The Adjacency-Jacobsthal-Hurwitz Type Numbers

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Abstract. In this paper, we define the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind. Then we give the exponential, combinatorial, permanental and determinantal representations and the Binet formulas of the adjacency-Jacobsthal-Hurwitz numbers of the first and second kind by the aid of the generating functions and the generating matrices of the sequences defined.

1. Introduction

It is well-known that Jacobsthal sequence \( \{J_n\} \) is defined recursively by the equation
\[
J_{n+1} = J_n + 2J_{n-1}
\]
for \( n > 0 \), where \( J_0 = 0 \), \( J_1 = 1 \).

In [5], Deveci and Artun defined the adjacency-Jacobsthal sequence as follows:
\[
J_{m,n} (mn+k) = J_{m,n} (mn - n + k + 1) + 2J_{m,n} (k)
\]
for \( k \geq 1 \), \( m \geq 2 \) and \( n \geq 4 \) with initial constants \( J_{m,n} (1) = \cdots = J_{m,n} (mn - 1) = 0 \) and \( J_{m,n} (mn) = 1 \).

It is easy to see that the characteristic polynomial of the adjacency-Jacobsthal sequence is
\[
f(x) = x^{mn} - x^{mn-n+1} - 2.
\]

Suppose that the \((n+k)\)th term of a sequence is defined recursively by a linear combination of the preceding \( k \) terms:
\[
a_{n+k} = c_0a_n + c_1a_{n+1} + \cdots + c_{k-1}a_{n+k-1}
\]
where \( c_0, c_1, \ldots, c_{k-1} \) are real constants. In [10], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

Let the matrix \( A \) be defined by
Let an nth degree real polynomial \( q \) be given by

\[
q(x) = c_0 x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n.
\]

In [9], the Hurwitz matrix \( H_n = [h_{ij}]_{n \times n} \) associated to the polynomial \( q \) was defined as shown:

\[
H_n = \begin{bmatrix}
0 & c_1 & c_3 & \cdots & \cdots & c_{n-3} & c_{n-1} & 0 \\
0 & c_0 & c_2 & \cdots & \cdots & c_{n-4} & c_{n-2} & c_n \\
0 & 0 & c_1 & \cdots & c_{n-5} & c_{n-3} & c_{n-1} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & c_{n-2} & c_{n} & 0 \\
\end{bmatrix}
\]

Recently, many authors have studied number theoretic properties such as these obtained from homogeneous linear recurrence relations relevant to this paper [3, 4, 6–8, 11, 12, 14, 16–19]. In this paper, we define the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind by using Hurwitz matrix for characteristic polynomial of the adjacency-Jacobsthal sequence of order \( 4m \). Then we develop some their properties such as the generating function, exponential representations, the generating matrices and the combinatorial representations. Also, we give relationships among the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind and the permanents and the determinants of certain matrices which are produced by using the generating matrices of the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind. Finally, we obtain the Binet formulas for the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind by the aid of the roots of characteristic polynomials of the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind.
2. The Main Results

It is readily seen that Hurwitz matrix for characteristic polynomial of the adjacency-Jacobsthal sequence of order 4\(m\), \(H_{J4m}^J\) is defined by
\[
h_{i,j} = \begin{cases} 
1 & \text{if } i = 2t \text{ and } j = t \text{ for } 1 \leq t \leq 2m, \\
-1 & \text{if } i = 1 + 2t \text{ and } j = 2 + t \text{ for } 1 \leq t \leq 2m - 1, \\
-2 & \text{if } i = 2t \text{ and } j = 2m + t \text{ for } 1 \leq t \leq 2m, \\
0 & \text{otherwise}.
\end{cases}
\]

By the aid of the matrix \(H_{J4m}^J\), we define the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind, respectively by:
\[
J_1^{(4m+k)} = J_1^{(2m+k)} - 2J_1^{(k)} \text{ for } k \geq 1 \text{ and } m \geq 4, \quad (1)
\]
where
\[
J_1^{(1)} = 1, J_1^{(2)} = \cdots = J_1^{(2m)} = 0, J_1^{(2m+1)} = 1, J_1^{(2m+2)} = \cdots = J_1^{(4m)} = 0
\]
and
\[
J_2^{(4m+k)} = J_2^{(2m+k)} - 2J_2^{(k)} \text{ for } k \geq 1 \text{ and } m \geq 4, \quad (2)
\]
where
\[
J_2^{(1)} = 1, J_2^{(2)} = \cdots = J_2^{(4m-1)} = 0, J_2^{(4m)} = 1.
\]

Clearly, the generating functions of the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind are given by
\[
g_1(x) = \frac{1}{1 - x^{2m} + 2x^{4m}}
\]
and
\[
g_2(x) = \frac{1 + 3x^{2m}}{1 + 2x^{2m} - x^{4m}}.
\]
respectively. It can be readily established that the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind have the following exponential representations, respectively:
\[
g_1(x) = \exp \left( \sum_{i=1}^{\infty} \frac{(x^{2m})^i}{i} \left( 1 - 2x^{2m} \right)^i \right)
\]
and
\[
g_2(x) = (1 + 3x^{2m}) \exp \left( \sum_{i=1}^{\infty} \frac{(x^{2m})^i}{i} \left( x^{2m} - 2 \right)^i \right).
\]
By equations (1) and (2), we can write the following companion matrices, respectively:

\[
C_m^1 = \begin{bmatrix}
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2 \\
1 & 0 & \cdots & 0 & 0 & 0 & 0 & \vdots \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \vdots \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & \vdots \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0
\end{bmatrix}_{4m \times 4m}
\]

and

\[
C_m^2 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
1 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0
\end{bmatrix}_{(2m + 1) \times 4m}
\]

The companion matrices \(C_m^1\) and \(C_m^2\) are called the adjacency-Jacobsthal-Hurwitz matrices of the first and second kind, respectively. For detailed information about the companion matrices, see [13, 15]. Let \(f_m^1(a)\) and \(f_m^2(a)\) be denoted by \(f_m^{1\alpha}\) and \(f_m^{2\alpha}\). By mathematical induction on \(a\), we derive

\[
(C_m^1)^\alpha = \begin{bmatrix}
f_m^{1\alpha+1} & f_m^{1\alpha+2} & \cdots & f_m^{1\alpha+2m} & -2f_m^{1\alpha-2m+1} & \cdots & -2f_m^{1\alpha-2m+1} & -2f_m^{1\alpha-2m+1} \\
f_m^{1\alpha+1} & f_m^{1\alpha+2} & \cdots & f_m^{1\alpha+2m} & -2f_m^{1\alpha-2m+1} & \cdots & -2f_m^{1\alpha-2m+1} & -2f_m^{1\alpha-2m+1} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
f_m^{1\alpha-4m+2} & f_m^{1\alpha-4m+3} & \cdots & f_m^{1\alpha-2m+1} & -2f_m^{1\alpha-6m+2} & \cdots & -2f_m^{1\alpha-4m+1} & -2f_m^{1\alpha-4m+1}
\end{bmatrix}_{4m \times 4m}
\]

and

\[
(C_m^2)^\alpha = \begin{bmatrix}
f_m^{2\alpha} & f_m^{2\alpha-1} & \cdots & f_m^{2\alpha-2m+1} & f_m^{2\alpha+2} & \cdots & f_m^{2\alpha+2} & f_m^{2\alpha+2} \\
f_m^{2\alpha+1} & f_m^{2\alpha} & \cdots & f_m^{2\alpha-2m+1} & f_m^{2\alpha} & \cdots & f_m^{2\alpha} & f_m^{2\alpha} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
f_m^{2\alpha+4m-1} & f_m^{2\alpha+4m-2} & \cdots & f_m^{2\alpha+2} & f_m^{2\alpha+6m-1} & \cdots & f_m^{2\alpha+4m+1} & f_m^{2\alpha+4m+1}
\end{bmatrix}_{4m \times 4m}
\]

for \(a \geq 1\). Note that \(\det(C_m^1)^\alpha = (2)^\alpha\) and \(\det(C_m^2)^\alpha = (-1)^\alpha\).

Let \(K(k_1, k_2, \ldots, k_v)\) be a \(v \times v\) companion matrix as follows:

\[
K(k_1, k_2, \ldots, k_v) = \begin{bmatrix}
k_1 & k_2 & \cdots & k_v \\
1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0
\end{bmatrix}
\]
Theorem 2.1. (Chen and Louck [2]) The \((i, j)\) entry \(k_{ij}^{(n)}(k_1, k_2, \ldots, k_v)\) in the matrix \(K^n(k_1, k_2, \ldots, k_v)\) is given by the following formula:

\[
k_{ij}^{(n)}(k_1, k_2, \ldots, k_v) = \sum_{t_1, t_2, \ldots, t_v} \frac{t_1 + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} x_1^{t_1} \cdots x_v^{t_v},
\]

where the summation is over nonnegative integers satisfying \(t_1 + 2t_2 + \cdots + vt_v = n - i + j\), \((t_1 + \cdots + t_v) = \frac{(t_1 + \cdots + t_v)}{v}\) is a multinomial coefficient, and the coefficients in (5) are defined to be 1 if \(n = i - j\).

Now we concentrate on finding combinatorial representations for the adjacency-Jacobsthal-Hurwitz numbers of the first and second kind.

Corollary 2.2. The following hold:

(i) \(J_m(n) = \sum_{t_1, t_2, \ldots, t_{2m}} \frac{t_1 + \cdots + t_{2m}}{t_1 + t_2 + \cdots + t_{2m}} x_1^{t_1} \cdots x_{2m}^{t_{2m}} (-2)^{t_{2m}}\) for \(1 \leq \alpha \leq 2m\),

where the summation is over nonnegative integers satisfying \(t_1 + 2t_2 + \cdots + (4m) t_{4m} = n - 1\).

(ii) \(J_m(n) = \sum_{t_1, t_2, \ldots, t_{2m}} \frac{t_1 + \cdots + t_{2m}}{t_1 + t_2 + \cdots + t_{2m}} x_1^{t_1} \cdots x_{2m}^{t_{2m}} (-2)^{t_{2m}}\) for \(1 \leq \alpha \leq 2m\),

where the summation is over nonnegative integers satisfying \(t_1 + 2t_2 + \cdots + (4m) t_{4m} = n - 4m - 1\).

(iii) \(J_m(n) = \sum_{t_1, t_2, \ldots, t_{2m}} \frac{t_1 + \cdots + t_{2m}}{t_1 + t_2 + \cdots + t_{2m}} x_1^{t_1} \cdots x_{2m}^{t_{2m}} (-2)^{t_{2m}}\) for \(1 \leq \alpha \leq 2m\),

where the summation is over nonnegative integers satisfying \(t_1 + 2t_2 + \cdots + (4m) t_{4m} = n\).

Proof. If we take \(i = \alpha + 1\), \(j = \alpha\) such that \(1 \leq \alpha \leq 2m\) for case (i), \(i = 1, j = 4m\) for case (ii) and \(i = j = \alpha\) such that \(1 \leq \alpha \leq 2m\) for case (iii) in Theorem 2.1, then we can directly see the conclusions from equations (3) and (4). □

Definition 2.3. A \(u \times v\) real matrix \(M = \begin{bmatrix} m_{ij} \end{bmatrix}\) is called a contractible matrix in the \(k\)th column (resp. row) if the \(k\)th column (resp. row) contains exactly two nonzero entries.

Let \(u_1, u_2, \ldots, u_{mn}\) be row vectors of the matrix \(M\). If \(M\) is contractible in the \(k\)th column such that \(m_{ij} \neq 0, m_{jk} \neq 0\) and \(i \neq j\), then the \((u - 1) \times (v - 1)\) matrix \(M_{ij}\) obtained from \(M\) by replacing the \(i\)th row with \(m_{ij}x_i + m_{jk}x_j\) and deleting the \(j\)th row. The \(k\)th column is called the contraction in the \(k\)th column relative to the \(i\)th row and the \(j\)th row.

If \(M\) is a real matrix of order \(a > 1\) and \(N\) is a contraction of \(M\), then \(\text{per}(M) = \text{per}(N)\) which was proved in [1].

Now we consider relationships between the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind and the permanents of certain matrices which are obtained by using the generating matrices of these sequences.

Let \(u \geq 4m\) be a positive integer and suppose that \(M_m^{1, u} = \begin{bmatrix} m_{ij}^{1, u} \end{bmatrix}\) and \(M_m^{2, u} = \begin{bmatrix} m_{ij}^{2, u} \end{bmatrix}\) are the \(u \times u\) super-diagonal matrices, defined by

\[
M_m^{1, u} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & \cdots \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 0 & -2 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 0 & -2 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\end{bmatrix}_{u \times u}
\]
and

\[
M^2_{m,u} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & -2 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
\]

\[
\uparrow \quad (u - 4m + 1) \text{ th} \quad \uparrow \quad (u - 2m + 1) \text{ th}
\]

Theorem 2.4. For \( u \geq 4m \),

\[
\operatorname{per}(M^1_{m,u}) = f^1_m(u + 1) \quad \text{and} \quad \operatorname{per}(M^2_{m,u}) = f^2_m(u + 4m).
\]

Proof. Let us consider the matrix \( M^1_{m,u} \) and the adjacency-Jacobsthal-Hurwitz sequence of the first kind. We use induction on \( u \). Now we assume that the equation holds for \( u \geq 4m \), then we show that the equation holds for \( u + 1 \). If we expand the \( \operatorname{per}(M^1_{m,u+1}) \) by the Laplace expansion of permanent according to the first row, then we obtain

\[
\operatorname{per}(M^1_{m,u+1}) = \operatorname{per}(M^1_{m,u-2m+1}) - 2\operatorname{per}(M^1_{m,u-4m+1}).
\]

Since \( \operatorname{per}(M^1_{m,u-2m+1}) = f^1_m(u - 2m + 2) \) and \( \operatorname{per}(M^1_{m,u-4m+1}) = f^1_m(u - 4m + 2) \), it is easy to see that \( \operatorname{per}(M^1_{m,u+1}) = f^1_m(u - 2m + 2) - 2f^1_m(u - 4m + 2) = f^1_m(u + 2) \). So we have the conclusion.

There is a similar proof for the matrix \( M^2_{m,u} \) and the adjacency-Jacobsthal-Hurwitz sequence of the second kind. \( \square \)

Let \( v \geq 4m \) be a positive integer and suppose that the matrices \( A^1_{m,v} = [a^1_{i,j,v}]_{v \times v} \) and \( A^2_{m,v} = [a^2_{i,j,v}]_{v \times v} \) are defined, respectively, by

\[
a^1_{i,j,v} = \begin{cases} 
1 & \text{if } i = j \text{ and } i = j + 2m - 1 \text{ for } 1 \leq i \leq v - 2m + 1 \\
-1 & \text{if } i = j + i + 1 \text{ and } j = i \text{ for } 1 \leq i \leq v - 2m, \\
-2 & \text{if } i = j + i + 1 \text{ and } j = i \text{ for } v - 2m + 1 \leq i \leq v - 1, \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
a^2_{i,j,v} = \begin{cases} 
1 & \text{if } i = j + 2m - 1 \text{ and } j = i + 2m \text{ for } 1 \leq i \leq v - 2m \\
-1 & \text{if } i = j + 4m - 1 \text{ and } j = i \text{ for } 1 \leq i \leq v - 4m + 1, \\
-2 & \text{if } i = j + 2m - 1 \text{ and } j = i \text{ for } 1 \leq i \leq v - 2m + 1, \\
0 & \text{otherwise}.
\end{cases}
\]
Then we can give the permanental representations other than the above by the following theorem.

**Theorem 2.5.** For \( v \geq 4m \),

\[
\text{per}(A_{m}^{v,1}) = -j_{m}^{1}(v + 1) \quad \text{and} \quad \text{per}(A_{m}^{2,v}) = -j_{m}^{2}(v + 4m).
\]

**Proof.** Let us consider the matrix \( A_{m}^{2,v} \) and the adjacency-Jacobsthal-Hurwitz sequence of the second kind. The assertion may be proved by induction on \( v \). Let the equation be hold for \( v \geq 4m \), then we show that the equation holds for \( v + 1 \).

If we expand the \( \text{per}(A_{m}^{2,v+1}) \) by the Laplace expansion of permanent according to the first row, then we obtain

\[
\text{per}(A_{m}^{2,v+1}) = \text{per}(A_{m}^{2,v-4m+1}) - 2\text{per}(A_{m}^{2,v-2m+1})
\]

\[
= -j_{m}^{2}(v + 1) - 2(-j_{m}^{2}(v + 4m + 1)) = -j_{m}^{2}(v + 4m + 1).
\]

Thus we have the conclusion.

There is a similar proof for the matrix \( A_{m}^{1,v} \) and the adjacency-Jacobsthal-Hurwitz sequence of the first kind.

Now we define a \( v \times v \) matrix \( B_{m}^{v} \) as in the following form:

\[
B_{m}^{v} = \begin{bmatrix}
\vdots & \vdots & \vdots & \vdots \\
-1 & \cdots & -1 & 0 & \cdots & 0 \\
-1 & & & 0 & & \vdots \\
\vdots & & & & & \\
0 & & & & A_{m}^{1,v-1}
\end{bmatrix},
\]

then we have the following result:

**Corollary 2.6.** For \( v > 4m + 1 \),

\[
\text{per}B_{m}^{v} = \sum_{i=1}^{v-1} j_{i}^{1}(i).
\]

**Proof.** If we extend the \( \text{per}B_{m}^{v} \) with respect to the first row, we obtain

\[
\text{per}B_{m}^{v} = \text{per}B_{m}^{v-1} + \text{per}A_{m}^{1,v-1}.
\]

From Theorem 2.4, Theorem 2.5 and induction on \( v \), the proof follows directly.

A matrix \( M \) is called convertible if there is an \( n \times n \) \((1, -1)\)-matrix \( K \) such that \( \det(M \circ K) = \text{per}M \), where \( M \circ K \) denotes the Hadamard product of \( M \) and \( K \).

Now assume that the matrices \( T = [t_{i,j}]_{m \times n} \) and \( S = [s_{i,j}]_{p \times q} \) are defined by

\[
T = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
-1 & 1 & 1 & \cdots & 1 & 1 \\
1 & -1 & 1 & \cdots & 1 & 1 \\
\vdots & \ddots & \ddots & \cdots & \ddots & \vdots \\
1 & \cdots & 1 & -1 & 1 & 1 \\
1 & \cdots & 1 & 1 & -1 & 1
\end{bmatrix},
\]

\[
S = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
-1 & 1 & \cdots & 1 & 1 \\
1 & -1 & \cdots & 1 & 1 \\
\vdots & \ddots & \ddots & \cdots & \ddots \\
1 & \cdots & 1 & -1 & 1 \\
1 & \cdots & 1 & 1 & -1
\end{bmatrix}.
\]
and

\[
S = \begin{bmatrix}
1 & -1 & 1 & \ldots & 1 \\
1 & 1 & -1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \ldots & 1 & -1 \\
1 & 1 & \ldots & 1 & 1
\end{bmatrix}
\]

Then we give relationships between the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind and the determinants of the Hadamard products \(M^{1,n}_m \circ T\), \(A^{1,n}_m \circ T\), \(M^{2,n}_m \circ S\) and \(A^{2,n}_m \circ S\).

**Theorem 2.7.** Let \(u, v \geq 4m\), then

\[
\det (M^{1,n}_m \circ T) = j^1_m (u + 1), \\
\det (A^{1,n}_m \circ T) = -j^1_m (v + 1), \\
\det (M^{2,n}_m \circ S) = j^2_m (u + 4m)
\]

and

\[
\det (A^{2,n}_m \circ S) = -j^2_m (v + 4m).
\]

**Proof.** Since \(\det (M^{1,n}_m \circ T) = \det (A^{1,n}_m \circ T) = \det (A^{2,n}_m \circ S) = \det (M^{2,n}_m \circ S)\) and \(A^{2,n}_m \circ S\) for \(u, v \geq 4m\), by Theorem 2.4 and Theorem 2.5, we have the conclusion.

Now we concentrate on finding the Binet formulas for the adjacency-Jacobsthal numbers.

Clearly, the characteristic equations of the matrices \(M^{1,n}_m\) and \(M^{2,n}_m\) are

\[
x^{4m} - x^{2m} + 2 = 0
\]

and

\[
x^{4m} + 2x^{2m} - 1 = 0,
\]

respectively. It is easy to see that the above equations do not have multiple roots. Let \(\{\rho^{(1)}_1, \rho^{(1)}_2, \ldots, \rho^{(1)}_{4m}\}\) and \(\{\rho^{(2)}_1, \rho^{(2)}_2, \ldots, \rho^{(2)}_{4m}\}\) be the sets of the eigenvalues of the matrices \(M^{1,n}_m\) and \(M^{2,n}_m\), respectively and let \(V^{(1)}_m\) be \((4m) \times (4m)\) Vandermonde matrix as follows:

\[
V^{(1)}_m = \begin{bmatrix}
\rho^{(1)}_1^{4m-1} & \rho^{(1)}_2^{4m-1} & \cdots & \rho^{(1)}_{4m}^{4m-1} \\
\rho^{(1)}_1^{4m-2} & \rho^{(1)}_2^{4m-2} & \cdots & \rho^{(1)}_{4m}^{4m-2} \\
\vdots & \vdots & \ddots & \vdots \\
\rho^{(1)}_1 & \rho^{(1)}_2 & \cdots & \rho^{(1)}_{4m}
\end{bmatrix},
\]

where \(\lambda = 1, 2\). Now assume that

\[
W^{(1)}_m(i) = \begin{bmatrix}
\rho^{(1)}_1^{a+4m-i} \\
\rho^{(1)}_2^{a+4m-i} \\
\vdots \\
\rho^{(1)}_{4m}^{a+4m-i}
\end{bmatrix}
\]

and \(V^{(1)}_m(i, j)\) is a \((4m) \times (4m)\) matrix obtained from \(V^{(1)}_m\) by replacing the \(j\)th column of \(V^{(1)}_m\) by \(W^{(1)}_m(i)\).
Theorem 2.8. For \( \alpha \geq 1 \) and \( \lambda = 1, 2 \),
\[
\gamma_{i,j}^{m,\lambda,\alpha} = \frac{\det(V_m^{(i)}(i,j))}{\det(V_m^{(1)})},
\]
where \( (C_m)_{i,j}^{(\alpha)} = \left[c_{i,j}^{m,\lambda,\alpha}\right] \).

Proof. Let us consider \( \lambda = 1 \). Since the equation \( x^{4m} - x^{2m} + 2 = 0 \) does not have multiple roots, \( \beta_1^{(1)}, \beta_2^{(1)}, \ldots, \beta_{4m}^{(1)} \) are distinct and so the matrix \( M_{1,\alpha} \) is diagonalizable. Then, it is readily seen that \( C_m^{1} V_m^{(1)} = V_m^{(1)} \Omega_m^1 \) where \( \Omega_m^1 = (\beta_1^{(1)}, \beta_2^{(1)}, \ldots, \beta_{4m}^{(1)}) \). Since the Vandermonde matrix \( V_m^{(1)} \) is invertible, we can write \( (V_m^{(1)})^{-1} C_m^{1} V_m^{(1)} = \Omega_m^1 \).

Thus, we easily see that the matrix \( C_m^{1} \) is similar to \( \Omega_m^1 \). Then, we have \( (C_m^{1})^{\alpha} V_m^{(1)} = V_m^{(1)} (\Omega_m^1)^{\alpha} \) for \( \alpha \geq 1 \). So we obtain the following linear system of equations:
\[
\begin{align*}
\det(V_m^{(1)}) & = \det(V_m^{(1)}(i,j)).
\end{align*}
\]
There is a similar proof for \( \lambda = 2 \). □

As an immediate consequence of this we have

Corollary 2.9. For \( \alpha \geq 1 \),
\[
\begin{align*}
f_m^{1}(\alpha) &= \frac{\det(V_m^{(k+1)}(k+1,k))}{\det(V_m^{(1)})} \text{ for } 1 \leq k \leq 2m, \\
f_m^{1}(\alpha) &= -\frac{\det(V_m^{(1)}(1,4m))}{2 \det(V_m^{(1)})},
\end{align*}
\]
and
\[
\begin{align*}
f_m^{2}(\alpha) &= \frac{\det(V_m^{(k,k)})}{\det(V_m^{(2)})} \text{ for } 1 \leq k \leq 2m.
\end{align*}
\]

References


