}^{n+2} \text{int}(d_i) = X$. Now, using Proposition 3.1, we obtain a regular closed shrinking $C \stackrel{\text{df}}{=} \{c_1, \ldots, c_{n+2}\}$ of the regular open cover $\{\text{int}(d_i) \mid i = 1, \ldots, n+2\}$ of X. Then $c_i \ll d_i \ll a_i$ for every $i = 1, \ldots, n+2$, $\bigvee_{i=1}^{n+2} c_i = 1$ and $\bigwedge_{i=1}^{n+2} d_i = \text{cl}(\text{int}(\bigcap_{i=1}^{n+2} d_i)) = 0$.

(⇐) Let $\mathcal{U} \stackrel{\text{df}}{=} \{U_1, \ldots, U_{n+2}\}$ be a regular open cover of X. Then, by Theorem 2.19, $\mathcal{U}$ has a closed shrinking $\mathcal{F} \stackrel{\text{df}}{=} \{F_1, \ldots, F_{n+2}\}$. By Theorem 2.20, $\mathcal{F}$ has an open swelling $\mathcal{V} \stackrel{\text{df}}{=} \{V_1, \ldots, V_{n+2}\}$ such that $\text{cl}(V_i) \subseteq U_i$, for every $i = 1, \ldots, n+2$. Set $a_i \stackrel{\text{df}}{=} \text{cl}(U_i)$ and $b_i \stackrel{\text{df}}{=} \text{cl}(V_i)$, for every $i = 1, \ldots, n+2$. Then $b_i \subseteq \text{int}(a_i)$, i.e. $b_i \ll a_i$ for every $i = 1, \ldots, n+2$. Since $\bigcup\{\text{cl}(V_i) \mid i = 1, \ldots, n+2\} = X$, we obtain that $\bigvee_{i=1}^{n+2} b_i = 1$. Thus, by our hypothesis, there exist $c_1, \ldots, c_{n+2}, d_1, \ldots, d_{n+2} \in B$ such that $c_i \ll d_i \ll a_i$ for every $i = 1, \ldots, n+2$, $\bigvee_{i=1}^{n+2} c_i = 1$ and $\bigwedge_{i=1}^{n+2} d_i = 0$. Then $c_i \subseteq \text{int}(d_i)$, for every $i = 1, \ldots, n+2$. Furthermore, we have that $\text{cl}(\bigcap_{i=1}^{n+2} \text{int}(d_i)) = \text{cl}(\text{int}(\bigcap_{i=1}^{n+2} d_i)) = \bigwedge_{i=1}^{n+2} d_i = \emptyset$. Thus $\bigcap_{i=1}^{n+2} \text{int}(d_i) = \emptyset$. Now we obtain that $\bigcap_{i=1}^{n+2} c_i \subseteq \bigcap_{i=1}^{n+2} \text{int}(d_i) = \emptyset$, and hence $\bigcap_{i=1}^{n+2} c_i = \emptyset$. Therefore, Proposition 3.1 implies that $\dim(X) \leq n$. □

Corollary 3.5. *(a) If $\langle B, \rho, \mathbb{B}\rangle$ is an LCA such that $\Lambda^a(\langle B, \rho, \mathbb{B}\rangle)$ is a normal space, then* $\dim_a(\langle B, \rho, \mathbb{B}\rangle) = \dim(\Lambda^a(\langle B, \rho, \mathbb{B}\rangle))$. *In particular, for every NCA $\langle B, \rho\rangle$, we have that* $\dim_a(\langle B, \rho\rangle) = \dim(\Lambda^a(\langle B, \rho\rangle))$.

(b) If X is a normal locally compact T_1-space, then $\dim(X) = \dim_a(\Lambda^t(X))$. *In particular, for every compact Hausdorff space X,* $\dim(X) = \dim_a(\Lambda^t(X))$.

Proof. This follows from Theorems 3.4 and 2.10. □

The next notion is analogous to the notions of "dense subset" and "dV-dense subset" regarded, respectively, in [14] and [15, 17].

Definition 3.6. Let $\langle B, \rho\rangle$ be a precontact algebra. A subset D of B is said to be *DV-dense in* $\langle B, \rho\rangle$ if it satisfies the following condition:

(DV) If $a, b \in B$ and $a \ll b$ then there exists $c \in D$ such that $a \ll c \ll b$.

Lemma 3.7. *Let $\langle B, \rho\rangle$ be a precontact algebra, D be a Boolean subalgebra of B which is DV-dense in $\langle B, \rho\rangle$ and ρ' be the restriction of the relation ρ on $D \times D$. Then $\langle D, \rho'\rangle$ is a precontact algebra and* $\dim_a(\langle B, \rho\rangle) = \dim_a(\langle D, \rho'\rangle)$.

Proof. Clearly, $\langle D, \rho'\rangle$ is a precontact algebra.

If $\dim_a(\langle D, \rho'\rangle) = \infty$ then $\dim_a(\langle B, \rho\rangle) \leq \dim_a(\langle D, \rho'\rangle)$. Suppose that $\dim_a(\langle D, \rho'\rangle) = n$, where $n \in \mathbb{N}^-$, and let $a_1, \ldots, a_{n+2}, b_1, \ldots, b_{n+2} \in B$ be such that $\bigvee_{i=1}^{n+2} b_i = 1$ and $b_i \ll a_i$ for all $i = 1, \ldots, n+2$. Then, by (DV), there exist $c_1, \ldots, c_{n+2}, d_1, \ldots, d_{n+2} \in D$ such that $b_i \ll c_i \ll d_i \ll a_i$ for all $i = 1, \ldots, n+2$. Obviously, we have that $\bigvee_{i=1}^{n+2} c_i = 1$. Thus there exist $c'_1, \ldots, c'_{n+2}, d'_1, \ldots, d'_{n+2} \in D$ such that $c'_i \ll d'_i \ll d_i$ for all $i = 1, \ldots, n+2$, $\bigvee_{i=1}^{n+2} c'_i = 1$ and $\bigwedge_{i=1}^{n+2} d'_i = 0$. Since $c'_i \ll d'_i \ll a_i$ for all $i = 1, \ldots, n+2$ and $D \subseteq B$, we obtain that $\dim_a(\langle B, \rho\rangle) \leq n$. So, we have proved that $\dim_a(\langle B, \rho\rangle) \leq \dim_a(\langle D, \rho'\rangle)$.

For the other direction, let us prove that $\dim_a(\langle B, \rho\rangle) \geq \dim_a(\langle D, \rho'\rangle)$. Obviously, if $\dim_a(\langle B, \rho\rangle) = \infty$ then $\dim_a(\langle D, \rho'\rangle) \leq \dim_a(\langle B, \rho\rangle)$. Now, suppose that $\dim_a(\langle B, \rho\rangle) = n$, where $n \in \mathbb{N}^-$, and let $a_1, \ldots, a_{n+2}, b_1, \ldots, b_{n+2} \in D$ be such that $\bigvee_{i=1}^{n+2} b_i = 1$ and $b_i \ll a_i$ for all $i = 1, \ldots, n+2$. Then there exist $c'_1, \ldots, c'_{n+2}, d'_1, \ldots, d'_{n+2} \in B$ such that $c'_i \ll d'_i \ll a_i$ for all $i = 1, \ldots, n+2$, $\bigvee_{i=1}^{n+2} c'_i = 1$ and $\bigwedge_{i=1}^{n+2} d'_i = 0$. Now, by (DV), there exist $c_1, \ldots, c_{n+2}, d_1, \ldots, d_{n+2} \in D$ such that $c'_i \ll c_i \ll d_i \ll d'_i$ for all $i = 1, \ldots, n+2$. Obviously, we have $\bigvee_{i=1}^{n+2} c_i = 1$ and $\bigwedge_{i=1}^{n+2} d_i = 0$. Since $c_i \ll d_i \ll a_i$ for all $i = 1, \ldots, n+2$, we obtain $\dim_a(\langle D, \rho'\rangle) \leq n$. So, we have proved that $\dim_a(\langle D, \rho'\rangle) \leq \dim_a(\langle B, \rho\rangle)$, and therefore, $\dim_a(\langle B, \rho\rangle) = \dim_a(\langle D, \rho'\rangle)$. □

Theorem 3.8. *Let $\langle B, C\rangle$ be an normal contact algebra and $(\varphi, \langle B', C'\rangle)$ be the NCA-completion of it. Then* $\dim_a(\langle B, C\rangle) = \dim_a(\langle B', C'\rangle)$.

Proof. By Definition 2.16 and Fact 2.15, $\varphi(B)$ is a DV-dense subset of B'. Thus, by Lemma 3.7, $\dim_a(\langle B', C'\rangle) = \dim_a(\langle \varphi(B), C''\rangle)$, where C'' is the restriction of the relation C' to $\varphi(B) \times \varphi(B)$. Hence, our assertion follows from the fact that $\dim_a(\langle B, C\rangle) = \dim_a(\langle \varphi(B), C''\rangle)$. □

Proposition 3.9. *Let B be a non-degenerate Boolean algebra (i.e., $|B| > 1$). Then* $\dim_a(\langle B, \rho_s\rangle) = 0 = \dim_a(\langle B, \rho_l\rangle)$ *(see Example 2.6 for ρ_s and ρ_l).*

Proof. Since $|B| > 1$, we have $\dim_a(\langle B, \rho_s\rangle) > -1$ and $\dim_a(\langle B, \rho_l\rangle) > -1$. So, we need to show that $\dim_a(\langle B, \rho_s\rangle) \leq 0$ and $\dim_a(\langle B, \rho_l\rangle) \leq 0$.

We will first prove that $\dim_a(\langle B, \rho_s\rangle) \leq 0$. Recall that in $\langle B, \rho_s\rangle$, $a \ll b$ if and only if $a \leq b$. So, let $a_1, a_2, b_1, b_2 \in B$, $b_1 \vee b_2 = 1$ and $b_i \leq a_i$ for $i = 1, 2$. Then $a_1 \vee a_2 = 1$. Set $a \stackrel{\text{df}}{=} a_1 \wedge a_2$, $c_1 = d_1 \stackrel{\text{df}}{=} a^* \wedge a_1$ and $c_2 = d_2 \stackrel{\text{df}}{=} a_2$. Then $c_1 \leq d_1 \leq a_1$, $c_2 \leq d_2 \leq a_2$, $c_1 \vee c_2 = (a^* \wedge a_1) \vee a_2 = ((a_1^* \vee a_2^*) \wedge a_1) \vee a_2 = (a_2^* \wedge a_1) \vee a_2 = (a_1 \vee a_2) \wedge (a_2 \vee a_2^*) = 1$ and $d_1 \wedge d_2 = (a^* \wedge a_1) \wedge a_2 = (a_1 \wedge a_2)^* \wedge (a_1 \wedge a_2) = 0$. Thus, $\dim_a(\langle B, \rho_s\rangle) \leq 0$, and altogether $\dim_a(\langle B, \rho_s\rangle) = 0$.

Next, we will prove that $\dim_a(\langle B, \rho_l\rangle) \leq 0$. It is easy to see that in $\langle B, \rho_l\rangle$, $a \ll b$ if and only if $a = 0$ or $b = 1$. So, let $a_1, a_2, b_1, b_2 \in B$, $b_1 \vee b_2 = 1$ and $b_i \ll a_i$ for $i = 1, 2$. Then $b_i = 0$ or $a_i = 1$, for $i = 1, 2$. We will consider all possible cases.

Case 1. Let $b_1 = 0$. Then $b_2 = 1$ and hence $a_2 = 1$. However, a_1 could be equal to 0 or to 1. In both cases, setting $c_1 = d_1 \stackrel{\text{df}}{=} 0$ and $c_2 = d_2 \stackrel{\text{df}}{=} 1$, we obtain $\dim_a(\langle B, \rho_l\rangle) \leq 0$.

Case 2. Let $b_2 = 0$. Then we argue analogously (just interchange the indices).

Case 3. Let $a_1 = 1$. Since $a_1 \vee a_2 = 1$, a_2 could be equal to 0 or to 1.

Case 3a. Let $a_2 = 0$. Setting $c_1 = d_1 \stackrel{\text{df}}{=} 1$ and $c_2 = d_2 \stackrel{\text{df}}{=} 0$, we obtain $\dim_a(\langle B, \rho_l\rangle) \leq 0$.

Case 3b. Let $a_2 = 1$. Setting $c_1 = d_1 \stackrel{\text{df}}{=} 0$ and $c_2 = d_2 \stackrel{\text{df}}{=} 1$, we obtain $\dim_a(\langle B, \rho_l\rangle) \leq 0$.

Case 4. Let $a_2 = 1$. Then we argue analogously to Case 3 (just interchange the indices).

Thus, we have shown that $\dim_a(\langle B, \rho_l\rangle) = 0$. □

It is well known that for a normal T_1-space X and a regular closed subset M of X, $\dim(M) \leq \dim(X)$ holds (this is true even for closed subsets M of X, see e.g. [28]). According to Theorems 2.10 and 3.4, the dual of this assertion is the following one: if X is a normal locally compact T_1-space and $M \in RC(X)$, then $\dim_a(\Lambda^t(M)) \leq \dim_a(\Lambda^t(X))$. Theorem 2.11 describes the LCA $\Lambda^t(M)$ in terms of the LCA $\Lambda^t(X)$, so that we can reformulate the above statement in a purely algebraic terms. We will supply this new statement with an algebraic proof, obtaining in this way an algebraic generalization of the topological statement stated above. (Note that we will just take an LCA without requiring that it is dual to a *normal* locally compact T_1-space.)

Proposition 3.10. *Suppose that $\langle B, \rho, \mathbb{B}\rangle$ is an LCA, $m \in B^+$, and $\langle B_m, \rho_m, \mathbb{B}_m\rangle$ is the relative LCA of $\langle B, \rho, \mathbb{B}\rangle$, i.e.*

$$\rho_m \stackrel{\text{df}}{=} \rho \restriction B_m^2, \quad \mathbb{B}_m \stackrel{\text{df}}{=} \{b \wedge m : b \in \mathbb{B}\}.$$

Then, $\dim_a(\langle B_m, \rho_m, \mathbb{B}_m\rangle) \leq \dim_a(\langle B, \rho, \mathbb{B}\rangle)$.

Proof. Recall that $B_m \stackrel{\text{df}}{=} \{b \in B \mid b \leq m\}$. We denote the complement in B_m by ^{*_m}, i.e. $a^{*_m} \stackrel{\text{df}}{=} a^* \wedge m$. Note that, for $a, b \in B_m$, $a \ll_m b$ means that $a(-\rho)b^{*_m}$, i.e., $a(-\rho)(b^* \wedge m)$. Clearly, if $a, b \in B_m$ and $a \ll_m b$, then $b^{*_m} \ll_m a^{*_m}$.

Let $\dim_a(\langle B, \rho, \mathbb{B}\rangle) = n$, and suppose that $a_1, \ldots, a_{n+2}, b_1, \ldots, b_{n+2} \in B_m$ are such that $\bigvee_{i=1}^{n+2} b_i = m$ and $b_i \ll_m a_i$. Then $a_i^* \wedge m \ll m$. Indeed, we have that $b_i \ll_m a_i$ for $i = 1, \ldots, n+2$. Thus $a_i^{*_m} \ll_m b_i^{*_m}$, i.e. $(a_i^* \wedge m) \ll (b_i^* \wedge m)$ for $i = 1, \ldots, n+2$. Since $b_i^* \wedge m \leq m$ for $i = 1, \ldots, n+2$, ($\ll$3) implies that $a_i^* \wedge m \ll m$ for $i = 1, \ldots, n+2$. Now, set

$$a'_i \stackrel{\text{df}}{=} a_i \vee m^* \text{ and } b'_i \stackrel{\text{df}}{=} b_i \vee m^*$$

for all $1 \leq i \leq n+2$; clearly, $\sum_{i=1}^{n+2} b'_i = 1$. Furthermore, $b'_i \ll a'_i$. Indeed, assume not; then $b'_i \rho (a'_i)^*$, i.e. $(b_i \vee m^*)\rho(a_i^* \wedge m)$. If $b_i \rho (a_i^* \wedge m)$, then $b_i(-\ll_m)a_i)$, and if $m^* \rho (a_i^* \wedge m)$, then $(a_i^* \wedge m)(-\ll)m)$, a contradiction in both cases.

Since $\dim_a(\langle B, \rho, \mathbb{B}\rangle) = n$, there exist $c_1, \ldots, c_{n+2}, d_1 \ldots, d_{n+2} \in B$ such that

$$\bigvee_{i=1}^{n+2} c_i = 1, \ \bigwedge_{i=1}^{n+2} d_i = 0, \text{ and } c_i \ll d_i \ll a'_i$$

for every $i = 1, \ldots, n+2$. Set

$$s_i \stackrel{\text{df}}{=} c_i \wedge m \quad \text{and} \quad t_i \stackrel{\text{df}}{=} d_i \wedge m.$$

Clearly, $\bigvee_{i=1}^{n+2} s_i = m$ and $\bigwedge_{i=1}^{n+2} t_i = 0$. All that is left to show is $s_i \ll_m t_i \ll_m a_i$. We have that

$$\begin{aligned} s_i \ll_m t_i &\iff s_i(-\rho)t_i^{*_m} \\ &\iff s_i(-\rho)(t_i^* \wedge m), \\ &\iff (c_i \wedge m)(-\rho)((d_i \wedge m)^* \wedge m) \\ &\iff (c_i \wedge m)(-\rho)(d_i^* \wedge m). \end{aligned}$$

Now, $(c_i \wedge m)(-\rho)(d_i^* \wedge m)$ is implied by $c_i \ll d_i$.

Similarly,

$$t_i \ll_m a_i \iff t_i(-\rho)(a_i^* \wedge m),$$
$$\iff (d_i \wedge m)(-\rho)(a_i^* \wedge m).$$

Since $d_i \ll a_i'$, i.e. $d_i(-\rho)(a_i \vee m^*)^*$, we see that $(d_i \wedge m)\rho(a_i^* \wedge m)$ is impossible, and it follows that $t_i \ll_m a_i$. □

4. Weight of a local contact algebra

In this section, we are going to define the notions of base and weight of an LCA $\underline{B} \stackrel{\text{df}}{=} \langle B, \rho, \mathbb{B}\rangle$ in such a way that if $\underline{B}$ is complete, then the weight of $\underline{B}$ is equal to the weight of the space $\Lambda^a(\underline{B})$, equivalently, if X is a locally compact Hausdorff space, then the weight of X is equal to the weight of $\Lambda^t(X)$. Clearly, the main step is to define an adequate notion of base for a complete LCA $\underline{B}$. In doing this, we use the fact that the family $\mathrm{RO}(X) = \{\mathrm{int}(F) \mid F \in \mathrm{RC}(X)\}$ is an open base for X (because X is regular) and hence, by Theorem 2.21, $\mathrm{RO}(X)$ has a subfamily $\mathcal{B}$, with $|\mathcal{B}| = w(X)$, which is a base for X.

The next definition and theorem generalize the analogous definition and theorem of de Vries [14]. Note that our "base" (see the definition below) appears in [14] (for NCAs) as "dense set".

Definition 4.1. Let $\langle B, \rho, \mathbb{B}\rangle$ be an LCA and D be a subset of $\mathbb{B}$. Then D is called a *base* for $\langle B, \rho, \mathbb{B}\rangle$ if it is dV-dense in $\langle B, \rho, \mathbb{B}\rangle$. The cardinal number

$$w_a(\langle B, \rho, \mathbb{B}\rangle) \stackrel{\text{df}}{=} \min\{|D| \mid D \text{ is a base for } \langle B, \rho, \mathbb{B}\rangle\}$$

is called the *weight of* $\langle B, \rho, \mathbb{B}\rangle$.

Lemma 4.2. *Let $X \in |\mathbf{HLC}|$ and $\mathcal{D}$ be a base for the LCA $\Lambda^t(X)$. Then*

$$\mathcal{B}_{\mathcal{D}} \stackrel{\text{df}}{=} \{\mathrm{int}(F) \mid F \in \mathcal{D}\}$$

is a base for X.

Proof. Let $x \in X$ and U be a neighborhood of x. Since X is regular and locally compact, there exist $F, G \in \mathrm{CR}(X)$ such that $x \in \mathrm{int}(F) \subseteq F \subseteq \mathrm{int}(G) \subseteq G \subseteq U$. Then $F \ll_{\rho_X} G$. Hence, there exists $H \in \mathcal{D}$ such that $F \subseteq H \subseteq G$. It follows that $\mathrm{int}(H) \in \mathcal{B}_{\mathcal{D}}$ and $x \in \mathrm{int}(H) \subseteq U$. So, $\mathcal{B}_{\mathcal{D}}$ is a base for X. □

Lemma 4.3. *Let $X \in |\mathbf{HLC}|$, $\mathcal{B}$ be a base for X and $Cl(\mathcal{B}) \stackrel{\text{df}}{=} \{\mathrm{cl}(U) \mid U \in \mathcal{B}\} \subseteq \mathrm{CR}(X)$. Then, the sub-join-semilattice $\mathcal{L}_J(\mathcal{B})$ of $\mathrm{CR}(X)$ generated by $Cl(\mathcal{B})$ is a base for the LCA $\Lambda^t(X)$.*

Proof. Let $F, G \in \mathrm{CR}(X)$ and $F \ll_{\rho_X} G$, i.e. $F \subseteq \mathrm{int}(G)$. By regularity, for every $x \in F$ there exists $U_x \in \mathcal{B}$ such that $x \in U_x \subseteq \mathrm{cl}(U_x) \subseteq \mathrm{int}(G)$. Since F is compact, there exist $n \in \mathbb{N}^+$ and $x_1, \ldots, x_n \in F$ such that $F \subseteq \bigcup_{i=1}^n U_{x_i} \subseteq \bigcup_{i=1}^n \mathrm{cl}(U_{x_i}) \subseteq \mathrm{int}(G)$. Thus $H \stackrel{\text{df}}{=} \bigcup_{i=1}^n \mathrm{cl}(U_{x_i}) = \bigvee_{i=1}^n \mathrm{cl}(U_{x_i}) \in \mathcal{L}_J(\mathcal{B})$ and $F \subseteq H \subseteq G$. So, $\mathcal{L}_J(\mathcal{B})$ is a base for the LCA $\langle \mathrm{RC}(X), \rho_X, \mathrm{CR}(X)\rangle$. □

Theorem 4.4. *Let X be a locally compact Hausdorff space and $w(X) \geq \aleph_0$. Then $w(X) = w_a(\langle \mathrm{RC}(X), \rho_X, \mathrm{CR}(X)\rangle)$ (i.e., $w(X) = w_a(\Lambda^t(X))$).*

Proof. We know that the family $\mathcal{B}_0 \stackrel{\text{df}}{=} \{\mathrm{int}(F) \mid F \in \mathrm{CR}(X)\}$ is a base for X. Hence, by Theorem 2.21, there exists a base $\mathcal{B}$ of X such that $\mathcal{B} \subseteq \mathcal{B}_0$ and $|\mathcal{B}| = w(X)$. Let $\mathcal{L}_J(\mathcal{B})$ be the sub-join-semilattice of $\mathrm{CR}(X)$ generated by the set $\{\mathrm{cl}(U) \mid U \in \mathcal{B}\}$. Then, by Lemma 4.3, $\mathcal{L}_J(\mathcal{B})$ is a base for $\langle \mathrm{RC}(X), \rho_X, \mathrm{CR}(X)\rangle$. Clearly, $|\mathcal{L}_J(\mathcal{B})| = |\mathcal{B}| = w(X)$. Hence, $w(X) \geq w_a(\langle \mathrm{RC}(X), \rho_X, \mathrm{CR}(X)\rangle)$.

Conversely, let $\mathcal{D}$ be a base for $\langle \mathrm{RC}(X), \rho_X, \mathrm{CR}(X)\rangle$ such that

$$|\mathcal{D}| = w_a(\langle \mathrm{RC}(X), \rho_X, \mathrm{CR}(X)\rangle).$$

Then, by Lemma 4.2, $\mathcal{B}_{\mathcal{D}} \stackrel{\text{df}}{=} \{\mathrm{int}(F) \mid F \in \mathcal{D}\}$ is a base for X. Since $|\mathcal{B}_{\mathcal{D}}| = |\mathcal{D}|$, we obtain that $w(X) \leq w_a(\langle \mathrm{RC}(X), \rho_X, \mathrm{CR}(X)\rangle)$.

Altogether, we have shown that $w(X) = w_a(\langle \mathrm{RC}(X), \rho_X, \mathrm{CR}(X)\rangle)$. □

Lemma 4.5. *Let $\langle B, \rho, \mathbb{B}\rangle$ be an LCA and $(\varphi, \langle B', \rho', \mathbb{B}'\rangle)$ be its LCA-completion. Then:*
(a) if D is a base for $\langle B, \rho, \mathbb{B}\rangle$, then $\varphi(D)$ is a base for $\langle B', \rho', \mathbb{B}'\rangle$;
(b) if D' is a base for $\langle B', \rho', \mathbb{B}'\rangle$ and $D' \subseteq \varphi(\mathbb{B})$, then $\varphi^{-1}(D')$ is a base for $\langle B, \rho, \mathbb{B}\rangle$.

Proof. By definition, $\varphi(\mathbb{B})$ is dV-dense in $\langle A', \rho', \mathbb{B}'\rangle$.
(a) Let $a, c \in \mathbb{B}'$ and $a \ll_{\rho'} c$. Then, by Fact 2.15, there exist $b_1, b_2 \in \mathbb{B}$ such that $a \ll_{\rho'} \varphi(b_1) \ll_{\rho'} \varphi(b_2) \ll_{\rho'} c$; thus $b_1 \ll_\rho b_2$. Hence, there exists some $b \in D$ such that $b_1 \ll_\rho b \ll_\rho b_2$. Then, $a \ll_{\rho'} \varphi(b) \ll_{\rho'} c$, and therefore, $\varphi(D)$ is a base for $\langle A', \rho', \mathbb{B}'\rangle$.
(b) This is obvious. □

Theorem 4.6. *Let $\langle B, \rho, \mathbb{B}\rangle$ be an LCA, $(\varphi, \langle B', \rho', \mathbb{B}'\rangle)$ be its LCA-completion and $w_a(\langle B, \rho, \mathbb{B}\rangle) \geq \aleph_0$. Then $w_a(\langle B, \rho, \mathbb{B}\rangle) = w_a(\langle B', \rho', \mathbb{B}'\rangle)$.*

Proof. Let $X \stackrel{\text{df}}{=} \Lambda^a(\langle B', \rho', \mathbb{B}'\rangle)$. Then, by Theorem 2.17, we may suppose w.l.o.g. that $\langle B, \rho, \mathbb{B}\rangle$ is an LC-subalgebra of $\Lambda^t(X) = \langle \mathrm{RC}(X), \rho_X, \mathrm{CR}(X)\rangle$ and $(id, \Lambda^t(X))$ is an LCA-completion of $\langle B, \rho, \mathbb{B}\rangle$, where $id : \langle B, \rho, \mathbb{B}\rangle \longrightarrow \Lambda^t(X)$ is the inclusion map; also, $(id, \Lambda^t(X))$ and $(\varphi, \langle B', \rho', \mathbb{B}'\rangle)$ are equivalent LCA-completions of $\langle B, \rho, \mathbb{B}\rangle$ (recall also that, by Theorem 2.10, $\Lambda^t(X)$ and $\langle B', \rho', \mathbb{B}'\rangle$ are LCA-isomorphic). So, $\mathbb{B}$ is dV-dense in $\Lambda^t(X)$. Thus $\mathbb{B}$ is a base for $\Lambda^t(X)$. Let $\mathcal{D}$ be a base for $\langle B, \rho, \mathbb{B}\rangle$ and $|\mathcal{D}| = w_a(\langle B, \rho, \mathbb{B}\rangle)$. Then, by Lemma 4.5(a), $\mathcal{D}$ is a base for $\Lambda^t(X)$. Therefore, $w_a(\langle B', \rho', \mathbb{B}'\rangle) \leq |\mathcal{D}| = w_a(\langle B, \rho, \mathbb{B}\rangle)$. Further, by Lemma 4.2, $\mathcal{B}_{\mathcal{D}} \stackrel{\text{df}}{=} \{\mathrm{int}(F) \mid F \in \mathcal{D}\}$ is a base for X. Applying Theorem 2.21, we find a base $\mathcal{B}$ for X such that $\mathcal{B} \subseteq \mathcal{B}_{\mathcal{D}}$ and $|\mathcal{B}| = w(X)$. Then, Lemma 4.3 implies that the sub-join-semilattice $\mathcal{L}_J(\mathcal{B})$ of $\mathrm{CR}(X)$, generated by the set $Cl(\mathcal{B}) \stackrel{\text{df}}{=} \{\mathrm{cl}(U) \mid U \in \mathcal{B}\}$, is a base for $\Lambda^t(X)$. Since $\mathcal{B} \subseteq \mathcal{B}_{\mathcal{D}}$, we have $Cl(\mathcal{B}) \subseteq \mathcal{D}$. On the other hand, $\mathcal{D} \subseteq \mathbb{B}$ and $\mathbb{B}$ is a sub-join-semilattice of $\mathrm{CR}(X)$; hence $\mathcal{L}_J(\mathcal{B}) \subseteq \mathbb{B}$. Then, by Lemma 4.5(b), $\mathcal{L}_J(\mathcal{B})$ is a base for $\langle B, \rho, \mathbb{B}\rangle$. Thus, using Theorem 4.4, we obtain

$$w_a(\langle B, \rho, \mathbb{B}\rangle) \leq |\mathcal{L}_J(\mathcal{B})| = |\mathcal{B}| = w(X) = w_a(\langle B', \rho', \mathbb{B}'\rangle) \leq w_a(\langle B, \rho, \mathbb{B}\rangle).$$

So, $w_a(\langle B, \rho, \mathbb{B}\rangle) = w_a(\langle B', \rho', \mathbb{B}'\rangle)$. □

The next theorem is an analogue of Theorem 2.21.

Theorem 4.7. *Let D be a base for an LCA $\langle B, \rho, \mathbb{B}\rangle$ with infinite weight. Then there exists a subset D_1 of D such that $|D_1| = w_a(\langle B, \rho, \mathbb{B}\rangle)$ and the sub-join-semilattice L of $\mathbb{B}$, generated by D_1, is a base for $\langle B, \rho, \mathbb{B}\rangle$ with cardinality $w_a(\langle B, \rho, \mathbb{B}\rangle)$. If D is, in addition, a sub-join-semilattice of $\mathbb{B}$, then $L \subseteq D$.*

Proof. Let $(\varphi, \langle B', \rho', \mathbb{B}'\rangle)$ be the LCA-completion of $\langle B, \rho, \mathbb{B}\rangle$. As in the proof of Theorem 4.6, we set $X \stackrel{\text{df}}{=} \Lambda^a(\langle B', \rho', \mathbb{B}'\rangle)$ and suppose w.l.o.g. that $\langle B, \rho, \mathbb{B}\rangle$ is an LC-subalgebra of $\Lambda^t(X)$. Then, by Lemma 4.5(a), D is a base for $\Lambda^t(X)$. Thus, by Lemma 4.2, $\mathcal{B}_D \stackrel{\text{df}}{=} \{\mathrm{int}(F) \mid F \in D\}$ is a base for X. Using Theorem 2.21, we obtain a base $\mathcal{B}$ for X such that $\mathcal{B} \subseteq \mathcal{B}_D$ and $|\mathcal{B}| = w(X)$. Let $D_1 \stackrel{\text{df}}{=} \{\mathrm{cl}(U) \mid U \in \mathcal{B}\}$. Then $D_1 \subseteq D \subseteq \mathbb{B}$ and, by Lemma 4.3, the sub-join-semilattice L of $\mathrm{CR}(X)$, generated by D_1, is a base for $\Lambda^t(X)$. Since $L \subseteq \mathbb{B}$, Lemma 4.5(b) implies that L is a base for $\langle B, \rho, \mathbb{B}\rangle$. Clearly, L coincides with the sub-join-semilattice of $\mathbb{B}$, generated by D_1. Using Theorems 4.4 and 4.6, we obtain $|L| = |D_1| = |\mathcal{B}| = w(X) = w_a(\langle B', \rho', \mathbb{B}'\rangle) = w_a(\langle B, \rho, \mathbb{B}\rangle)$. □

Proposition 4.8. *If $\langle B, \rho, \mathbb{B}\rangle$ is an LCA and $|B| \geq \aleph_0$ then $w_a(\langle B, \rho, \mathbb{B}\rangle) \geq \aleph_0$.*

Proof. Let $(\varphi, \langle B', \rho', \mathbb{B}'\rangle)$ be the LCA-completion of $\langle B, \rho, \mathbb{B}\rangle$. As in the proof of Theorem 4.6, we set $X \stackrel{\text{df}}{=} \Lambda^a(\langle B', \rho', \mathbb{B}'\rangle)$ and suppose w.l.o.g. that $\langle B, \rho, \mathbb{B}\rangle$ is an LC-subalgebra of $\Lambda^t(X)$. Then $B \subseteq \mathrm{RC}(X)$, and thus $|\mathrm{RC}(X)| \geq \aleph_0$. Assume that $w(X)$ is finite. Then X is a discrete space and $w(X) = |X|$. Thus $RC(X)$ is finite, a contradiction. Therefore, $w(X) \geq \aleph_0$. From Theorems 4.4 and 4.6, we obtain $w_a(\langle B, \rho, \mathbb{B}\rangle) = w_a(\langle B', \rho', \mathbb{B}'\rangle) = w_a(\Lambda^t(X)) = w(X) \geq \aleph_0$. □

Theorem 4.9. *Let $X \in |\mathbf{HLC}|$. Then X is metrizable iff there exists a set Γ and a family $\{\langle B_\gamma, \rho_\gamma, \mathbb{B}_\gamma\rangle \mid \gamma \in \Gamma\}$ of complete LCAs such that*

$$\Lambda^t(X) = \prod\{\langle B_\gamma, \rho_\gamma, \mathbb{B}_\gamma\rangle \mid \gamma \in \Gamma\}$$

and, for each $\gamma \in \Gamma$, $w_a(\langle B_\gamma, \rho_\gamma, \mathbb{B}_\gamma\rangle) \leq \aleph_0$.

Proof. It is well known that a locally compact Hausdorff space is metrizable if and only if it is a topological sum of locally compact Hausdorff spaces with countable weight (see, e.g., [3, p. 315] or [27, Theorem 5.1.27]). Since Λ^t is a duality functor, it converts the **HLC**-sums in **DLC**-products. Hence, our assertion follows from the theorem cited above and Theorems 2.13 and 4.4. □

Corollary 4.10. *If $\langle B, \rho, \mathbb{B}\rangle$ is a complete LCA and $w_a(\langle B, \rho, \mathbb{B}\rangle) \leq \aleph_0$, then $\Lambda^a(\langle B, \rho, \mathbb{B}\rangle)$ is a metrizable, separable, locally compact space.*

Notation 4.11. Let $\langle A, \rho, \mathbb{B}\rangle$ be an LCA. We set

$$\langle A, \rho, \mathbb{B}\rangle_S \stackrel{\text{df}}{=} \{a \in A \mid a \ll_\rho a\}.$$

We will write simply "A_S" instead of "$\langle A, \rho, \mathbb{B}\rangle_S$" when this does not lead to an ambiguity.

Theorem 4.12. *Let $\langle B, \rho, \mathbb{B}\rangle$ be an LCA and $(\varphi, \langle B', \rho', \mathbb{B}'\rangle)$ be its LCA-completion. Then the space $\Lambda^a(\langle B', \rho', \mathbb{B}'\rangle)$ is zero-dimensional if and only if the set $B_S \cap \mathbb{B}$ is a base for $\langle B, \rho, \mathbb{B}\rangle$.*

Proof. Set $X \stackrel{\text{df}}{=} \Lambda^a(\langle B', \rho', \mathbb{B}'\rangle)$. As in the proof of Theorem 4.6, we may suppose w.l.o.g. that $\langle B, \rho, \mathbb{B}\rangle$ is an LC-subalgebra of $\Lambda^t(X)$, and that $\mathbb{B}$ is dV-dense in $\Lambda^t(X)$. Then, it follows from Lemma 4.2 that the set $\mathcal{B}_{\mathbb{B}} \stackrel{\text{df}}{=} \{\text{int}(F) \mid F \in \mathbb{B}\}$ is a base for X.
(⇒) Let X be zero-dimensional. Then there exists a base $\mathcal{B}$ for X consisting of clopen compact sets. Clearly, for every $U \in \mathcal{B}$, we have $U \ll_{\rho_X} U$. Since $\mathbb{B}$ is dV-dense in $\Lambda^t(X)$, we obtain $\mathcal{B} \subseteq B_S \cap \mathbb{B}$. Therefore, $B_S \cap \mathbb{B}$ is a base for X. Since $B_S \cap \mathbb{B}$ is closed under joins, Lemma 4.3 implies that $B_S \cap \mathbb{B}$ is a base for $\Lambda^t(X)$. Then, using Lemma 4.5(b), we obtain that $B_S \cap \mathbb{B}$ is a base for $\langle B, \rho, \mathbb{B}\rangle$.
(⇐) Let $x \in X$ and U be a neighborhood of x. Since $\mathcal{B}_{\mathbb{B}}$ is a base for X, there exist $a, b \in \mathbb{B}$ such that $x \in \text{int}(a) \subseteq a \subseteq \text{int}(b) \subseteq b \subseteq U$; hence, $a \ll_\rho b$. Thus, there exists some $c \in B_S \cap \mathbb{B}$ such that $a \leq c \leq b$. Since c is clopen in X and $x \in c \subseteq U$, it follows that X has a base consisting of clopen sets, i.e. X is zero-dimensional. □

In the sequel, we will denote by K the Cantor set.

Note that $\text{RC}(K)$ is isomorphic to the completion A of a free Boolean algebra A_0 with $\aleph_0$ generators, Equivalently, $\text{RC}(K)$ is the unique (up to isomorphism) atomless complete Boolean algebra A containing a countable dense subalgebra A_0 (see, e.g., [33, Example 7.24]). Defining in A a relation ρ by $a(-\rho)b$ if and only if there exists some $c \in A_0$ such that $a \leq c \leq b^*$, we obtain (as we will see below) that $\langle A, \rho\rangle$ is a complete NCA which is NCA-isomorphic to the complete NCA $\langle \text{RC}(K), \rho_K\rangle$. We will now present a generalization of this construction.

We denote by **Bool** the category of all Boolean algebras and Boolean homomorphisms, by **Stone** the category of all compact zero-dimensional Hausdorff spaces and continuous maps, and by

$$S^a : \mathbf{Bool} \longrightarrow \mathbf{Stone}$$

the Stone duality functor (see, e.g., [33]).

Theorem 4.13. *Let A_0 be a dense Boolean subalgebra of a Boolean algebra A. For all $a, b \in A$, set $a \ll_\rho b$ if there exists some $c \in A_0$ such that $a \leq c \leq b$. Then the following holds:*

(a) $\langle A, \rho\rangle$ is an NCA, $\langle A, \rho\rangle_S = A_0$, A_0 is the smallest base for $\langle A, \rho\rangle$ and $w(\langle A, \rho\rangle) = |A_0|$.

(b) If A is complete, then $\Lambda^a(\langle A, \rho\rangle)$ is homeomorphic to $S^a(A_0)$, and $(i_0, \langle A, \rho\rangle)$ is an NCA-completion of the NCA $\langle A_0, \rho_s^{A_0}\rangle$, where $i_0 : A_0 \longrightarrow A$ is the inclusion map.

Proof. (a) It is easy to check that the relation ρ satisfies conditions ($\ll$ 1)-($\ll$ 7). To establish ($\ll$ 5) and ($\ll$ 6), use the fact that for every $c \in A_0$ we have, by the definition of the relation $\ll_\rho$, that $c \ll_\rho c$. Hence, $\langle A, \rho\rangle$ is an NCA. By definition of the relation $\ll_\rho$, we obtain for $c \in A$, $c \ll_\rho c$ if and only if $c \in A_0$; thus, $\langle A, \rho\rangle_S = A_0$. Obviously, A_0 is the smallest base for $\langle A, \rho\rangle$; hence, $w(\langle A, \rho\rangle) = |A_0|$.

(b) Let A be complete and set $X \stackrel{\text{df}}{=} S^a(A_0)$. Then, the Stone map $s : A_0 \longrightarrow \mathrm{CO}(X)$ is a Boolean isomorphism. Let $i : \mathrm{CO}(X) \longrightarrow \mathrm{RC}(X)$ be the inclusion map. Then $(i \circ s, \mathrm{RC}(X))$ is a completion of A_0. We know that (i_0, A) is a completion of A_0. Thus, there exists a Boolean isomorphism $\varphi : A \longrightarrow \mathrm{RC}(X)$ such that $\varphi \circ i_0 = i \circ s$. We will show that $\varphi : \langle A, \rho\rangle \longrightarrow \langle \mathrm{RC}(X), \rho_X\rangle$ is an NCA-isomorphism. Let $a, b \in A$ and $a \ll_\rho b$. Then, there exists some $c \in A_0$ such that $a \le c \le b$. Thus, $\varphi(a) \le \varphi(c) \le \varphi(b)$. We have $\varphi(A_0) = CO(X)$; hence, $\varphi(c) \in \mathrm{CO}(X)$. Therefore, $\varphi(a) \subseteq \mathrm{int}(\varphi(b))$, i.e. $\varphi(a) \ll_{\rho_X} \varphi(b)$. Conversely, let $F, G \in \mathrm{RC}(X)$ and $F \ll_{\rho_X} G$, i.e. $F \subseteq \mathrm{int}(G)$. Since $\mathrm{CO}(X)$ is a base of X, F is compact and $\mathrm{CO}(X)$ is closed under finite unions, we obtain that there exists some $U \in \mathrm{CO}(X)$ such that $F \subseteq U \subseteq \mathrm{int}(G) \subseteq G$. Then, $\varphi^{-1}(U) \in A_0$ and $\varphi^{-1}(F) \le \varphi^{-1}(U) \le \varphi^{-1}(G)$. Thus, by the definition of ρ, we obtain $\varphi^{-1}(F) \ll_\rho \varphi^{-1}(G)$. Therefore, $\varphi : \langle A, \rho\rangle \longrightarrow \langle \mathrm{RC}(X), \rho_X\rangle$ is an NCA-isomorphism. Since $\langle \mathrm{RC}(X), \rho_X\rangle = \Lambda^t(X)$ and $\Lambda^a(\varphi) : \Lambda^a(\Lambda^t(X)) \longrightarrow \Lambda^a(\langle A, \rho\rangle)$ is a homeomorphism, we obtain that $\Lambda^a(\langle A, \rho\rangle)$ is homeomorphic to $S^a(A_0)$, using Theorem 2.10.

As we have seen in (a), A_0 is a base for $\langle A, \rho\rangle$, and thus, A_0 is dV-dense in $\langle A, \rho\rangle$. Hence, for proving that $(i_0, \langle A, \rho\rangle)$ is an NCA-completion of $\langle A_0, \rho_s^{A_0}\rangle$, we need only show that $\rho \cap (A_0 \times A_0) = \rho_s^{A_0}$. So, let $a, b \in A_0$. Then,

$$a(-\rho)b \Longleftrightarrow (\exists c \in A_0)(a \le c \le b^*).$$

Clearly, $a(-\rho)b$ implies that $a \wedge b = 0$, i.e., $a(-\rho_s^{A_0})b$. Conversely, if $a(-\rho_s^{A_0})b$, then $a \wedge b = 0$; hence, $a \le b^*$. Since $a \le a \le b^*$ and $a \in A_0$, we obtain that $a(-\rho)b$. Therefore, for every $a, b \in A_0$, we have $a\rho_s^{A_0}b$ if and only if $a\rho b$. □

5. Algebraic density and weight

One may wonder why we do not define the notion of weight of a local contact algebra, or, more generally, of a Boolean algebra, in a much simpler way, based on the following reasoning: if X is a semiregular space, then $\mathrm{RO}(X)$ is a base for X; thus, by Theorem 2.21, $\mathrm{RO}(X)$ contains a subfamily $\mathcal{B}$ such that $\mathcal{B}$ is a base for X and $|\mathcal{B}| = w(X)$; clearly, if X is semiregular, then a subfamily $\mathcal{B}$ of $\mathrm{RO}(X)$ is a base for X if and only if for any $U \in \mathrm{RO}(X)$, we have $U = \bigcup\{V \in \mathcal{B} \mid V \subseteq U\}$.

Having this in mind, it would be natural to define the weight of a Boolean algebra B as the smallest cardinality of subsets M of B such that for each $b \in B$,

$$b = \bigvee\{x \in M \mid x \le b\}.$$

The obtained cardinal invariant is well known in the theory of Boolean algebras as the *density* or *π-weight* (and even *pseudoweight*) of B and is denoted by $\pi w(B)$ (see, e.g., [24, 33, 36]), but we will denote it by $\pi w_a(B)$. So,

$$\pi w_a(B) \stackrel{\text{df}}{=} \min\{|M| \mid (\forall b \in B)(b = \bigvee\{x \in M \mid x \le b\})\}.$$

It is easy to see that $\pi w_a(B)$ is equal to the smallest cardinality of a dense subset of B (see [33, Lemma 4.9.]). Clearly, if B is a dense subalgebra of A, then $\pi w_a(B) = \pi w_a(A)$; in particular, B has the same density as its completion. Observe that a Boolean algebra has infinite π-weight if and only if it is infinite.

However, owing to the fact that in $\mathrm{RO}(X)$ the union is not equal to the join, $\pi w_a(\mathrm{RO}(X))$ may be strictly smaller than the weight of a space X, even when X is semiregular. It is well known that πw_a corresponds to the topological notion of π-weight. Recall that a *π–base* for a topological space $(X, \mathcal{T})$ is a subfamily $\mathcal{P}$ of $\mathcal{T} \setminus \{\emptyset\}$ such that for every $U \in \mathcal{T} \setminus \{\emptyset\}$ there exists some $V \in \mathcal{P}$ with $V \subseteq U$. The cardinal invariant *π-weight* is defined as

$$\pi w(X) \stackrel{\text{df}}{=} \min\{|\mathcal{P}| \mid \mathcal{P} \text{ is a } \pi\text{–base for } X\}.$$

It is easy to see that for a semiregular space X,

$$\pi w(X) = \pi w_a(\mathrm{RO}(X)) = \pi w_a(\mathrm{RC}(X)). \tag{1}$$

Clearly, $\pi w(X) \leq w(X)$, and, as is well known, the inequality may be strict, even for compact Hausdorff spaces. For example, consider $\mathbb{N}$ with the discrete topology, and its Stone-Čech compactification $\beta\mathbb{N}$. Since $\{\{n\} \mid n \in \mathbb{N}\}$ is a π–base for $\beta\mathbb{N}$, we obtain $\pi w(\beta\mathbb{N}) = \pi w_a(\mathrm{RC}(\beta\mathbb{N})) = \aleph_0$. On the other hand, it is well known that $w(\beta\mathbb{N}) = 2^{\aleph_0}$ [27]. The same example shows that πw is not isotone, since $\beta\mathbb{N} \setminus \mathbb{N} \subseteq \beta\mathbb{N}$, and

$$\aleph_0 = \pi w(\beta\mathbb{N}) \lneq \pi w(\beta\mathbb{N} \setminus \mathbb{N}) = 2^{\aleph_0}.$$

Algebraically, the situation is as follows. Let B be the finite–cofinite algebra over $\mathbb{N}$, and $\overline{B}$ its completion; then, $\pi w_a(\overline{B}) = \pi w_a(B) = \aleph_0$. Now, $\overline{B}$ is isomorphic to the set algebra $2^{\mathbb{N}}$ which, in turn, is isomorphic to $\mathrm{RC}(\beta\mathbb{N})$.

In the rest of the section we shall investigate the connections among $w_a, \pi w_a$, and their corresponding topological notions.

Suppose that $\langle B, \rho, \mathbb{B}\rangle$ is an LCA. Obviously, (LC3) implies that $\mathbb{B}$ is dense in B. If D is a dense subset of B, then $D \cap \mathbb{B}$ is a dense subset of B, since $\mathbb{B}$ is an ideal of B. Furthermore, every base for $\langle B, \rho, \mathbb{B}\rangle$ is a dense subset of B; hence,

$$\pi w_a(B) \leq w_a(B, \rho, \mathbb{B}).$$

Proposition 5.1. *Let $\langle B, \rho, \mathbb{B}\rangle$ be an LCA and M be a subset of B. Then the following conditions are equivalent:*

1. *M is a dense subset of $\langle B, \rho, \mathbb{B}\rangle$.*
2. *For each $a \in B^+$ there exists $b \in M^+$ such that $b \ll_\rho a$.*
3. *For each $a \in \mathbb{B}^+$, $a = \bigvee\{b \in M \mid b \ll_\rho a\}$;*
4. *For each $a \in B^+$, $a = \bigvee\{b \in M \mid b \ll_\rho a\}$.*

Proof. The implications

$$1. \iff 2., 3. \iff 4., \text{and } 4. \Rightarrow 1.$$

can be easily obtained using (LC3) or [33, Lemma 4.9.], or the fact that $\mathbb{B}$ is a dense subset of B. So we only show 1. $\Rightarrow$ 4. Let $a \in B^+$; then $a = \bigvee\{b \in M \mid b \leq a\}$ since M is dense in B. Let $a_1 \in B$ and $b \leq a_1$ for every $b \in M$ such that $b \ll_\rho a$. Assume that $a \nleq a_1$. Then $a \wedge a_1^* > 0$. By (LC3) there exists some $c \in M^+$ such that $c \ll_\rho a \wedge a_1^*$, and the density of M implies that there is some $b \in M^+$ with $b \leq c$. Then $b \ll_\rho a \wedge a_1^*$. Thus $b \ll_\rho a$; hence, $b \leq a_1$ by the definition of b. Altogether, we obtain $b \leq a_1 \wedge a_1^* = 0$, a contradiction. It follows that $a \leq a_1$; therefore, $a = \bigvee\{b \in M \mid b \ll_\rho a\}$. □

Definition 5.2. A topological space $(X, \mathcal{T})$ is called *π-semiregular* if the family $\mathrm{RO}(X)$ is a π-base for X.

Clearly, every semiregular space is π-semiregular. The converse is not true. Indeed, the *half–disc topology* from [42, Example 78] is a π-semiregular $T_{2\frac{1}{2}}$-space which is not semiregular. On the other hand, there exist spaces which are not π-semiregular: if X is an infinite set with the cofinite topology then X is not a π-semiregular space since $\mathrm{RO}(X) = \{\emptyset, X\}$.

The following lemma from [24] is an analogue of Theorem 2.21:

Lemma 5.3. ([24]) *If $\mathcal{B}$ is a π-base for a space X then there exists a π-base $\mathcal{B}'$ of X such that $\mathcal{B}' \subseteq \mathcal{B}$ and $|\mathcal{B}'| = \pi w(X)$.*

The next proposition is a generalisation of (1).

Proposition 5.4. *If X is π-semiregular, then $\pi w(X) = \pi w_a(\mathrm{RC}(X))$.*

Proof. Since X is π-semiregular, $\mathrm{RO}(X)$ is a π-base for X. Hence, by Lemma 5.3, there exists a π-base $\mathcal{B}$ of X such that $\mathcal{B} \subseteq \mathrm{RO}(X)$ and $|\mathcal{B}| = \pi w(X)$; obviously, $\mathcal{B}$ is a dense subset of $\mathrm{RO}(X)$ as well. Hence, $\pi w(X) \geq \pi w_a(\mathrm{RO}(X))$, and, clearly, $\pi w(X) \leq \pi w_a(\mathrm{RO}(X)) = \pi w_a(\mathrm{RC}(X))$. □

Proposition 5.5. *Let A be an infinite Boolean algebra. Then there exists a normal contact relation ρ on A such that $w_a(\langle A, \rho\rangle) = \pi w_a(A)$ and $\langle A, \rho\rangle_S$ is a base for $\langle A, \rho\rangle$.*

Proof. There exists a dense subset D of A with $|D| = \pi w_a(A)$. Note that $\pi w_a(A) \geq \aleph_0$. Let B be the Boolean subalgebra of A generated by D. Now, Proposition 4.13 implies that there exists a normal contact relation ρ on A such that $B \stackrel{\text{df}}{=} \langle A, \rho\rangle_S$ is a base for $\langle A, \rho\rangle$ and $w_a(\langle A, \rho\rangle) = |B|$. Since $|B| = |D|$, we obtain $w_a(\langle A, \rho\rangle) = \pi w_a(A)$. $\square$

Theorem 5.6. *Let X be a π-semiregular space and $\pi w(X) \geq \aleph_0$. Then there exists a zero-dimensional compact Hausdorff space Y with $w(Y) = \pi w(X)$ such that the Boolean algebras $\mathrm{RC}(X)$ and $\mathrm{RC}(Y)$ are isomorphic.*

Proof. Set $\tau \stackrel{\text{df}}{=} \pi w(X)$ and $A \stackrel{\text{df}}{=} \mathrm{RC}(X)$. Then, by Proposition 5.4, $\pi w_a(A) = \tau$. Now, by Proposition 5.5, there exists a normal contact relation ρ on A such that $w_a(\langle A, \rho\rangle) = \tau$ and $\langle A, \rho\rangle_S$ is a base for $\langle A, \rho\rangle$. Using Theorems 4.12 and 4.4, we see that $Y \stackrel{\text{df}}{=} \Lambda^a(\langle A, \rho\rangle)$ is a zero-dimensional compact Hausdorff space with $w(Y) = \tau$. Finally, by de Vries' duality theorem, $\mathrm{RC}(Y)$ is isomorphic to A, i.e. to $\mathrm{RC}(X)$. $\square$

Theorem 5.6 is not true for general spaces with infinite π-weight. Indeed, let X be countably infinite with the cofinite topology; then, $\pi w(X) = \aleph_0$, and $\mathrm{RC}(X) = \{\emptyset, X\}$. On the other hand, if Y is a zero-dimensional compact Hausdorff space with $\mathrm{RC}(Y) = \{\emptyset, Y\}$ then $1 = w(Y) < \aleph_0 = \pi w(X)$.

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